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THE METHOD OF LOEWNER CHAINS IN THE STUDY OF THE UNIVALENCE OF C^2 MAPPINGS

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Abstract. We continue the work of W.C. Royster [26], P.T. Mocanu [20, 21], M. Cristea [4-7], G. Kohr [19], H. Hamada and G. Kohr [14] of extending univalence criteria for holomorphic mappings to C^1 mappings and we continue our work from [7] of improving the method of Loewner chains which is used in complex univalence theory. We show that the method remains valid even for C^2 mappings which are not necessarily holomorphic and we give further applications of our results.

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Key words. Loewner chain, Loewner differential equation, univalent mapping, quasiconformal mapping.

1. INTRODUCTION

We set B the unit ball in $\mathbb{R}^n$ and if $f : B \to \mathbb{R}^n$ is Fréchet differentiable in z, we set $Df(z)$ the real Fréchet derivative of f in z. We shall have in mind the usual identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and also two scalar products on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, namely a real scalar product

$$\langle z, w\rangle_1 = \sum_{k=1}^{2n} z_k w_k \text{ for } z = (z_1, \ldots, z_{2n}) \in \mathbb{R}^{2n},\ w = (w_1, \ldots, w_{2n}) \in \mathbb{R}^{2n}$$

and

$$\langle z, w\rangle_2 = \sum_{k=1}^{n} z_k \overline{w}_k \text{ for } z = (z_1, \ldots, z_n) \in \mathbb{C}^n,\ w = (w_1, \ldots, w_n) \in \mathbb{C}^n,$$

and we see that $\langle a, b\rangle_1 = \mathrm{Re}\langle a, b\rangle_2$ for $a, b \in \mathbb{R}^{2n} \simeq \mathbb{C}^n$. If $D \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ is a domain, $f : D \to \mathbb{C}^n$ is holomorphic,

$$f = (f_1, \ldots, f_n),\ f_k = u_k + \mathrm{i}v_k,\ z = (z_1, \ldots, z_n) \in D,\ z_k = x_k + \mathrm{i}y_k,$$

$k = 1, \ldots, n$, we have the usual identification of f given by

$$F(x_1, y_1, \ldots, x_n, y_n) = (u_1, v_1, \ldots, u_n, v_n)$$

and if $f'(z)$ is the complex derivative of f in z, we have

$$f'(z)(u) = DF(x_1, y_1, \ldots, x_n, y_n)(a_1, b_1, \ldots, a_n, b_n)$$

and

$$||f'(z)||_2 = ||DF(x_1, y_1, \ldots, x_n, y_n)||_1$$

if

$z=(z_1,\dots,z_n),\ u=(u_1,\dots,u_n),\ z_k=x_k+\mathrm{i}y_k,\ u_k=a_k+\mathrm{i}b_k,\ k=1,\dots,n.$
In this way, even if we work with complex functions (see for instance Theorem 6 from this paper), we reduce the problem to real functions.

The other theorems from this paper are given for real functions and using their identification we have analogue enounces for holomorphic functions. We deduce in this way that our results generalize the corresponding theorems from complex univalence theory to C^2 mappings and also that our results hold even on $\mathbb{R}^n$, with $n=2k+1,\ k\in\mathbb{N}$.

We denote by $e_1,\dots,e_n$ the canonical base in $\mathbb{R}^n$, by

$$H_i=\{x\in\mathbb{R}^n|\langle x,e_i\rangle=0\},\ i=1,\dots,n$$

and by $P_i:\mathbb{R}^n\to H_i$ the canonical projection on H_i for $i=1,\dots,n$. If $D\subset\mathbb{R}^n$ is a domain, we say that f is ACL if for every cube $Q\subset\subset D$ with the sides parallel to coordinate axis it results that $f|P_i^{-1}(y)\cap Q:P_i^{-1}(y)\cap Q\to\mathbb{R}^n$ is absolutely continuous for a.e. $y\in Q_i,\ i=1,\dots,n$, where Q_i is the face of Q which is perpendicular on e_i for $i=1,\dots,n$. An ACL map has a.e. partial derivatives and if $\frac{\partial f}{\partial x_i}\in L^p_{loc}(D)$ for $i=1,\dots,n,\ p\geq 1$, we say that f is ACL^p on D. We denote by $W^{1,p}_{loc}(D,\mathbb{R}^n)$ the Sobolev space of all functions $f:D\subset\mathbb{R}^n\to\mathbb{R}^n$ which are locally in L^p together with their first order partial derivatives. Using Proposition 1.2, page 6 from [25] we see that if $f\in C(D,\mathbb{R}^n)$ and $p>1$, then the weak and classical partial derivatives coincide a.e. and f is ACL^p on D if and only if $f\in W^{1,p}_{loc}(D,\mathbb{R}^n)$. We say that $f:D\to\mathbb{R}^n$ is quasiregular if f is ACL^n on D and there exists $K\geq 1$ so that $||f'(x)||^n\leq K\cdot J_f(x)$ a.e. in D. Here $f'(x)$ denotes the weak derivative of f in x and $J_f(x)$ denotes the weak jacobian of f in x. A nonconstant quasiregular map f is a.e. differentiable and $J_f(x)\neq 0$ a.e. We recommend the monographs [25], [28], [29] for the basic properties of quasiregular mappings. If $A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $\det A\neq 0$, we set $l(A)=\inf\limits_{||x||=1}||A(x)||,\ ||A||=\sup\limits_{||x||=1}||A(x)||,$

$$H(A)=\frac{||A||}{l(A)},\ K_0(A)=\frac{||A||^n}{|\det A|},\ K_I(A)=\frac{|\det A|}{l(A)^n},$$

and we see that $H(A)\leq K_0(A)$. If $D\subset\mathbb{R}^n$ is a domain and $f:D\to\mathbb{R}^n$ is a.e. differentiable and $J_f(x)\neq 0$ a.e. we set $K_0(f)=\operatorname{ess\,sup}K_0(f'(x)),\ K_I(f)=\operatorname{ess\,sup}K_I(f'(x))$. If $f:D\to\mathbb{R}^n$ is quasiregular and $K_0(f)\leq K,\ K_I(f)\leq K$, we say that f is K-quasiregular and if in addition $f:D\to f(D)$ is a homeomorphism, we say that f is K quasiconformal. If $f\in C^1(D,D')$ is K quasiconformal with $J_f(x)\neq 0$ on D and we set $H(x,f)=\frac{||f'(x)||}{l(f'(x))}$ for $x\in D$ we see that $H(x,f)\leq K$ for every $x\in D$. We set $S^n=\{x\in\mathbb{R}^n|||x||=1\}$.

The following generalization of Loewner's equation was proved in [7]:

Theorem A. *Let $K=\mathbb{R},\mathbb{C},E$ a Hilbert space over the field K, $b\in(0,\infty]$, $h:B\times(0,\infty)\to E$ continuous so that:*

(1) *For every* $0 < s < a < b$ *and every* $0 < r < 1$ *there exists* $K(s, a, r)$ *so that* $||h(z, t)|| \leq K(s, a, r)$ *for every* $s \leq t \leq a$ *and every* $z \in \overline{B}(0, r)$.

(2) *For every* $0 < s < a < b$ *and every* $0 < r < 1$ *there exists* $M(s, a, r)$ *so that* $||h(z, t) - h(w, t)|| \leq M(s, a, r) \cdot ||z - w||$ *for every* $s \leq t \leq a$ *and every* $z, w \in \overline{B}(0, r)$.

(3) *For every* $0 < s < b$ *there exists* $0 \leq r_s < 1$ *so that* $\mathrm{Re}\langle h(z, t), z\rangle \geq 0$ *for every* $s \leq t < b$ *and every* $z \in B \backslash B(0, r_s)$.

Then the Loewner equation

$$(*) \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -h(v, t),\ v(s) = z,\ 0 < s < b,\ z \in B$$

has an unique solution v_z *on* $[0, b)$. *If* $r_s = 0$, *then* $||v_z(t)|| \leq ||z||$ *for* $z \in B$ *and every* $s \leq t < b$ *and if* $\mathrm{Re}\langle h(z, t), z\rangle \geq c \cdot ||z||^2$ *for* $s \leq t \leq a$, *then* $||v_z(t)|| \leq ||z|| \cdot \mathrm{e}^{-c(t-s)}$ *for* $s \leq t \leq a$.

The result extends known facts from the method of Loewner chains used in complex univalence theory (see Theorem 8.1.3, page 298 from [12]). We recommend the monograph of I. Graham and G. Kohr [12] for the applications of the method of Loewner chains to complex univalence theory. See also the strong contributions of Ch. Pommerenke [22] and J.A. Pfalzgraff [23]. The main instrument used in [7] for proving univalence theorems for C^1 mappings was the following theorem:

Theorem B. *Let* $n \geq 2$, $g : B \to \mathbb{R}^n$ *a continuous, light map,* $f \in C^1(B \times (0, \infty), \mathbb{R}^n)$, $f_t : B \to \mathbb{R}^n$ *given by* $f_t(z) = f(z, t)$ *for* $(z, t) \in B \times (0, \infty)$ *so that*

(4) $\frac{\partial f}{\partial t}(z, t) = Df_t(z)(h_t(z))$ *for* $(z, t) \in B \times (0, \infty)$, *where* $h : B \times (0, \infty) \to \mathbb{R}^n$ *satisfies conditions* (1), (2), (3) *and* $h_t(z) = h(z, t)$ *for* $(z, t) \in B \times (0, \infty)$.

(5) *There exists continuous mappings* $\lambda_t : B \to \mathbb{R}^n$ *for* $0 < t < \infty$ *so that for every* $0 < r < 1$ *there exists* $t_r > 0$ *so that the mappings* λ_t *are injective on* $\overline{B}(0, r)$ *for* $t_r < t < \infty$ *and for every* $\varepsilon > 0$ *there exists* $t_r < \delta_{\varepsilon,r}$ *so that* $||f_t(z) - \lambda_t(z)|| \leq \varepsilon$ *on* $\overline{B}(0, r)$ *for* $\delta_{\varepsilon,r} < t < \infty$.

(6) $f_t \to g$ *uniformly on the compact subsets of* B.

Then g *is injective on* B. *If the following conditions hold:*

(7) *There exists* $c > 0$ *so that* $\mathrm{Re}\langle h(z, t), z\rangle \geq c||z||^2$ *for every* $(z, t) \in B \times (0, \infty)$.

(8) *There exists* $M > 0$ *so that* $||h(z, t)|| \leq M \cdot ||z||$ *for every* $(z, t) \in B \times (0, \infty)$.

(9) f *extends by continuity on* $\overline{B} \times (0, \infty)$.

(10) *There exists* $K \geq 1$ *so that all the mappings* f_t *are* K *quasiconformal on* B.

Then there exists $F : \mathbb{R}^n \to \mathbb{R}^n$ Q *quasiconformal so that* $F|B = g$.

Since there is a gap in the proof in Theorem 4 from [7] which says that if $g : B \to \mathbb{R}^n$ is a C^1 quasiconformal map, then there exists $F : \mathbb{R}^n \to \mathbb{R}^n$

Q quasiconformal so that $F|B = g$, it results that also Theorem 5 from [7] remains partially true. Indeed, using word by word the proof from Theorem 5 from [7], we have:

THEOREM C. *Let $n \geq 2$, $k \geq 1$, $g \in C^2(B, \mathbb{R}^n)$ a light map so that $g(0) = 0$, $G : B \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ a C^{k+1} map so that $\det G(z) \neq 0$ on B, $G(0) = I$, $||G^{-1}(0) \circ Dg(0) - I|| < 1$ and*

$$||||z||^{k+1} \cdot (G(z)^{-1} \circ Dg(z) - I) + (1 - ||z||^{k+1}) \cdot G(z)^{-1} \circ DG(z)(z, \cdot)|| < 1 \text{ on } B.$$

Then g is injective on B.

2. APPLICATIONS OF LOEWNER'S METHOD TO UNIVALENCE CRITERIA

Theorem C extends some results from [24] and [16]. If $G(z) = Dh(z)$ on B, we extend a result from [9].

THEOREM 1. *Let $n \geq 2$, $k \geq 1$, $g \in C^2(B, \mathbb{R}^n)$ a light map so that $g(0) = 0$, let $h \in C^{k+2}(B, \mathbb{R}^n)$ be so that $J_h(z) \neq 0$ on B,*

$$Dh(0) = I, \ ||Dh(0)^{-1} \circ Dg(0) - I|| < 1$$

and

$$||||z||^{k+1} \cdot (Dh(z)^{-1} \circ Dg(z) - I) + (1 - ||z||^{k+1}) Dh(z)^{-1} \circ D^2h(z)(z, \cdot)|| < 1$$

on B. Then g is injective on B.

If $G(z) = Dg(z)$, Theorem C extends the known univalence result of Becker.

THEOREM 2. *Let $n \geq 2$, $k \geq 1$, $g \in C^{k+2}(B, \mathbb{R}^n)$ be so that $g(0) = 0$, $Dg(0) = I$, $Jg(z) \neq 0$ on B and*

$$||(1 - ||z||^{k+1}) \cdot (Dg(z)^{-1} \circ D^2g(z)(z, \cdot))|| < 1 \text{ on } B.$$

Then g is injective on B.

We can also prove in this case a quasiconformal extension result.

THEOREM 3. *Let $n \geq 2$, $k \geq 1$, $0 < c < 1$, $g \in C^{k+2}(B, \mathbb{R}^n)$ a light map so that $g(0) = 0$, $Dg(0) = I$, g is K quasiconformal on B and*

$$||(1 - ||z||^{k+1}) \cdot Dg(z)^{-1} \circ D^2g(z)(z, \cdot)|| \leq c \text{ on } B.$$

Then there exists $F : \mathbb{R}^n \to \mathbb{R}^n$ Q quasiconformal so that $F|B = g$.

Proof. We use the method from Theorem 5 from [7].

Let $f : B \times (0, \infty) \to \mathbb{R}^n$ be given by

$$f(z,t) = g(ze^{-t}) + (e^{kt} - e^{-t}) Dg(ze^{-t})(z) \text{ for } (z,t) \in B \times (0, \infty).$$

We see that $f_t \to g$ uniformly on the compact subsets of B, hence f satisfies condition (6) and as in Theorem 5 from [7] we show that it also satisfies condition (5). Let

$$H(z,t) = (1 - e^{-(k+1)t}) Dg(ze^{-t})^{-1} \circ D^2g(ze^{-t})(ze^{-t}, \cdot) \text{ for } (z,t) \in B \times (0, \infty).$$

We see that $D(f_t)(z) = \mathrm{e}^{kt} Dg(z\mathrm{e}^{-t})(I - H(z,t))$,

$$\frac{\partial f}{\partial t}(z,t) = \mathrm{e}^{kt} Dg(z\mathrm{e}^{-t})(kI + H(z,t))(z) \text{ for } (z,t) \in B \times (0,\infty).$$

Let $E(z) = (1 - |z|^{k+1})(kI + Dg(z)^{-1} \circ D^2 g(z)(z,\cdot))$ for $z \in B$. We see that $||H(z,t)|| \leq E(z\mathrm{e}^{-t}) \leq c < 1$ for $(z,t) \in B \times (0,\infty)$, hence there exists $(I - H(z,t))^{-1}$ for $(z,t) \in B \times (0,\infty)$. Let $h : B \times (0,\infty) \to \mathbb{R}^n$,

$$h(z,t) = (I - H(z,t))^{-1} \circ (kI + H(z,t))(z) \text{ for } (z,t) \in B \times (0,\infty)$$

and let $h_t(z) = h(z,t)$ for $(z,t) \in B \times (0,\infty)$. We see that

$$\frac{\partial f}{\partial t}(z,t) = D(f_t)(z)(h_t(z)) \text{ for } (z,t) \in B \times (0,\infty),$$

hence f satisfies condition (4). Using relations (12) and (13) from [7] we see that $\mathrm{Re}\langle h_t(z), z\rangle \geq \frac{k^2-c^2}{2(k+c^2)} \cdot ||z||^2$ and $||h(z,t)|| \leq \frac{k+c}{1-c} \cdot ||z||$ for $(z,t) \in B \times (0,\infty)$, hence f also satisfies conditions (1), (2), (3), (7), (8). We also see that f satisfies condition (9).

We see that if $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\det A \neq 0$, $\det B \neq 0$, then $l(A) \cdot l(B) \leq l(A \circ B)$ and $||A \circ B|| \leq ||A|| \circ ||B||$. We see that $||I - H(z,t)|| \leq 1 + c$ and $||(I - H(z,t))(u)|| \geq ||u|| - ||H(z,t)(u)|| \geq 1 - ||H(z,t)|| \cdot ||u|| \geq 1 - c$ for $u \in S^n$, hence $l(I - H(z,t) \geq 1 - c$ for $(z,t) \in B \times (0,\infty)$. Then

$$H(z, f_t) = \frac{||Df_t(z)||}{l(Df_t(z))} = \frac{||\mathrm{e}^{kt} \cdot Dg(z\mathrm{e}^{-t}) \circ (I - H(z,t))||}{l(\mathrm{e}^{kt} Dg(z\mathrm{e}^{-t})) \circ (I - H(z,t))}$$

$$\leq \frac{\mathrm{e}^{kt} \cdot ||Dg(z\mathrm{e}^{-t})|| \cdot ||I - H(z,t)||}{\mathrm{e}^{kt} \cdot l(Dg(z\mathrm{e}^{-t})) \cdot l(I - H(z,t))} \leq H(Dg(z\mathrm{e}^{-t})) \cdot \frac{1+c}{1-c}$$

$$\leq K_0(Dg(z\mathrm{e}^{-t})) \cdot \frac{1+c}{1-c} \leq K \cdot \frac{1+c}{1-c},$$

for every $z \in B$ and every $t > 0$. It results that f satisfies condition (10) and using Theorem B, we find $F : \mathbb{R}^n \to \mathbb{R}^n$ Q quasiconformal so that $F|B = g$. □

We can also extend a result of Brodskii [2] and some results of P. Curt [8] and G. Kohr and H. Hamada [16].

THEOREM 4. *Let* $g \in C^2(B, \mathbb{R}^n)$ *be so that* $g(0) = 0$, $J_g(z) \neq 0$ *on* B *and there exists* $0 < c \leq 1$ *so that* $||Dg(z) - I|| < c$ *on* B. *Then* g *is injective on* B, *and if* $c < 1$ *and* g *is* K *quasiconformal, there exist* $F : \mathbb{R}^n \to \mathbb{R}^n$ Q *quasiconformal so that* $F|B = g$.

Proof. Suppose that $c < 1$. Let $f : B \times (0,\infty) \to \mathbb{R}^n$ be given by $f(z,t) = g(z\mathrm{e}^{-t}) + (\mathrm{e}^t - \mathrm{e}^{-t})z$ for $(z,t) \in B \times (0,\infty)$ and let

$$H(z,t) = \mathrm{e}^{-2t}(I - Dg(z\mathrm{e}^{-t})) \text{ for } (z,t) \in B \times (0,\infty).$$

We see that $||H(z,t)|| \leq c < 1$, hence these exists $(I - H(z,t))^{-1}$ for $(z,t) \in B \times (0,\infty)$. Let $h : B \times (0,\infty) \to \mathbb{R}^n$, $h(z,t) = (I - H(z,t))^{-1} \circ (I + H(z,t))(z)$ for $(z,t) \in B \times (0,\infty)$ and let $h_t(z) = h(z,t)$ for $(z,t) \in B \times (0,\infty)$.

Then $Df_t(z) = \mathrm{e}^t(I - H(z,t))$, $\frac{\partial f}{\partial t}(z,t) = \mathrm{e}^t(I + H(z,t))(z)$ and $\frac{\partial f}{\partial t}(z,t) = Df_t(z)(h_t(z))$ for $(z,t) \in B \times (0,\infty)$.

Since $||H(z,t)|| \leq c$, we use relations (13) and (14) from [7] to see that $\mathrm{Re}\langle h(z,t), z\rangle \geq \frac{1-c^2}{2(1+c^2)} \cdot ||z||^2$ and $||h(z,t)|| \leq \frac{1+c}{1-c}||z||$ for $(z,t) \in B \times (0,\infty)$. Also $H(z, f_t) = \frac{||\mathrm{e}^t(I-H(z,t))||}{l(\mathrm{e}^t(I-H(z,t)))} \leq \frac{1+c}{1-c}$ for $(z,t) \in B \times (0,\infty)$. We apply now Theorem B. □

If $c = 1$, the result is given by the following more generally and quite elementary theorem:

THEOREM 5. *Let $D \subset \mathbb{R}^n$ be a convex domain and $g \in C^1(D, \mathbb{R}^n)$ so that $J_g(z) \neq 0$ on D and $||Dg(z) - I|| < 1$ on D. Then g is injective on D.*

Proof. We see that $||Dg(z)(u)||^2 - 2\mathrm{Re}\langle Dg(z)(u), u\rangle + 1 = ||Dg(z)(u) - u||^2 < 1$ if $z \in D$ and $u \in S^n$, hence $\mathrm{Re}\langle Dg(z)(u), u\rangle > 0$ for every $z \in D$ and every $u \in S^n$. Let $z, w \in D$ be so that $g(z) = g(w)$ and let $h : [0,1] \to \mathbb{R}^n$ be given by $h(t) = g((1-t)w + tz)$ for $t \in [0,1]$. Then $0 = \mathrm{Re}\langle g(z) - g(w), z - w\rangle = \mathrm{Re}\langle h(1) - h(0), z - w\rangle = \mathrm{Re}\left\langle \int_0^1 h'(t)\mathrm{d}t, z - w \right\rangle = \int_0^1 \mathrm{Re}\langle h'(t), z - w\rangle \mathrm{d}t$ $= \int_0^1 \mathrm{Re}\langle Dg((1-t)z + tw)(z-w), z - w\rangle \mathrm{d}t > 0$ if $z \neq w$. It results that $z = w$ and hence g is injective on D. □

If $g \in H(B)$, we have a quasiconformal extension result in the case of Theorem C.

THEOREM 6. *Let $n \geq 2$, $k \geq 1$, $g \in H(B)$ be quasiregular, nonconstant with $g(0) = 0$, let $G : B \to \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ holomorphic so that $G(0) = I$, $\det G(z) \neq 0$ for $z \in B$, there exists $K \geq 1$ so that $||G(z)||^n \leq K \cdot |\det G(z)|$ for every $z \in B$ and there exists $0 < c < 1$ so that*

$$||||z||^{k+1} \cdot (G(z)^{-1} \circ Dg(z) - I) + (1 - ||z||^{k+1}) \cdot (G(z)^{-1} \circ DG(z)(z, \cdot))|| \leq c \text{ on } B.$$

Then there exists $F : \mathbb{C}^n \to \mathbb{C}^n$ Q - quasiconformal so that $F|B = g$.

Proof. We see from Theorem C that g is quasiconformal and let $f : B \times (0,\infty) \to \mathbb{R}^n$, $f(z,t) = g(z\mathrm{e}^{-t}) + (\mathrm{e}^{kt} - \mathrm{e}^{-t})G(z\mathrm{e}^{-t})(z)$ for $(z,t) \in B \times (0,\infty)$. Let $H : \overline{B} \times (0,\infty) \to \mathbb{C}^n$, $H(z,t) = -((\mathrm{e}^{-(k+1)t}(G(z\mathrm{e}^{-t})^{-1} \circ Dg(z\mathrm{e}^{-t}) - I)$ $+(1 - \mathrm{e}^{-(k+1)t})G(z\mathrm{e}^{-t})^{-1} \circ DG(z\mathrm{e}^{-t})(z\mathrm{e}^{-t}, \cdot)$ for $z \in \overline{B}$, $t \geq 0$ and

$$E(z) = ||z||^{k+1} \cdot (G^{-1}(z) \circ Dg(z) - I) + (1 - ||z||^{k+1}) \cdot G(z)^{-1} \circ DG(z)(z, \cdot)$$

for $z \in B$. Then $||H(z,t)|| = ||E(z\mathrm{e}^{-t})|| \leq c < 1$ if $z \in S^n$, $t > 0$ and applying the maximum principle we see that $||H(z,t)|| \leq c$ on B for every $t > 0$. It results that there exists $(I - H(z,t))^{-1}$ for $(z,t) \in B \times (0,\infty)$ and let $h(z,t) = (I - H(z,t))^{-1} \circ (kI + H(z,t))(z)$ for $(z,t) \in B \times (0,\infty)$. Then $Df_t(z) = \mathrm{e}^{kt} \cdot G(z\mathrm{e}^{-t})(I - H(z,t))$, $\frac{\partial f}{\partial t}(z,t) = \mathrm{e}^{kt} \cdot G(z\mathrm{e}^{-t})(kI + H(z,t))(z)$ and $\frac{\partial f}{\partial t}(z,t) = D(f_t)(z)(h(z,t))$ for $(z,t) \in B \times (0,\infty)$ and using relations

(12) and (13) from [7] we see that $\mathrm{Re}\langle h(z,t),z\rangle \geq \frac{k^2-c^2}{2(k+c^2)}||z||^2$ and $||h(z,t)|| \leq \frac{k+c}{1-c}||z||$ for $(z,t) \in B \times (0,\infty)$. We have that

$$H(z,f_t) = \frac{||Df_t(z)||}{l(Df_t(z))} = \frac{||\mathrm{e}^{kt}\cdot G(z\mathrm{e}^{-t})(I-H(z,t))||}{l(\mathrm{e}^{kt}\cdot G(z\mathrm{e}^{-t}))\circ(I-H(z,t))}$$

$$\leq \frac{||G(z\mathrm{e}^{-t})||}{l(G(z\mathrm{e}^{-t}))}\cdot\frac{||I-H(z,t)||}{l(I-H(z,t))} \leq H(G(z\mathrm{e}^{-t}))\cdot\frac{1+c}{1-c}$$

$$\leq K_0(G(z\mathrm{e}^{-t}))\cdot\frac{1+c}{1-c} \leq K\cdot\frac{1+c}{1-c} \text{ for } (z,t)\in B\times(0,\infty).$$

We apply now Theorem B to find $F:\mathbb{C}^n \to \mathbb{C}^n$ Q quasiconformal so that $F|B = g$. □

REMARK 1. The result extends a similar one from [24] and [16] established in the case $k=1$ (see also Example 8.5.4 from [12]). The important instrument we used in the case of holomorphic mappings was the maximum principle and this allowed us to find that $||H(z,t)|| \leq c$ for $z\in B,\ t>0$.

3. APPLICATIONS OF LOEWNER'S DIFFERENTIAL EQUATION TO THE STUDY OF THE GROWTH OF THE MODULUS OF C^2 MAPPINGS

Let $n\geq 2$, $D\subset\mathbb{R}^n$ a set with $0\in D$, $\Phi: D\to\mathbb{R}^n$ a C^2 map so that $\Phi(0)=0$. We say that D is Φ like if the equation $\frac{dw}{\mathrm{d}t} = -\Phi(w),\ w(0)=z$ has an unique solution $w_z:[0,\infty)\to D$ for every $z\in D$. If $A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, then the equation $\frac{dw}{\mathrm{d}t} = -A(w),\ w(0)=z$ has the unique solution $w_z(t)=\mathrm{e}^{-tA}\cdot z$ for $z\in B$, $t\geq 0$. If $m(A)=\inf\limits_{||z||=1}\mathrm{Re}\langle A(z),z\rangle>0$, then $\mathrm{Re}\langle A(z),z\rangle \geq m(A)\cdot ||z||^2$ for every $z\in B$ and we see from Remark 3 from [7] that $\lim\limits_{t\to\infty} w_z(t) = \lim\limits_{t\to\infty}\mathrm{e}^{-tA}\cdot z = 0$. A set $D\subset\mathbb{R}^n$ with $0\in D$ which is A like is called of spirallike type and this is equivalent with the fact that $\mathrm{e}^{-tA}\cdot z\in D$ for every $z\in D$ and every $t\geq 0$. If $A=I$, a set $D\subset\mathbb{R}^n$ with $0\in D$ is I like if and only is starlike, i.e. if $[0,z]\subset D$ for every $z\in D$. If $X\subset\mathbb{R}^n$ is a C^1 manifold with boundary, $\dim X = n$ and $x\in\partial X$, we set $I-TX_x = \{u\in\mathbb{R}^n\backslash T(\partial X)_x|$ there exists $\gamma: [0,1]\to X$ a C^1 path so that $\gamma(0)=x$ and $\gamma'(0)=u\}$.

Let $n\geq 2$, $g\in C^2(B,\mathbb{R}^n)$ so that $g(0)=0$, $J_g(z)\neq 0$ on B and let $\Phi\in C^1(g(B),\mathbb{R}^n)$ be so that $\Phi(0)=0$. We say that g is Φ like if $\mathrm{Re}\langle Dg(z)^{-1}\circ\Phi(g(z)),z\rangle > 0$ on $B\backslash\{0\}$. We say that g is asymptotic Φ like if g is injective, $g(\overline{B}(0,r))$ is Φ like for every $0<r<1$ and the unique solution $w_z:[0,\infty)\to g(\overline{B}(0,||z||))$ of the equation $\frac{dw}{\mathrm{d}t} = -\Phi(w),\ w(0)=g(z)$ is so that $w_z'(0)\in I-T(g(\overline{B}(0,||z||)))_{g(z)}$ for every $z\in B$. The relation $w_z'(0)\in I-T(g(\overline{B}(0,||z||)))_{g(z)}$ says that the path $w_z:[0,\infty)\to g(\overline{B}(0,||z||))$ and the $n-1$ manifold $g(S(0,||z||))$ are transversal in the point z. If $\Phi(z)=A$ for $z\in B$, where $A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ and g is asymptotic Φ like, we say that g is asymptotic spirallike, and if $A=I$, we say that g is asymptotic starlike. These definitions are similar with those from [12], Definition 6.4.1 and from [13], Definition 2.1 and the next theorems extend similar results from the theory of holomorphic mappings (see Theorem 6.4.5 and 6.4.7 from [12]).

THEOREM 7. *Let* $n \geq 2$, $g \in C^2(B, \mathbb{R}^n)$ *be so that* $g(0) = 0$, $J_g(z) \neq 0$ *on* B *and let* $\Phi \in C^1(g(B), \mathbb{R}^n)$ *be so that* $\Phi(0) = 0$ *and* g *is* Φ *like. Then* $g(B)$ *is* Φ *like. Let* $h : B \to \mathbb{R}^n$ *be defined by* $h(z) = Dg(z)^{-1} \circ \Phi(g(z))$ *for* $z \in B$ *and the equations*

$$(*) \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -h(v), v(0) = z$$

$$(**) \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = -\Phi(w), w(0) = g(z), z \in B$$

Suppose that one of the following conditions hold:

a) *Every solution* $v_z : [0, \infty) \to \mathbb{R}^n$ *of the equation* $(*)$ *is so that*

$$\lim_{t \to \infty} v_z(t) = 0 \text{ for every } z \in B.$$

b) *There exists* $c > 0$ *so that* $\mathrm{Re}\langle h(z), z\rangle \geq c||z||^2$ *for every* $z \in B$.

c) $g^{-1}(g(0)) = \{0\}$ *and every solution* $w_z : [0, \infty) \to \mathbb{R}^n$ *of the equation* $(**)$ *is so that* $\lim_{t \to \infty} w_z(t) = 0$.

d) $g^{-1}(g(0)) = \{0\}$ *and there exists* $c > 0$ *so that* $\mathrm{Re}\langle \Phi(w), w\rangle \geq c \cdot ||w||^2$ *for every* $w \in g(B)$.

Then g *is univalent on* B *and* g *is asymptotic* Φ *like.*

Proof. Let $z \in B$. Since $\mathrm{Re}\langle h(x), x\rangle > 0$ on $B \setminus \{0\}$, we see from Theorem A that there exists an unique solution v_z of equation $(*)$ and $||v_z(t)|| \leq ||z||$ for $t \geq 0$. Let $w_z : [0, \infty) \to \mathbb{R}^n$, $w_z = g \circ v_z$. Then w_z is well defined, $\mathrm{Im} w_z \subset g(B)$, $w_z(0) = g(z)$ and

$$\frac{\mathrm{d}w_z}{\mathrm{d}t} = Dg(v_z(t)) \cdot \frac{\mathrm{d}v_z}{\mathrm{d}t} = Dg(v_z(t))(-h(v_z(t))$$

$$= -Dg(v_z(t)) \circ (Dg(v_z(t)))^{-1} \cdot \Phi(g(v_z(t))) = -\Phi(w_z(t)) \text{ for } t \geq 0,$$

hence w_z is the unique solution of equation $(**)$ and hence $g(B)$ is Φ like.

Suppose that condition a) holds. Let $a, b \in B$ be so that $g(a) = g(b)$ and let w_a, w_b be the solutions of equation $(**)$. Since $w_a(0) = g(a) = g(b) = w_b(0)$, it results that $w_a(t) = w_b(t)$ for $t \geq 0$. Let $\varepsilon > 0$ be so that g is univalent on $B(0, \varepsilon)$ and let $t_\varepsilon > 0$ be so that $v_a(t) \in B(0, \varepsilon)$, $v_b(t) \in B(0, \varepsilon)$ for $t \geq t_\varepsilon$. Then $g(v_a(t_\varepsilon)) = w_a(t_\varepsilon) = w_b(t_\varepsilon) = g(v_b(t_\varepsilon))$, $v_a(t_\varepsilon), v_b(t_\varepsilon) \in B(0, \varepsilon)$ and g is injective on $B(0, \varepsilon)$, hence $v_a(t_\varepsilon) = v_b(t_\varepsilon)$. Since $g \circ (v_a|[0, t_\varepsilon]) = g \circ (v_b|[0, t_\varepsilon]) = w_a[0, t_\varepsilon]$ and g is a local homeomorphism, we use the property of the uniqueness of path lifting to find that $v_a(t) = v_b(t)$ for $t \in [0, t_\varepsilon]$. It results that $a = v_a(0) = v_b(0) = b$, hence g is injective on B.

Suppose that condition b) holds and let $c > 0$ be so that $\mathrm{Re}\langle h(z), z\rangle \geq c \cdot ||z||^2$ for $z \in B$. Using Theorem A, we see that the unique solution v_z :

$[0,\infty)\to\mathbb{R}^n$ of the equation $(*)$ is so that $||v_z(t)||\leq||z||\cdot\mathrm{e}^{-ct}$ for $t\geq 0$, hence $\lim\limits_{t\to\infty}v_z(t)=0$ and we apply the preceding step.

Suppose that condition c) holds. Let $z\in B$ be fixed, let $v_z:[0,\infty)\to\mathbb{R}^n$ be the unique solution of equation $(*)$ and let $w_z=g\circ v_z$. Then w_z is the unique solution of equation $(**)$, hence $\lim\limits_{t\to\infty}w_z(t)=0$. Since $||v_z(t)||\leq||z||$ for $t\geq 0$, we see that $v_z:[0,\infty)\to B(0,||z||)$ has at least a limit point, and if $b\in B$ is such a limit point than $g(b)=0$ and hence $b=0$. It results that $\lim\limits_{t\to\infty}v_z(t)=0$ and using condition a), we see that g is injective on B.

Suppose now that condition d) holds. Using Theorem A, we see that the equation $(**)$ has a unique solution $w_z:[0,\infty)\to\mathbb{R}^n$ so that

$$||w_z(t)||\leq||z||\mathrm{e}^{-ct}\text{ for every }z\in B\text{ and every }t\geq 0,$$

hence $\lim\limits_{t\to\infty}w_z(t)=0$. We use now condition c) to see that g is injective on B.

Suppose now that one of the conditions a), b), c), d) hold. Then g is injective on B. Let $z\in B$, $r=||z||$, let $v_z:[0,\infty)\to\mathbb{R}^n$ be the unique solution of equation $(*)$ and let $w_z=g\circ v_z$. Then w_z is the unique solution of equation $(**)$ and $w_z'(0)=Dg(z)(v_z'(0))$. Let $\rho:[0,\infty)\to\mathbb{R}_+$, $\rho(t)=||v_z(t)||^2$ for $t\geq 0$. Then

$$\rho'(t)=2\mathrm{Re}\langle v_z'(t),v_z(t)\rangle=-2\mathrm{Re}\langle h(v_z(t)),v_z(t)\rangle\leq 0\text{ for }t\geq 0,$$

hence ρ is decreasing on $[0,\infty)$. Suppose that there exists $t_0>0$ so that $\rho(t_0)=0$. Then $\rho(t)=0$ for $t\geq t_0$ and let $t_1=\inf\{t>0|\rho(t)=0\}$. Since $\rho(0)=r>0$, we see that $t_1>0$ and $\rho(t)>0$ on $[0,t_1)$, $\rho(t)=0$ on $[t_1,\infty)$. Also, $\rho'(t)=-2\mathrm{Re}\langle h(v_z(t)),v_z(t)\rangle<0$ on $[0,t_1)$, hence ρ is strictly decreasing on $[0,t_1)$. If $\rho(t)>0$ for every $t>0$, then ρ is strictly decreasing on $[0,\infty)$ and in both cases we see that $v_z(t)\in B(0,r)$ for $t>0$, hence

$$w_z(t)=g(v_z(t))\in g(B(0,r)),\text{ for }t>0.$$

Since $2\mathrm{Re}\langle v_z'(0),v_z(0)\rangle=-2\mathrm{Re}\langle h(z),z\rangle<0$, we see that $\mathrm{Re}\langle v_z'(0),z\rangle\neq 0$, hence $v_z'(0)\in I-T(\overline{B}(0,r))_z$ and since $w_z'(0)=Dg(z)(v_z'(0))$, we see that $w_z'(0)\in I-T(g(\overline{B}(0,r)))_{g(z)}$. It results that g is asymptotic Φ like. Moreover, if $z\in B$ and $r=||z||$, then every solution $w_z:[0,\infty)\to\mathbb{R}^n$ of equation $(**)$ is so that $w_z(t)\in g(B(0,r))$ for $t>0$ and $w_z'(0)\in I-T(g(\overline{B}(0,r)))_{g(z)}$. □

THEOREM 8. *Let $n\geq 2$, $g\in C^2(B,\mathbb{R}^n)$ be injective so that $g(0)=0$, $J_g(z)\neq 0$ on B, let $\Phi\in C^1(g(B),\mathbb{R}^n)$ be so that $\Phi(0)=0$ and g is asymptotic Φ like. Then g is Φ like.*

Proof. Let $h:B\to\mathbb{R}^n$, $h(z)=Dg(z)^{-1}\circ\Phi(g(z))$ for $z\in B$. Let $z\in B$, $r=||z||$ and let $w_z:[0,\infty)\to g(\overline{B}(0,r))$ be so that $w_z(0)=g(z)$, $\frac{\mathrm{d}w_z}{\mathrm{d}t}=-\Phi(w_z(t))$ for $t\geq 0$ and $w_z'(0)\in I-T(g(\overline{B}(0,r)))_{g(z)}$. Let $v_z=g^{-1}\circ w_z$. Then $v_z(0)=g^{-1}(w_z(0))=g^{-1}(g(z))=z$ and

$$\frac{\mathrm{d}v_z}{\mathrm{d}t}=D(g^{-1})(w_z(t))(w_z'(t))=Dg(v_z(t))^{-1}(-\Phi(w_z(t)))$$

$$= -Dg(v_z(t))^{-1} \circ \Phi(g(v_z(t))) = -h(v_z(t)) \text{ for } t \geq 0.$$

Let $\rho : [0, \infty) \to [0, \infty)$, $\rho(t) = ||v_z(t)||^2$ for $t \geq 0$. Since

$$v_z(t) = g^{-1}(w_z(t)) \in g^{-1}(g(\overline{B}(0, r))) \subset \overline{B}(0, r)$$

for $t \geq 0$, we see that $\rho(t) = ||v_z(t)|| \leq r = \rho(0)$ for $t \geq 0$, hence $\rho'(0) \leq 0$. Since

$$\rho'(0) = 2\mathrm{Re}\langle v_z'(0), v_z(0)\rangle = -2\mathrm{Re}\langle h(z), z\rangle,$$

we find that $\mathrm{Re}\langle h(z), z\rangle \geq 0$. If $\mathrm{Re}\langle h(z), z\rangle = 0$, then $h(z) \in T(S(0, r))_z$ and $v_z'(0) = -h(v_z(0)) = -h(z) \in T(S(0, r))_z$. Then

$$w_z'(0) = Dg(z)(v_z'(0)) \in Dg(z)(T(S(0, r))_z) = T(g(S(0, r)))_{g(z)}$$

and we reached a contradiction. We proved that $\mathrm{Re}\langle h(z), z\rangle > 0$ on $B\backslash\{0\}$, hence g is Φ like. □

We immediately obtain:

Theorem 9. *Let $n \geq 2$, $g \in C^2(B, \mathbb{R}^n)$ so that $g^{-1}(g(0)) = \{0\}$, $J_g(z) \neq 0$ on B and let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be so that $\mathrm{Re}\langle A(x), z\rangle > 0$ on $B\backslash\{0\}$. Then g is A like if and only if is asymptotic A like.*

Theorem 10. *Let $n \geq 2$, $g \in C^2(B, \mathbb{R}^n)$ be such that $g^{-1}(g(0)) = \{0\}$ and $J_g(z) \neq 0$ on B. Then $\mathrm{Re}\langle Dg(z)^{-1}(g(z)), z\rangle > 0$ on $B\backslash\{0\}$ if and only if g is asymptotic starlike.*

If $g \in C^2(B, \mathbb{R}^n)$, $g^{-1}(g(0)) = \{0\}$, $J_g(z) \neq 0$ on B and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\det A \neq 0$ is so that $\mathrm{Re}\langle A(z), z\rangle > 0$ on $B\backslash\{0\}$ and g is A like, we can define $m(A) = \inf\limits_{||z||=1} \mathrm{Re}\langle A(z), z\rangle$, $K(A) = \sup\limits_{||z||=1} \mathrm{Re}\langle A(z), z\rangle$, $m_g(r) = \inf\limits_{||z||=r} \mathrm{Re}\langle Dg(z)^{-1} \circ A(g(z)), z\rangle / r^2$, $M_g(r) = \sup\limits_{||z||=r} \mathrm{Re}\langle Dg(z)^{-1} \circ A(g(z)), z\rangle / r^2$ for $0 < r < 1$. We see that $0 < m(A) < K(A) \leq ||A||$, $0 < m_g(r) \leq M_g(r) < \infty$ for $0 < r < 1$.

We have the following estimate of the growth of the modules of a A like map.

Theorem 11. *Let $n \geq 2$, $g \in C^2(B, \mathbb{R}^n)$ be so that*

$$g^{-1}(g(0)) = \{0\},\ Dg(0) = I,\ J_g(z) \neq 0 \text{ on } B$$

and there exists $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ so that $0 < m(A) = K(A)$ and g is A-like. Then

$$||z|| \cdot \exp\left(\int_0^{||z||} \frac{1}{x}\left(\frac{m(A)}{M_g(x)} - 1\right) \mathrm{d}x\right) \leq ||g(z)|| \leq ||z|| \cdot \exp\left(\int_0^{||z||} \frac{1}{x}\left(\frac{K(A)}{m_g(x)} - 1\right) \mathrm{d}x\right)$$

for every $z \in B$.

Proof. We see from Theorem 7 that g is injective on B. Let $h : B \to \mathbb{R}^n$,

$$h(z) = Dg(z)^{-1} \circ A(g(z)) \text{ for } z \in B.$$

Let $z \in B$ and let v_z be the unique solution of the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -h(v),\ v(0) = z.$$

Since $w_z(t) = \mathrm{e}^{-tA} \cdot g(z)$ is the unique solution of the equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -A(w),\ w(0) = g(z),$$

we see that $v_z = g^{-1} \circ w_z$.

Indeed, let $v : [0,\infty) \to \mathbb{R}^n$, $v(t) = g^{-1}(\mathrm{e}^{-tA}g(z))$ for $t \geq 0$. Then $g(v(t)) = \mathrm{e}^{-tA}g(z)$ for $t \geq 0$ and

$$\frac{\mathrm{d}v}{\mathrm{d}t} = D(g^{-1})(w_z(t))\left(\frac{\mathrm{d}w_z}{\mathrm{d}t}\right) = D(g^{-1})(\mathrm{e}^{-tA} \cdot g(z))(-A(w_z(t)))$$

$$= -D(g^{-1})(g(v(t)))(A(g(v(t))) = -Dg(v(t))^{-1} \circ A(g(v(t)) = -h(v(t))$$

and $v(0) = g^{-1}(g(z)) = z$, hence $v = v_z$.

We show that $\mathrm{e}^{tA}v_z(t) \to g(z)$ if $t \to \infty$. Since $||\mathrm{e}^{-tA}g(z)|| \leq \mathrm{e}^{-m(A)\cdot t}||g(z)||$ for $z \in B$, $t \geq 0$, we see that $\mathrm{e}^{-tA}g(z) \to 0$ if $t \to \infty$. We also see from Lemma 2.1 from [10] that $\mathrm{e}^{m(A)\cdot t} \cdot ||u|| \leq ||\mathrm{e}^{tA}u|| \leq \mathrm{e}^{K(A)\cdot t} \cdot ||u||$ and $\mathrm{e}^{-K(A)\cdot t} \cdot ||u|| \leq ||\mathrm{e}^{-tA}u|| \leq \mathrm{e}^{-m(A)\cdot t} \cdot ||u||$ for $u \in \mathbb{R}^n$ and $t \geq 0$. Since $D(g^{-1})(0) = Dg(0)^{-1} = I$, we see that for $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ so that $||g^{-1}(u) - u|| \leq \varepsilon \cdot ||u||$ for $||u|| \leq \delta_\varepsilon$. Let $t_\varepsilon > 0$ be so that $||\mathrm{e}^{-tA}g(z)|| \leq \delta_\varepsilon$ for $t \geq t_\varepsilon$. Then

$$\begin{aligned} &||\mathrm{e}^{tA} \cdot v_z(t) - g(z)|| = ||\mathrm{e}^{tA}(g^{-1}(\mathrm{e}^{-tA}g(z)) - \mathrm{e}^{-tA}g(z))|| \\ \leq &\mathrm{e}^{K(A)t} \cdot ||g^{-1}(\mathrm{e}^{-tA}g(z)) - \mathrm{e}^{-tA}g(z)|| \leq \varepsilon \cdot \mathrm{e}^{K(A)t} \cdot ||\mathrm{e}^{-tA}g(z)|| \\ \leq &\varepsilon \cdot \mathrm{e}^{(K(A)-m(A))\cdot t} \cdot ||g(z)|| = \varepsilon||g(z)|| \text{ for } t \geq t_\varepsilon, \end{aligned}$$

hence $\mathrm{e}^{tA} \cdot v_z(t) \to g(z)$ if $t \to \infty$. Also, $||v_z(t)|| \leq \mathrm{e}^{-m(A)\cdot t} \cdot ||\mathrm{e}^{tA}v_z(t)||$, hence $v_z(t) \to 0$ if $t \to \infty$.

Let $\rho : [0,\infty) \to [0,\infty)$, $\rho(t) = ||v_z(t)||^2$ for $t \geq 0$. Then

$$2\rho(t) \cdot \rho'(t) = \rho^2(t)' = 2\mathrm{Re}\langle v_z'(t), v_z(t)\rangle = -2\mathrm{Re}\langle h(v_z(t)), v_z(t)\rangle \leq 0 \text{ for } t \geq 0,$$

hence ρ is decreasing on $(0,\infty)$.

Using the substitution $x = \rho(u)$, we have

$$\int_{\rho(t)}^{||z||} \frac{\mathrm{d}x}{x \cdot m_g(x)} = \int_t^0 \frac{\rho'(u)\mathrm{d}u}{\rho(u) \cdot m_g(\rho(u))} = -\int_0^t \frac{\rho(u) \cdot \rho'(u)\mathrm{d}u}{\rho^2(u) \cdot m_g(\rho(u))}$$

$$= \int_0^t \frac{\mathrm{Re}\langle h(v_z(u)), v_z(u)\rangle \mathrm{d}u}{||v_z(u)||^2 \cdot m_g(||v_z(u)||)} \geq t.$$

Then $\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{K(A)}{m_g(x)} - 1\right)\mathrm{d}x \geq K(A)\cdot t - \ln\frac{||z||}{\rho(t)}$, hence

$$\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{K(A)}{m_g(x)} - 1\right)\mathrm{d}x\right) \geq \mathrm{e}^{K(A)\cdot t}\cdot\frac{\rho(t)}{||z||}.$$

We have

$$\rho(t) \leq^{-K(A)\cdot t} \cdot ||z||\cdot\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{K(A)}{m_g(x)} - 1\right)\mathrm{d}x\right) \text{ for } t \geq 0.$$

We also have

$$\int\limits_{\rho(t)}^{||z||}\frac{\mathrm{d}x}{x\cdot M_g(x)} = \int\limits_{t}^{0}\frac{\rho'(u)\mathrm{d}u}{\rho(u)\cdot M_g(\rho(u))} = \int\limits_{0}^{t}\frac{\mathrm{Re}\langle h(v_z(u)), v_z(u)\rangle\mathrm{d}u}{||v_z(u)||^2\cdot M_g(||v_z(u)||)} \leq t,$$

hence $\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{m(A)}{M_g(x)} - 1\right)\mathrm{d}x \leq m(A)\cdot t - \ln\frac{||z||}{\rho(t)}$ and $\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{m(A)}{M_g(x)} - 1\right)\mathrm{d}x\right)$ $\leq \mathrm{e}^{m(A)\cdot t}\cdot\frac{\rho(t)}{||z||}$. We obtained that

$$\mathrm{e}^{-m(A)\cdot t}\cdot||z||\cdot\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{m(A)}{M_g(x)} - 1\right)\mathrm{d}x\right)$$

$$\leq \rho(t) \leq \mathrm{e}^{-K(A)t}\cdot||z||\cdot\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{K(A)}{m_g(x)} - 1\right)\mathrm{d}x\right).$$

We have that

$$||\mathrm{e}^{tA}\cdot v_z(t)|| \leq \mathrm{e}^{K(A)\cdot t}\cdot||v_z(t)|| \leq ||z||\cdot\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{K(A)}{m_g(x)} - 1\right)\mathrm{d}x\right)$$

and $||\mathrm{e}^{tA}v_z(t)|| \geq \mathrm{e}^{m(A)t}\cdot||v_z(t)|| \geq ||z||\exp\left(\int\limits_{\rho(t)}^{||z||} \frac{1}{x}\left(\frac{m(A)}{M_g(x)} - 1\right)\mathrm{d}x\right)$, hence

$$||z||\cdot\exp\Big(\int\limits_{\rho(t)}^{||z||}\frac{1}{x}\Big(\frac{m(A)}{M_g(x)} - 1\Big)\mathrm{d}x\Big) \leq ||\mathrm{e}^{tA}v_z(t)|| \leq ||z||\exp\Big(\int\limits_{\rho(t)}^{||z||}\frac{1}{x}\Big(\frac{K(A)}{m_g(x)} - 1\Big)\mathrm{d}x\Big),$$

for every $t > 0$. Letting $t \to \infty$, we find that

$$||z||\cdot\exp\Big(\int_0^{||z||}\frac{1}{x}\Big(\frac{m(A)}{M_g(x)}-1\Big)\mathrm{d}x\Big) \le ||g(z)|| \le ||z||\cdot\exp\Big(\int_0^{||z||}\frac{1}{x}\Big(\frac{K(A)}{m_g(x)}-1\Big)\mathrm{d}x\Big),$$

for every $z \in B$. □

REMARK 2. If $g \in H(B)$ then g is A like if and only if g is injective and $\mathrm{e}^{-tA}g(z) \in g(B)$ for every $(z,t) \in B \times [0,\infty)$ and g is I like if and only if $g(B)$ is starlike and g is injective. We have for A like holomorphic mappings some estimates of the growth of the modulus of $g(z)$ in Lemma 2.11 from [10] and for starlike mappings we have the well known formulae:

$$\frac{||z||}{(1+||z||)^2} \le ||g(z)|| \le \frac{||z||}{(1-||z||)^2} \text{ for } z \in B.$$

In fact, for holomorphic starlike mappings we see from Lemma 6.1.32 from [12] that $m_g(r) = \frac{1-r}{1+r}$, $M_g(r) = \frac{1+r}{1-r}$ for $0 < r < 1$ and using Theorem 11 we find this formulae.

Some of the result also hold on arbitrary Hilbert spaces.

THEOREM 12. *Let* $K = \mathbb{R}, \mathbb{C}$, E *a Hilbert space over the field* K, $g \in C^2(B,E)$ *so that* $g^{-1}(g(0)) = \{0\}$, $g'(z) \in \mathrm{Isom}(E,E)$ *for every* $z \in B$ *and suppose that* $\mathrm{Re}\langle Dg(z)^{-1}(g(z)), z\rangle > 0$ *on* $B\backslash\{0\}$. *Then* g *is univalent and* $g(B)$ *is starlike.*

THEOREM 13. *Let* $K = \mathbb{R}, \mathbb{C}$, E *be a Hilbert space over the field* K, $b \in (0,\infty]$, $h : B \times (0,b) \to E$ *continuous, satisfying conditions (1), (2), (3) so that there exists* $c, d : (0,b) \to \mathbb{R}_+$ *continuous so that* $c(||z||)\cdot||z||^2 \le \mathrm{Re}\langle h(z,t), z\rangle \le d(||z||)\cdot||z||^2$ *for every* $z \in B\backslash\{0\}$ *and every* $0 < t < b$. *Then, if* $z \in B$, $0 < s < b$ *and* $\phi(\cdot, s, z)$ *is the solution of Loewner's differential equation* $\frac{\mathrm{d}v}{\mathrm{d}t} = -h(v,t)$, $v(s) = z$ *and* $\rho(t) = ||\phi(t,s,z)||$ *for* $s \le t \le b$, *we have*

$$\mathrm{e}^s\cdot||z||\cdot\exp\Big(\int_{\rho(t)}^{||z||}\frac{1}{x}\Big(\frac{1}{d(x)}-1\Big)\mathrm{d}x\Big) \le \mathrm{e}^t\rho(t) \le \mathrm{e}^s\cdot||z||\cdot\exp\Big(\int_{\rho(t)}^{||z||}\frac{1}{x}\Big(\frac{1}{c(x)}-1\Big)\mathrm{d}x\Big).$$

4. QUASICONFORMAL EXTENSION OF A LIKE MAPPINGS

The following theorem extends some results of Chuaqui [3] and Hamada and Kohr [16, 17] established for holomorphic mappings:

THEOREM 14. *Let* $n \ge 2$, $g \in C^2(B,\mathbb{R}^n)$ K *quasiconformal so that* $g(0) = 0$, $J_g(z) \ne 0$ *on* B *and let* $A \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ *and* $c, M > 0$ *be so that* $0 < m(A) = K(A)$, $\mathrm{Re}\langle Dg(z)^{-1} \circ A(g(z)), z\rangle \ge c\cdot||z||^2$ *on* B *and* $||Dg(z)^{-1} \circ A(g(z))|| \le M$ *on* B. *Then* g *is a Lipschitz map on* B *and there exists* $F : \mathbb{R}^n \to \mathbb{R}^n$ Q *quasiconformal so that* $F|B = g$.

Proof. We see from Theorem 7 that g is injective on B. Since $A \circ g$ is differentiable in 0, there exists $\varepsilon_1 > 0$, $M_1 > 0$ so that $||A(g(z))|| \leq M_1 \cdot ||z||$ for $||z|| \leq \varepsilon_1$. Since $g(0) = 0$ and $(g^{-1})'$ is continuous in 0, there exists $0 < \varepsilon < \varepsilon_1$ and $M_2 > 0$ so that $||(g^{-1})'(g(z)) - (g^{-1})'(0)|| \leq M_2$ for $||z|| \leq \varepsilon$. Let $z \in B(0, \varepsilon)$. Then

$$\frac{||Dg(z)^{-1}(A(g(z)))||}{||z||} = \frac{||(g^{-1})'(g(z))(A(g(z)))||}{||z||}$$

$$\leq \frac{||(g^{-1})'(g(z))(A(g(z))) - (g^{-1})'(0)(A(g(z)))||}{||z||} + \frac{||(g^{-1})'(0)(A(g(z)))||}{||z||}$$

$$\leq ||(g^{-1})'(g(z)) - (g^{-1})'(0)|| \cdot \frac{||A(g(z))||}{||z||} + ||(g^{-1})'(0)|| \cdot \frac{||A(g(z))||}{||z||}$$

$$\leq M_1(M_2 + ||(g^{-1})'(0)||).$$

Let $M_3 = M_1(M_2 + ||(g^{-1})'(0)||)$ and $M_0 = \max\left\{M_3, \frac{M}{\varepsilon}\right\}$. We showed that $||Dg(z)^{-1}(A(g(z)))|| \leq M_0 \cdot ||z||$ for every $z \in B$. Also,

$$c \cdot ||z||^2 \leq \mathrm{Re}\langle Dg(z)^{-1}(A(g(z))), z\rangle$$

$$\leq |\langle Dg(z)^{-1}(A(g(z))), z\rangle| \leq ||Dg(z)^{-1}(A(g(z)))|| \cdot ||z||,$$

hence $c \cdot ||z|| \leq ||Dg(z)^{-1}(A(g(z)))||$ for every $z \in B$. Let $h : B \to \mathbb{R}^n$, $h(z) = Dg(z)^{-1}(A(g(z)))$ for $z \in B$ and the initial value problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -h(v),\ v(0) = z \text{ for } z \in B.$$

Since $\mathrm{Re}\langle h(z), z\rangle \geq c \cdot ||z||^2$ for every $z \in B$ we see from Theorem A that there exists an unique solution $v_z : [0, \infty) \to \mathbb{R}^n$ of this equation and $||v_z(t)|| \leq ||z|| \cdot \mathrm{e}^{-ct}$ for $z \in B$, $t \geq 0$. As in Theorem 11 we see that $v_z(t) = g^{-1}(\mathrm{e}^{-tA} g(z))$ for $z \in B$, $t \geq 0$, hence $||g^{-1}(\mathrm{e}^{-tA} g(z))|| \leq ||z|| \cdot \mathrm{e}^{-ct}$ for $z \in B$, $t \geq 0$. Let $t > 0$, $r > 0$ and $z \in \overline{B}(0, r)$. We see from Theorem 7 that g is asymptotic A like, hence there exists $w \in \overline{B}(0, r)$ so that $\mathrm{e}^{-tA} g(z) = g(w)$ and since $||w|| = ||g^{-1}(g(w))|| = ||g^{-1}(\mathrm{e}^{-tA} g(z))|| \leq ||z|| \cdot \mathrm{e}^{-ct} \leq r \cdot \mathrm{e}^{-ct}$, we see that $\mathrm{e}^{-tA} g(z) = g(w) \in g(\overline{B}(0, r\mathrm{e}^{-ct})) \in g(\overline{B}(0, \mathrm{e}^{-ct}))$. Let $K_t > 0$ be so that $g(\overline{B}(0, \mathrm{e}^{-ct})) \subset B(0, K_t)$. Then $||g(z)|| \leq \mathrm{e}^{K(A) \cdot t} ||\mathrm{e}^{-tA} g(z)||$, hence $g(z) \in B(0, K_t \cdot \mathrm{e}^{K(A) \cdot t})$. It results that $g(B) \subset B(0, K_t \cdot \mathrm{e}^{K(A) \cdot t})$, hence g is bounded on B and let $K_0 > 0$ be so that $||A(g(z))|| \leq K_0$ for every $z \in B$.

We show that there exists $\delta > 0$ so that $||Dg(z)^{-1}|| \geq \delta$ for every $z \in B$. Indeed, otherwise we can find $z_p \in B$ and $u_p \in S^n$ so that $||Dg(z_p)^{-1}(u_p)|| \to 0$.

Let $\lambda : g(B) \to B$ be the inverse of g. Then λ is also K quasiconformal and

$$K \geq \frac{||\lambda'(g(z_p))||}{l(\lambda'(g(z_p)))} \geq \frac{||\lambda'(g(z_p))(A(g(z_p))/||A(g(z_p))||)}{||\lambda'(g(z_p))(u_p)||}$$

$$= \frac{||Dg(z_p)^{-1}(A(g(z_p)))||}{||A(g(z_p))|| \cdot ||Dg(z_p)^{-1}(u_p)||} \geq \frac{c \cdot ||z_p||}{||A(g(z_p))||} \cdot \frac{1}{||Dg(z_p)^{-1}(u_p)||}$$

$$\geq c \cdot \min\left\{\frac{1}{M_1}, \frac{\varepsilon}{K_0}\right\} \cdot \frac{1}{||Dg(z_p)^{-1}(u_p)||} \to \infty \text{ if } p \to \infty.$$

We reached a contradiction, hence we proved that there exists $\delta > 0$ so that $||Dg(z)^{-1}|| \geq \delta$ for every $z \in B$. Then

$$||g'(z)|| = H(z,g) \cdot l(g'(z)) \leq K \cdot l(g'(z)) = \frac{K}{||Dg(z)^{-1}||} \leq \frac{K}{\delta},$$

for every $z \in B$, and this implies that g is a Lipschitz map on B and hence it extends continuously at $\overline{B}$.

Let $f_t : B \to \mathbb{R}^n$, $f_t(z) = \mathrm{e}^{tA} g(z)$ for $z \in B$, $t \geq 0$. We see that $Df_t(z) = \mathrm{e}^{tA} \circ Dg(z)$, $\frac{\partial f}{\partial t}(z,t) = A \circ \mathrm{e}^{tA} g(z)$ for $(z,t) \in B \times [0,\infty)$, hence

$$Df_t(z)(h(z)) = \mathrm{e}^{tA} Dg(z)(Dg(z)^{-1}(A(g(z))))$$
$$= \mathrm{e}^{tA} \circ A(g(z)) = A \circ \mathrm{e}^{tA}(g(z)) = \frac{\partial f}{\partial t}(z,t),$$

for $z \in B$ and $t \geq 0$. Also, $f_t \to g$ uniformly on the compact subsets of B, every map f_t is injective on B, f extends continuously on $\overline{B} \times (0,\infty)$ and $\mathrm{Re}\langle h(z), z\rangle \geq c \cdot ||z||^2$ on B, $||h(z)|| \leq M_0 \cdot ||z||$ for $z \in B$ and h is a C^1 map.

Also,

$$H(z,f_t) = \frac{||Df_t(z)||}{l(Df_t(z))} \leq \frac{||\mathrm{e}^{tA} \circ Dg(z)||}{l(\mathrm{e}^{tA} \circ Dg(z))} \leq \frac{||\mathrm{e}^{tA}|| \cdot ||Dg(z)||}{l(\mathrm{e}^{tA}) \cdot l(Dg(z))}$$
$$\leq \frac{\mathrm{e}^{K(A)t} \cdot ||Dg(z)||}{\mathrm{e}^{m(A)\cdot t} \cdot l(Dg(z))} = H(z,g) \leq K$$

for every $z \in B$ and every $t \geq 0$, hence all the mappings f_t are K quasiconformal. We apply now Theorem B to find $F : \mathbb{R}^n \to \mathbb{R}^n$ Q quasiconformal so that $F|B = g$. □

Remark 3. If $a > 0$ and $A + A^* = 2aI$, then $0 < a = m(A) = K(A)$. Also, the condition $\mathrm{Re}\langle Dg(z)^{-1}(A(g(z))), z\rangle \geq c \cdot ||z||^2$ for every $z \in B\backslash\{0\}$ is satisfied if f is strongly starlike (see Definition 8.3.22 in [12]) or if f is strongly starlike of order α (see Definition 8.5.12 from [12]). Indeed, in both cases there exists $0 < c < 1$ so that $\left|\frac{\langle h(z),z\rangle}{||z||^2} - \frac{1+c^2}{1-c^2}\right| \leq \frac{2c}{1-c^2}$ for every $B\backslash\{0\}$, hence $\frac{\langle h(z),z\rangle}{||z||^2} \in B\left(\frac{1+c^2}{1-c^2}, \frac{2c}{1-c^2}\right)$ for every $z \in B\backslash\{0\}$ and we see that $\frac{\mathrm{Re}\langle h(z),z\rangle}{||z||^2} \geq \frac{1-c}{1+c}$ on $B\backslash\{0\}$. It results that Theorem 14 extends the results from [3], [16], [17] even in the case of holomorphic mappings.

Finally we give the proof of the eliminability result for quasiregular mappings from Theorem 2 from [7], which was omitted in [7].

Theorem 15. *Let $n \geq 2$, $D \subset \mathbb{R}^n$ a domain, $E \subset D$ closed in D so that $\mu_n(E) = 0$ and let $f : D \to \mathbb{R}^n$ be continuous, open, discrete on D and K-quasiregular on $D\backslash E$. Let $H_i = \{x \in \mathbb{R}^n | \langle x, e_i\rangle = 0\}$ for $i = 1, \ldots, n$ and let $P_i : \mathbb{R}^n \to H_i$ be the projections on H_i for $i = 1, \ldots, n$ and suppose that*

$P_i^{-1}(y)\cap E$ is at most countable for a.e. $y\in H_i$, $i=1,\dots,n$. Then f is K quasiregular on D.

Proof. We see from Proposition 1.2 page 6 from [25] that the weak partial derivatives and the ordinary partial derivatives of f coincide a.e. in $D\backslash E$. We denote by $\frac{\partial f}{\partial x_i}(x)$ the ordinary partial derivatives of f in x, $i=1,\dots,n$, while $f'(x)$ and $J_f(x)$ will denote the weak derivative of f in x, respectively the weak Jacobian of f in x.

Let $x\in D$ be fixed. Since f is continuous, open, discrete on D, there exists $r_x>0$, $N_x\geq 1$ and $U_x\in V(x)$ so that $U_x\subset\subset D$, $f(U_x)=B(f(x),r_x)$ and $N(f,U_x)\leq N_x$. Since $f\in W^{1,1}_{loc}(D\backslash E)$, we use the change of variable formulae (3) from [18] to see that $\int\limits_{U_x}|J_f(z)|\mathrm{d}z=\int\limits_{U_x\backslash E}|J_f(z)|\mathrm{d}z\leq\int\limits_{\mathbb{R}^n}N(y,f,U_x\backslash E)dy$ $\leq N_x\cdot\mu_n(B(f(x),r_x))<\infty$. We therefore proved that $J_f\in\mathcal{L}^1_{loc}(D)$ and since $||f'(z)||^n\leq K\cdot J_f(z)$ a.e. in D, we see that $\int\limits_Q||f'(z)||^n\mathrm{d}z<\infty$ for every case $Q\subset\subset D$ with the sides parallel to coordinate axes. Let $Q\subset\subset D$ be such a cube, let $i\in\{1,\dots,n\}$ and let Q_i be the face of Q which is parallel to H_i and let $J_y=P_i^{-1}(y)\cap Q$ for $y\in Q_i$. Since $\int\limits_Q||f'(z)||\mathrm{d}z<\infty$, we use Fubini's theorem to see that $\int\limits_{J_y}||\frac{\partial f}{\partial x_i}(z)||\mathrm{d}z<\infty$ for a.e. $y\in Q_i$. Since f is quasiregular on $D\backslash E$, we see that $f|J_y:J_y\to\mathbb{R}^n$ is absolutely continuous on every closed internal $J\subset J_y\cap(D\backslash E)$ for a.e. $y\in Q_i$, and since $J_y\cap E$ is at most-countable for a.e. $y\in Q_i$, it results that all the components of the map $f|J_y:J_y\to\mathbb{R}^n$ satisfy condition (N) for a.e. $y\in Q_i$. Using Barry's theorem (see [27], page 285), we see that all the components of $f|J_y:J_y\to\mathbb{R}^n$ are absolutely continuous on J_y for a.e. $y\in Q_i$, $i=1,\dots,n$, hence f is ACL on D. Since $\int\limits_Q||f'(z)||^n\mathrm{d}z\leq K\cdot\int\limits_Q|J_f(z)|\mathrm{d}z<\infty$ for every cube $Q\subset\subset D$ with the sides parallel to coordinate axes, we see that f is ACL^n on D and $||f'(z)||^n\leq K\cdot J_f(z)$ a.e. in D. We proved that f is K-quasiregular on D. $\square$

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