MATHEMATICA, Tome 55 (78), N° 1, 2013, pp. 15–21

AN IMPROVED LOCAL CONVERGENCE ANALYSIS FOR SECANT–LIKE METHOD

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Abstract. We provide a local convergence analysis for Secant–like algorithm for solving nonsmooth variational inclusions in Banach spaces. An existence–convergence theorem and an improvement of the ratio of convergence of this algorithm are given under center–conditioned divided difference and Aubin's continuity concept. Our result compare favorably with related obtained in [18].

MSC 2010. 65K10, 65G99, 47H04, 49M15.

Key words. Banach space, Secant-like method, generalized equation, Aubin continuity, ratio of convergence, divided difference, set-valued map.

1. INTRODUCTION

This paper considers the problem of approximating a locally unique solution of nondifferentiable generalized equations using an uniparametric secant–type algorithm. Let X, Y be two Banach spaces, F is a continuous function from X into Y and G is a set–valued map from X to the subsets of Y with closed graph. We consider a generalized equation in the form

$$0 \in F(x) + G(x). \tag{1}$$

Generalized equations (1) was introduced by Robinson [21], [22]. (1) is an abstract model including mathematical programming problems, variational inequalities, optimal control, complementarity problems and other fields [10].

For approximating locally the unique solution x^* of (1), we consider the sequence [11], [18], [12]:

$$\begin{cases} x_0 \text{ and } x_1 \text{ are given starting points} \\ y_k = \beta\, x_k + (1-\beta)\, x_{k-1};\ \beta \text{ is fixed in } [0,1[\\ 0 \in F(x_k) + [y_k, x_k; F]\ (x_{k+1} - x_k) + G(x_{k+1}), \end{cases} \tag{2}$$

where $[x, y; F] \in \mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y is called a divided difference of F of order one at the points x and y, satisfying

$$[x, y; F]\ (y - x) = F(y) - F(x), \text{ for all } x, y \text{ in } X \text{ with } x \neq y. \tag{3}$$

Note that if F is Fréchet–differentiable, then $[x, x; F] = \nabla F(x)$ (see [5], [9]). For $G = \{0\}$ in (1), (1) becomes a nonlinear equation in the form

$$F(x) = 0. \tag{4}$$

To solve (4), a Secant method is considered in [1] assuming only that the nonlinear operator F has a Hölder continuous Fréchet derivative at the unique solution of (4). In [2] a Lipschitz–type condition on the first order divided difference is used for approximating the soltion of (4). A semilocal convergence of the Secant method under relaxed conditions is investigated in [6]. Using center–Lipschitz–type conditions, an existence–convergence results are given in [7]. A flexible and precise point–based approximation is provided in [8] for Secant–type iterative procedures for solving (4). Hernández and Rubio [15] consider a similar iterative method like (2) with $\beta = 0$ and $G = \{0\}$. In [16], [17] the authors studied the semilocal convergence for nondifferentiable equations using ω–conditioned divided difference for β fixed in $(0, 1)$. For $G \neq \{0\}$, some semilocal convergence results of Newton's method for solving (1) are developed in [3], [4] using certain assumptions on the first Fréchet derivative of F. In [12] a study of the existence and the convergence of the algorithm (2) is presented using a (ν, p)–Hölder continuous divided difference condition. In [18] we show the existence and the q–linear convergence of the sequence defined by (2) using ω–conditioned divided difference.

The purpose of this paper is to refine the convergence analysis of method (2) under weaker hypothesis and less computational cost than [18]. Using some ideas given in [5], [9] for nonlinear equations, we provide a local convergence with the following advantages over related in [18]: finer error bounds on the distances involved, and a larger radius of convergence. This observation is very important in computational mathematics [1]–[9].

The structure of this paper is the following. In section 2, we collect a number of basic definitions and recall a fixed points theorem for set–valued maps. In section 3, we show the existence and the q–linear convergence of the sequence defined by (2). Finally, we give some remarks on our method.

2. PRELIMINARIES AND ASSUMPTIONS

In order to make the paper as self–contained as possible we reintroduce some definitions and some results on fixed point theorem [7]–[14], [18]–[24]. We let Z be a Banach space equiped with the norm $\| \cdot \|$. The distance from a point x to a set A in Z is defined by $\text{dist}\,(x, A) = \inf_{y \in A} \| x - y \|$ and the excess e from the set A to the set $C \subset Z$ is given by $e(C, A) = \sup_{x \in C} \text{dist}\,(x, A)$. For a set–mapping $\Lambda : X \rightrightarrows Y$, we denote by $\text{gph}\,\Lambda$ the set $\{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y)$ the set $\{x \in X,\ y \in \Lambda(x)\}$. The norms in the Banach spaces X and Y will both be denoted by $\| \cdot \|$ and the closed ball centered at x with radius r by $B_r(x)$.

Definition 1. *A set–valued Λ is pseudo–Lipschitz around $(x_0, y_0) \in \text{gph}\,\Lambda$ with modulus M if there exist constants a and b such that*

$$\sup_{z \in \Lambda(y') \cap B_a(y_0)} \text{dist}\,(z, \Lambda(y'')) \leq M \| y' - y'' \|, \tag{5}$$

for all y' and y'' in $B_b(x_0)$.

In the term of excess, we have an equivalent definition of pseudo–Lipschitzness replacing the inequality (5) by

(6) $e(\Lambda(y') \cap B_a(y_0), \Lambda(y'')) \leq M \parallel y' - y'' \parallel, \quad$ for all y' and y'' in $B_b(x_0)$.

The pseudo–Lipschitzness concept has been introduced by Aubin [13]. Let us note that the pseudo–Lipschitzness of Λ is equivalent to the metric regularity of Λ^{-1} which is a basic well–posedness property in optimization problems. For some characterizations and applications of this concept we refer the reader to [13], [14], [20], [23], [24] and the references given there.

Definition 2. *A sequence (x_n) in X is said to be q–linearly convergent to x^* with parameter $\sigma \in]0,1[$ if we have the following inequality*

$$\parallel x_{n+1} - x^* \parallel \leq \sigma \parallel x_n - x^* \parallel .$$

We need the following fixed point theorem [19], [14].

Lemma 1. *Let ϕ be a set–valued map from X into the closed subsets of X. We suppose that for $\eta_0 \in X$, $r \geq 0$ and $0 \leq \lambda < 1$ the following properties hold*

(a) $\mathrm{dist}(\eta_0, \phi(\eta_0)) \leq r(1-\lambda)$.
(b) $e(\phi(y) \cap B_r(\eta_0), \phi(z)) \leq \lambda \parallel y - z \parallel, \ \forall y, z \in B_r(\eta_0)$.

Then ϕ has a fixed point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

We suppose that for every distinct points x and y in a convex neighborhood V of x^*, there exists a first order divided difference of f at these points. We will make the following assumptions on V:
($\mathcal{H}1$) For x, u and v in V, $\parallel [x, x^*; F] - [u, v; F] \parallel \leq \omega(\parallel x - u \parallel, \parallel x^* - v \parallel)$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous nondecreasing function in both arguments.
($\mathcal{H}2$) The set–valued map $(F+G)^{-1}$ is pseudo–Lipschitz with constants M, a and b around $(0, x^*)$ (this constants are given by Definition 1).
($\mathcal{H}3$) For all $x, y \in V$, we have $\parallel [x, x^*; F] \parallel \leq d_0$, $\parallel [x, y; F] \parallel \leq d$, $M\, d < 1$ and $M\ [d_0 + \omega(2\, a\, (1-\beta), a)] \ < 1$. Before proving the main result of this study, we need to introduce some notations [12]. First, define the set–valued maps $Q : X \rightrightarrows Y$ and $\psi_k : X \rightrightarrows X$ by $(k \in \mathbb{N}^*)$

(7) $$Q(x) = F(x^*) + G(x); \quad \psi_k(x) = Q^{-1}(Z_k(x)),$$

where Z_k is a mapping from X to Y defined by

(8) $$Z_k(x) = F(x^*) - F(x_k) - [y_k, x_k; F](x - x_k).$$

3. CONVERGENCE STUDY

In this section we will be concerned with the existence and the convergence of the sequence defined by (2) to the solution x^* of (1) under the previous assumptions. The main result of this study is as follows.

Theorem 1. *We suppose that assumptions ($\mathcal{H}1$)–($\mathcal{H}3$) are satisfied. For every constant c such that* $c_0 = \dfrac{M\ \omega(2\,a\,(1-\beta),a)}{1-M\,d_0} < c < 1$*, there exist* $\delta > 0$ *such that for every distinct starting points* x_0 *and* x_1 *in* $B_\delta(x^*)$ *(with* $x_0 \neq x^*$ *and* $x_1 \neq x^*$*), and a sequence* (x_k) *defined by* (2) *which is q–linearly convergent to* x^**, i.e.;*

$$\| x_{k+1} - x^* \| \leq c \| x_k - x^* \| . \tag{9}$$

The prove of theorem 1 in by induction on k. we first state a result which the starting points (x_0, x_1). Let us note that the point x_2 is a fixed point of ψ_1 if and only if $0 \in F(x_1) + [y_1, x_1; F](x_2 - x_1) + G(x_2)$.

Proposition 1. *Under the assumptions of Theorem* 1*, there exist* $\delta > 0$ *such that for every distinct starting points* x_0 *and* x_1 *in* $B_\delta(x^*)$ *(with* $x_0 \neq x^*$ *and* $x_1 \neq x^*$*), the set–valued map* ψ_1 *has a fixed point* x_2 *in* $B_\delta(x^*)$ *satisfying*

$$\| x_2 - x^* \| \leq c \| x_1 - x^* \|, \tag{10}$$

where c is given by Theorem 1*.*

Proof. By hypothesis ($\mathcal{H}2$) we have

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \| y' - y'' \|,\ \forall y', y'' \in B_b(0). \tag{11}$$

Fix $\delta > 0$ such that

$$\delta < \delta_0 = \min\left\{ a\ ;\ \frac{b}{d_0 + 2\,\omega(2\,a\,(1-\beta), a)} \right\}. \tag{12}$$

According to the definition of excess e, we have

$$\text{dist } (x^*, \psi_1(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \psi_1(x^*) \Big). \tag{13}$$

Moreover, by assumption ($\mathcal{H}1$) we have the following

$$\begin{aligned} \| Z_1(x^*) \| &= \| \Big([x_1, x^*; F] - [y_1, x_1; F] \Big)(x^* - x_1) \| \\ &\leq \| [x_1, x^*; F] - [y_1, x_1; F] \| \| x^* - x_1 \| \\ &\leq \omega((1-\beta)\ \| x_1 - x_0 \|, \| x_1 - x^* \|)\ \ \| x_1 - x^* \| \\ &\leq \omega(2\,a\,(1-\beta), a)\ \ \| x_1 - x^* \| . \end{aligned} \tag{14}$$

By (12) we have $Z_1(x^*) \in B_b(0)$. Hence from (11) one has

$$\begin{aligned} e\Big(Q^{-1}(0) \cap B_\delta(x^*), \psi_1(x^*) \Big) &= e\Big(Q^{-1}(0) \cap B_\delta(x^*), Q^{-1}[Z_1(x^*)] \Big) \\ &\leq M\ \omega(2\,a\,(1-\beta), a)\ \ \| x_1 - x^* \| . \end{aligned} \tag{15}$$

Using (13) the following inequality hold

$$\text{dist } (x^*, \psi_1(x^*)) \ \leq\ M\ \omega(2\,a\,(1-\beta), a)\ \ \| x_1 - x^* \| . \tag{16}$$

Since $c\ (1 - M\,d_0) > M\ \omega(2\,a\,(1-\beta), a)$ there exists $\lambda \in [M\,d, 1[$ such that $c\ (1-\lambda) \geq M\ \omega(2\,a\,(1-\beta), a)$ and

$$\text{dist } (x^*, \psi_1(x^*)) \leq c\ (1-\lambda)\ \ \| x_1 - x^* \| . \tag{17}$$

Identifying η_0, ϕ and r in Lemma 1 by x^*, ψ_1 and $r_1 = c \parallel x_1 - x^* \parallel$ respectively, we can deduce from the inequality (17) that the assertion (a) in Lemma 1 is satisfied. By (12) we have $r_1 \leq \delta \leq a$ and moreover for $x \in B_\delta(x^*)$ we get in turn that

$$\begin{array}{rcl}\parallel Z_1(x) \parallel & = & \parallel F(x^*) - F(x_1) - [y_1, x_1; F]\,(x - x_1) \parallel \\ & = & \parallel [x_1, x^*; F]\,(x^* - x + x - x_1) - [y_1, x_1; F]\,(x - x_1) \parallel \\ & \leq & \parallel [x_1, x^*; F] \parallel \parallel x^* - x \parallel \\ & & + \parallel [x_1, x^*; F] - [y_1, x_1; F] \parallel \parallel x - x_1 \parallel . \end{array} \tag{18}$$

Using the assumptions $(\mathcal{H}1)$ and $(\mathcal{H}3)$ we obtain

(19)

$$\begin{array}{rcl}\parallel Z_1(x) \parallel & \leq & d_0 \parallel x^* - x \parallel + \omega(\parallel x_1 - y_1 \parallel, \parallel x^* - x_1 \parallel) \parallel x - x_1 \parallel \\ & \leq & d_0 \parallel x^* - x \parallel + \omega((1-\beta) \parallel x_1 - x_0 \parallel, \parallel x_1 - x^* \parallel) \parallel x - x_1 \parallel \\ & \leq & d_0\, \delta + 2\, \delta\, \omega(2\, a\, (1-\beta), a). \end{array}$$

Then by (12) we deduce that for all $x \in B_\delta(x^*)$ we have $Z_1(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$ we have

$$e(\psi_1(x') \cap B_{r_1}(x^*), \psi_1(x'')) \leq e(\psi_1(x') \cap B_\delta(x^*), \psi_1(x'')),$$

which yields by (11)

$$\begin{array}{rcl} e(\psi_1(x') \cap B_{r_1}(x^*), \psi_1(x'')) & \leq & M \parallel Z_1(x') - Z_1(x'') \parallel \\ & = & M \parallel [y_1, x_1; F](x'' - x') \parallel \\ & \leq & M\, d \parallel x'' - x' \parallel . \end{array} \tag{20}$$

Using $(\mathcal{H}3)$ and the fact that $\lambda \geq M\, d$, we obtain

$$e(\phi_0(x') \cap B_{r_1}(x^*), \psi_1(x'')) \leq M\, d \parallel x'' - x' \parallel \leq \lambda \parallel x'' - x' \parallel . \tag{21}$$

The condition (b) of Lemma 1 is satisfied. By Lemma 1 we can deduce the existence of a fixed point $x_2 \in B_{r_1}(x^*)$ for the map ψ_1. Then the proof of Proposition 1 is complete. □

Proof. (Proof of Theorem 1) Keeping $\eta_0 = x^*$ and setting

$$r := r_k = c \parallel x^* - x_k \parallel,$$

the application of Proposition 1 to the map ψ_k gives the desired result. □

Application 1. A simple example for generalized equations, we suppose that X is a Hilbert space with inner product $(\cdot;\cdot)$, C is a convex subset of X and f is a map from X to X. The variational inequality problem consists to

$$\text{find } x^* \text{ in } C \text{ such that } (f(x^*); x - x^*) \ \geq\ 0, \text{ for all } x \in X \tag{22}$$

By Robinson [21], the problem (22) is equivalent to generalized equation

$$\text{find } x^* \text{ in } C \text{ such that } 0 \in f(x^*) + G(x^*),$$

where $G : X \rightrightarrows X$ is a set–valued mapping defined by

$$G(x) = \begin{cases} \{z/\ (z; y - x)\ \leq 0 \text{ for all } y \in X\}, & \text{if } x \in C \\ \emptyset, & \text{otherwise.} \end{cases} \tag{23}$$

We can then approximate the solution x^* of problem (22) using our method (2).

Remark 1. *In order for us to compare our results with corresponding ones in* [18]*, let us introduce assumptions:*

$(\mathcal{H}1)^\star$ $\| [x,y;f]-[u,v;f] \| \leq \overline{\omega}(\| x-u \|, \| y-v \|)$ for x, y, u and v in V, *where* $\overline{\omega}$ *is as function* ω *defined in* $(\mathcal{H}1)$.

$(\mathcal{H}3)^\star$ *For all* $x,y \in V$, *we have* $\| \, [x,y;f] \, \| \leq d$ *and* $M\,[d+\omega(2\,a\,(1-\beta),2\,a)] \, < 1$.

Assumption $(\mathcal{H}1)$ *is weaker than* $(\mathcal{H}1)^\star$. *Using* $(\mathcal{H}1)^\star$, $(\mathcal{H}2)$ *and* $(\mathcal{H}3)^\star$, *similar result was shown in* [18]. *Let us define*

$$\overline{c_0} = \frac{M\ \omega(2\,a\,(1-\beta),a)}{1-M\,d} \tag{24}$$

and

$$\overline{\delta_0} = \min\left\{ a\ ;\ \frac{b}{d+2\,\omega(2\,a\,(1-\beta),a)} \right\}. \tag{25}$$

We clearly have that

$$\omega \leq \overline{\omega}, \tag{26}$$

$$d_0 \leq d, \tag{27}$$

$$c_0 \leq \overline{c_0}, \tag{28}$$

$$\overline{\delta_0} \leq \delta_0 \tag{29}$$

and $\dfrac{\overline{\omega}}{\omega}$, $\dfrac{d}{d_0}$, $\dfrac{\overline{c_0}}{c_0}$ *can be arbitrarily large* [5]–[9]. *It then follows that our radius of convergence is larger than the corresponding in* [18]. *Hence, the claims made in the introduction have been justified.*

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Received November 24, 2011
Accepted October 22, 2012

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