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COPURE (m,n)-INJECTIVE MODULES AND (α,m,n)-COTORSION MODULES

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Abstract. In this paper we introduce the notions of copure (m,n)-injective modules and (α,m,n)-cotorsion modules as generalizations of (m,n)-injective modules and (m,n)-cotorsion modules, respectively. We also obtain some properties of these modules.

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Key words. Cotorsion pair, copure (m,n)-injective module, (α,m,n)-cotorsion module.

1. INTRODUCTION

The notions of right (m,n)-injective modules and right (m,n)-injective rings were introduced and studied by Chen, Ding, Li, and Zhou in [3]. For fixed positive integers m and n, a right R-module M is called (m,n)*-injective* if every right R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M. So, a ring R is said to be *right* (m,n)*-injective* if R_R is an (m,n)-injective R-module (see [3]). In the present paper, we introduce copure (m,n)-injective modules and obtain some properties of them. This definition unifies several definitions of injectivity of modules. Among other results, we also prove that if R is a right noetherian ring, then every $\mathcal{I}_0$-syzygy of any right R-module is copure (m,n)-injective.

Let M be a right R-module and α a fixed nonnegative integer. The module M is called *cotorsion* if $\mathrm{Ext}_R^1(F,M)=0$ for any flat right R-module F [5]. M is called *an* α*-cotorsion module* if $\mathrm{Ext}_R^{\alpha+1}(N,M)=0$ for any flat right R-module N. M is said to be *an* α*-flat module* if $\mathrm{Ext}_R^1(M,N)=0$ for any α-cotorsion right R-module N. Note that M is 0-cotorsion (respectively, 0-flat) if and only if M is cotorsion (respectively, flat). By [11, Remark 3.4], if α and β are integers with $0\le\alpha\le\beta$, then any α-cotorsion right R-module is β-cotorsion, and any β-flat right R-module is α-flat (see [10, 11, 12, 13]). In Section 3, we define and study (α,m,n)-cotorsion modules.

We prove the following theorems.

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THEOREM. *Every right R-module is (α,m,n)-cotorsion if and only if every copure (m,n)-flat right R-module is (α,m,n)-cotorsion.*

THEOREM. *Every right R-module is copure (m,n)-flat if and only if every (α,m,n)-cotorsion right R-module is copure (m,n)-flat.*

Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called *a $\mathcal{C}$-preenvelope* of M if, for any homomorphism f: $M \to F^{'}$ with $F^{'} \in \mathcal{C}$, there is a homomorphism $g : F \to F^{'}$ such that $g\phi = f$ (see [4]). On the other hand, if the only such g are automorphisms of F when $F^{'} = F$ and $f = \phi$, the $\mathcal{C}$-preenvelope ϕ is called *a $\mathcal{C}$-envelope* of M. It is well known that $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to an isomorphism. According to [6, Definition 7.1.6], a monomorphism α: $M \to C$ with $C \in \mathcal{C}$ is said to be *a special $\mathcal{C}$-preenvelope* of M if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathcal{C}$.

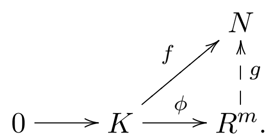
Dually we have the definitions of a (special) $\mathcal{C}$- precover and a $\mathcal{C}$-cover. Special $\mathcal{C}$-preenvelopes (resp. special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp. $\mathcal{C}$-precovers). $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to an isomorphism.

Throughout this paper, R is an associative ring with identity and all modules are unitary, and we freely use the conventions of the notions for homological algebra from the books Enochs-Jenda [6] and Rotman [14]. The symbols $E(M)$ and $C(M)$ will denote the injective envelope and cotorsion envelope of an R-module M, respectively. The latter always exists by a result of Bican, ElBashir and Enochs [1].

2. COPURE (m,n)-INJECTIVE MODULES

According to Enochs and Jenda [7], a left R-module M is called *copure injective* if $\mathrm{Ext}^1_R(N,M) = 0$ for all injective left R-modules N. Now we recall that an R-module M is called *(m,n)-presented* if there exists an exact sequence of right R-modules $0 \to K \to R^m \to M \to 0$, where K is n-generated. We call M a *copure (m,n)-injective module* if $\mathrm{Ext}^1_R(N,M) = 0$ for all (m,n)-injective modules N.

Let $\mathcal{I}_{(m,n)}$ denote the class of all (m,n)-injective modules. So a module N belongs to $\mathcal{I}_{(m,n)}$ if and only if the following diagram is commutative for any map $f : K \to N$, where K is n-generated

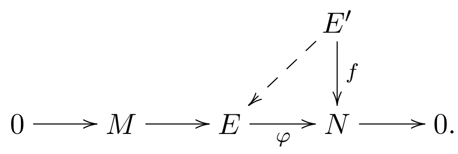


Theorem 2.1 below generalizes [9, Proposition 2.4] to (m,n)-injective modules.

THEOREM 2.1. *The following are equivalent for an R-module M:*

(1) *M is a copure (m,n)-injective module.*
(2) *For any exact sequence $0 \to M \to E \xrightarrow{\varphi} N \to 0$ with $E \in \mathcal{I}_{(m,n)}$, $E \xrightarrow{\varphi} N \to 0$ is a precover of N in $\mathcal{I}_{(m,n)}$.*
(3) *M is a kernel of an $\mathcal{I}_{(m,n)}$-precover $\pi : A \to A/M$ with A injective.*
(4) *M is an injective module with respect to every exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{I}_{(m,n)}$.*

Proof. (1) $\Rightarrow$ (2) Let $0 \to M \to E \xrightarrow{\varphi} N \to 0$ be any exact sequence with $E \in \mathcal{I}_{(m,n)}$ and let $E' \in \mathcal{I}_{(m,n)}$. Consider the following diagram:



If we apply $\mathrm{Hom}_R(E', -)$, we obtain from (1) that

$$\mathrm{Hom}_R(E', M) \to \mathrm{Hom}_R(E', E) \xrightarrow{\varphi^*} \mathrm{Hom}_R(E', N) \to \mathrm{Ext}^1_R(E', M) = 0.$$

Since E' is (m,n)-injective by assumption, there exists $g \in \mathrm{Hom}_R(E', E)$ such that $\varphi^*(g) = f$, so $g\varphi = f$.

(2) $\Rightarrow$ (3) Obviously the sequence $0 \to M \to E(M) \xrightarrow{\varphi} E(M)/M \to 0$ is exact. Since $E(M)$ is injective, it is (m,n)-injective. Using (2), we get that $E(M) \xrightarrow{\varphi} E(M)/M$ is an $\mathcal{I}_{(m,n)}$-precover.

(3) $\Rightarrow$ (1) By (3), M is a kernel of an $\mathcal{I}_{(m,n)}$-precover $\beta : E \to E/M$. So there is an exact sequence $0 \to M \xrightarrow{\alpha} E \xrightarrow{\beta} E/M \to 0$, where E is injective. Let N be any (m,n)-injective module. Then the sequence

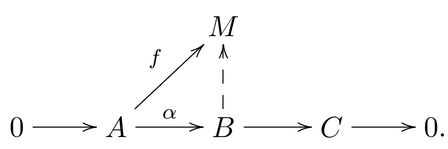
$$\mathrm{Hom}_R(N, E) \xrightarrow{\beta^*} \mathrm{Hom}_R(N, E/M) \xrightarrow{\alpha^*} \mathrm{Ext}^1_R(N, M) \to 0$$

is exact. Since $\beta : E \to E/M$ is an $\mathcal{I}_{(m,n)}$-precover with E injective, every map from N to E/M lifts to E. Therefore

$$\mathrm{Hom}_R(N, E) \xrightarrow{\beta^*} \mathrm{Hom}_R(N, E/M) \xrightarrow{\alpha^*} 0.$$

Hence $\mathrm{Ext}^1_R(N, M) = 0$, and so M is an (m,n)-injective module.

(1) $\Rightarrow$ (4) Consider the following diagram with $C \in \mathcal{I}_{(m,n)}$



If we apply $\mathrm{Hom}_R(-, M)$, we obtain the exact sequence

$$\mathrm{Hom}_R(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(C, M) = 0.$$

By (1), C is (m,n)-injective. Hence there exists $g \in \mathrm{Hom}_R(B,M)$ such that $\alpha^*(g)=f$, i.e., $g\alpha = f$.

(4) $\Rightarrow$ (1). Let $0 \to M \xrightarrow{f} E \to N \to 0$ be an exact sequence such that N is an (m,n)-injective module. By (4), the identity map 1_M of M extends to a map $f: E \to M$ such that $gf = 1_M$, so $\mathrm{Ext}^1_R(N,M)=0$. □

Corollary 2.2. *Let R be a right noetherian ring. Then every $\mathcal{I}_0$-syzygy of any right R-module is copure (m,n)-injective.*

Proof. Let $\cdots \to E_1 \to E_0 \to M \to 0$ be a right $\mathcal{I}_0$-resolution of a right R-module M. By [8, Lemma 8.4.34], $E_i \to K_i$ is an $\mathcal{I}_n$-precover, where K_i is the nth $\mathcal{I}_0$-syzygy of M, that is, $K_i = \ker(E_{i-1} \to E_{i-2})$, and so, by Theorem 2.1, the module K_i is copure (m,n)-injective for $i \geq 1$. □

Given a class $\mathcal{L}$ of right R-modules, the right orthogonal class of $\mathcal{L}$ is defined as

$$\mathcal{L}^{\perp} = \{C \in Mod-R \mid \mathrm{Ext}^1_R(L,C)=0 \text{ for all } L \in \mathcal{L}\},$$

and, similarly, the left orthogonal class of $\mathcal{L}$ is

$$^{\perp}\mathcal{L} = \{C \in Mod-R \mid \mathrm{Ext}^1_R(C,L)=0 \text{ for all } L \in \mathcal{L}\}.$$

According to Enochs and Jenda [6], a right R-module F is said to be *copure flat* if $\mathrm{Tor}^R_1(F,N)=0$ for every injective left R-module N. So we call the module M_R *copure (m,n)-flat* if $\mathrm{Tor}^R_1(M,N)=0$ for every (m,n)-injective left R-module N. Clearly, M_R is a copure (m,n)-flat module if and only if M_R is a copure$_1(m,n)$-flat module.

For any module M, we write $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. The following theorem generalizes Theorem 2.6 in [9].

Theorem 2.3. *The following are equivalent for any right R-module M:*

(1) *M is a copure (m,n)-flat module.*
(2) *M^+ is a copure (m,n)-injective module.*
(3) *M belongs to the class $^{\perp}(\mathcal{I}^+_{(m,n)})$ where $\mathcal{I}^+_{(m,n)} = \{N^+ \mid N \in \mathcal{I}_{(m,n)}\}$.*
(4) *For any exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with $C \in \mathcal{I}_{(m,n)}$, the functor $M \otimes_R -$ is exact.*

Proof. By [2, VI, 5.1] or [14, p. 360], there are the following standard isomorphisms: $\mathrm{Ext}^1_R(N,M^+) \cong \mathrm{Tor}^R_1(M,N)^+ \cong \mathrm{Ext}^1_R(M,N^+)$ for any (m,n)-injective left R-module N. Thus the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold.

(1) $\Rightarrow$ (4) Let $0 \to A \to B \to C \to 0$ be any exact sequence of left R-modules with $C \in \mathcal{I}_{(m,n)}$. We apply the functor $M \otimes_R -$ to this sequence to obtain the exact sequence

$$\cdots \to \mathrm{Tor}^R_1(M,C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0.$$

Since C is (m,n)-injective, we have $\mathrm{Tor}^R_1(M,C)=0$ for all $C \in \mathcal{I}_{(m,n)}$, by (1). Hence $M \otimes_R -$ is exact. This proves (4).

(4) $\Rightarrow$ (1) Let C be an (m,n)-injective module and consider a short exact sequence $0 \to A \to B \to C \to 0$ with B projective. By applying the functor $M \otimes_R -$ to this sequence, we obtain the exact sequence

$$0 = \mathrm{Tor}_1^R(M,B) \to \mathrm{Tor}_1^R(M,C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0.$$

Using the hypothesis we get $\mathrm{Tor}_1^R(M,C) = 0$. This completes the proof. □

3. (α, m, n)-COTORSION MODULES

Let M be a right R-module and α a fixed nonnegative integer. Recall that M is called a *cotorsion module* if $\mathrm{Ext}_R^1(F,M) = 0$ for any flat right R-module F (see [5]), and the module M is called an *α-cotorsion module* if $\mathrm{Ext}_R^{\alpha+1}(N,M) = 0$ for any flat right R-module N. We call the module M_R *(α, m, n)-cotorsion* if $\mathrm{Ext}_R^{\alpha+1}(N,M) = 0$ for any copure (m,n)-flat right R-module N.

For a right R-module M, $C_n(M)$ will denote an n-cotorsion envelope of M.

THEOREM 3.1. *The following are equivalent for a ring R:*

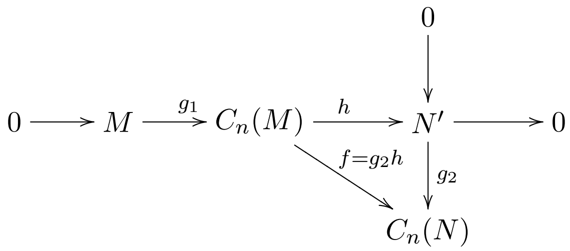
(1) *Every right R-module is (α, m, n)-cotorsion.*
(2) *Every copure (m,n)-flat right R-module is (α, m, n)-cotorsion.*
(3) *For any right R-homomorphism $f : M_1 \to M_2$ such that M_1 and M_2 are (α, m, n)-cotorsion modules,* $\ker(f)$ *is (α, m, n)-cotorsion.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [11, Theorem 4.1].

(1) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2) Let M be a copure (m,n)-flat module R-module. Then we have the sequences $0 \longrightarrow M \xrightarrow{g_1} C_n(M) \longrightarrow 0$ and $0 \longrightarrow N \xrightarrow{g_2} C_n(N)$.

Hence we obtain the following exact commutative diagram



and so

$$0 \longrightarrow M \longrightarrow C_n(M) \longrightarrow C_n(N)$$

is exact with $C_n(M)$ and $C_n(M')$ copure (m,n)-flat modules. By (3), this implies that $M = \ker(h) = \ker(f)$ is a copure (m,n)-flat module. □

For a right R-module M, $F_n(M)$ will denote an n-flat cover of M.

THEOREM 3.2. *The following are equivalent for a ring R:*

(1) *Every right R-module is copure (m,n)-flat.*

(2) *Every (α,m,n)-cotorsion right R-module is copure (m,n)-flat.*
(3) *Every (α,m,n)-cotorsion right R-module is injective.*
(4) *For any right R-homomorphism $f : M_1 \to M_2$ such that M_1 and M_2 are copure (m,n)-flat modules,* $\operatorname{coker}(f)$ *is copure (m,n)-flat.*
(5) *Every non-zero right R-module contains a non-zero copure (m,n)-flat submodule.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from [11, Theorem 4.1].

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are clear.

(4) $\Rightarrow$ (2) Let M be an (α,m,n)-cotorsion right R-module. Then we have the sequences $F_n(N) \xrightarrow{g_1} N \longrightarrow 0$ and $F_n(M) \xrightarrow{g_2} M \longrightarrow 0$.

Hence we obtain the following exact commutative diagram

$$\begin{array}{ccccccccc} & & F_n(N) & & & & & & \\ & & \downarrow g_1 & \searrow^{f=hg_1} & & & & & \\ 0 & \longrightarrow & N & \xrightarrow{h} & F_n(M) & \xrightarrow{g_2} & M & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

and so

$$F_n(K) \longrightarrow F_n(M) \longrightarrow M \longrightarrow 0$$

is exact with $F_n(N)$ and $F_n(M)$ n-flat. Thus $M \cong \operatorname{coker}(h) \cong \operatorname{coker}(f)$ is copure (m,n)-flat, by (4).

(5) $\Rightarrow$ (3). Assume that $0 \to A \to B \to C \to 0$ is any exact sequence and A is a submodule of B. Let M be an (α,m,n)-cotorsion right R-module and let $f : A \to M$ be any homomorphism. By Zorn's Lemma, we have $g : D \to M$, where $A \subseteq D \subseteq B$, $g|_A = f$, such that g cannot be extended to any submodule of B properly containing D. We claim that $D = B$. By (5), there exists a non-zero submodule N/D of B/D such that N/D is copure (m,n)-flat. Since M is (α,m,n)-cotorsion, there is a map $h : N \to M$ such that $h|_D = g$. It is obvious that h extends g, hence we get the desired contradiction, and so M is injective. ☐

Theorem 3.3. *Let $\mu : N \to M$ be a monomorphism. If* $\operatorname{coker}(\mu)$ *is copure (m,n)-flat, then $i\mu : N \to H$ is an (α,m,n)-cotorsion preenvelope of N whenever $i : M \to H$ is an (α,m,n)-cotorsion preenvelope of M.*

Proof. Let L be an (α,m,n)-cotorsion right R-module and $g : N \to L$ be any R-homomorphism. The exactness of the sequence

$$0 \longrightarrow N \xrightarrow{\mu} M \longrightarrow \operatorname{coker}(\mu) \longrightarrow 0$$

induces an exact sequence

$$\operatorname{Hom}_R(M,L) \longrightarrow \operatorname{Hom}_R(N,L) \longrightarrow \operatorname{Ext}^1_R(\operatorname{coker}(\mu),L) .$$

Since coker(μ) is copure (m,n)-flat, $\mathrm{Hom}_R(M,L) \longrightarrow \mathrm{Hom}_R(N,L)$ is epic. Therefore there exists $\beta : M \to L$ with $g = \beta\mu$. Then there exists $\gamma : H \to L$ such that $\beta = \gamma i$. Hence $\gamma(i\mu) = (\gamma i)\mu = \beta\mu = g$. □

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