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ULAM STABILITY GENERALIZATIONS OF 4^{th} ORDER TERNARY DERIVATIONS ASSOCIATED TO A JMRASSIAS QUARTIC FUNCTIONAL EQUATION ON FRÉCHET ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach ternary algebra over $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{X}$ be a ternary Banach $\mathcal{A}$-module. A quartic mapping $D : (\mathcal{A}, [\,]_{\mathcal{A}}) \to (\mathcal{X}, [\,]_{\mathcal{X}})$ is called a 4^{th} order ternary derivation if $D([x, y, z]) = [D(x), y^4, z^4]+[x^4, D(y), z^4]+[x^4, y^4, D(z)]$ for all $x, y, z \in \mathcal{A}$. We prove Ulam stability generalizations of 4^{th} order ternary derivations associated to the following JMRassias quartic functional equation on Fréchet algebras $f(2x+y)+f(2x-y) = 4f(x+y)+4f(x-y)+24f(x)-6f(y)$.

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Key words. Ulam stability, quartic functional equation, ternary Banach algebras, Fréchet algebras, 4^{th} order ternary derivation.

1. INTRODUCTION

Ternary algebraic operations were considered in the 19th century by several mathematicians such as A. Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([30]). The comments on physical applications of ternary structures can be found in [1, 31, 32, 33, 34, 48, 51].

A nonempty set G with a ternary operation $[\cdot, \cdot, \cdot] : G^3 \to G$ is called a *ternary groupoid* and is denoted by $(G, [\cdot, \cdot, \cdot])$. The ternary groupoid $(G, [\cdot, \cdot, \cdot])$ is called *commutative* if $[x_1, x_2, x_3] = [x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)}]$ for all $x_1, x_2, x_3 \in G$ and all permutations δ of $\{1, 2, 3\}$. If a binary operation $\circ$ is defined on G such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[\cdot, \cdot, \cdot]$ is derived from $\circ$. We say that $(G, [\cdot, \cdot, \cdot])$ is a *ternary semigroup* if the operation $[\cdot, \cdot, \cdot]$ is associative, i.e., if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see [4], [34] and [51]).

Let $\mathcal{A}$ be a Banach ternary algebra and $\mathcal{X}$ be a Banach space. Then $\mathcal{X}$ is called a *ternary Banach $\mathcal{A}$-module* if the module operations $\mathcal{A} \times \mathcal{A} \times \mathcal{X} \to \mathcal{X}$, $\mathcal{A} \times \mathcal{X} \times \mathcal{A} \to \mathcal{X}$, and $\mathcal{X} \times \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ are $\mathbb{C}$-linear in every variable, and satisfy the following relations

$$[[xab]_{\mathcal{X}}\ cd]_{\mathcal{X}} = [x[abc]_{\mathcal{A}}\ d]_{\mathcal{X}} = [xa[bcd]_{\mathcal{A}}]_{\mathcal{X}},$$
$$[[axb]_{\mathcal{X}}\ cd]_{\mathcal{X}} = [a[xbc]_{\mathcal{X}}\ d]_{\mathcal{X}} = [ax[bcd]_{\mathcal{A}}]_{\mathcal{X}},$$
$$[[abx]_{\mathcal{X}}\ cd]_{\mathcal{X}} = [a[bxc]_{\mathcal{X}}\ d]_{\mathcal{X}} = [ab[xcd]_{\mathcal{X}}]_{\mathcal{X}},$$
$$[abc]_{\mathcal{A}}\ xd]_{\mathcal{X}} = [a[bcx]_{\mathcal{X}}\ d]_{\mathcal{X}} = [ab[cxd]_{\mathcal{X}}]_{\mathcal{X}},$$
$$[[abc]_{\mathcal{A}}\ dx]_{\mathcal{X}} = [a[bcd]_{\mathcal{A}}\ x]_{\mathcal{X}} = [ab[cdx]_{\mathcal{X}}]_{\mathcal{X}},$$

for all $x \in \mathcal{X}$ and all $a, b, c, d \in \mathcal{A}$, respectively,

$$\max\{\|xab\|, \|axb\|, \|abx\|\} \leq \|a\|\|b\|\|x\|,$$

for all $x \in \mathcal{X}$ and all $a, b \in \mathcal{A}$.

In functional analysis and related areas of mathematics, Fréchet spaces, named after Maurice Fréchet, are special topological vector spaces. They are generalizations of Banach spaces (normed vector spaces which are complete with respect to the metric induced by the norm). Fréchet spaces can be defined in two equivalent ways: the first employs a translation invariant metric, the second a countable family of semi-norms. Thus we recall that a topological vector space X is a *Fréchet space* if it satisfies the following three properties:

1) it is complete as a uniform space;

2) it is locally convex;

3) its topology can be induced by a translation invariant metric, i.e., a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$. For more details we refer to [16].

The study of stability problems has its origin in a famous talk given by S. M. Ulam [50] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?". In the next year, in 1941, D. H. Hyers [28] answered affirmatively the question of Ulam. This stability phenomenon is called the *Ulam stability* of the additive functional equation $g(x + y) = g(x) + g(y)$. A generalized version of the theorem proved by Hyers was given by Th. M. Rassias [43] for approximately linear mappings, leading to the new concept of *generalized Ulam stability* of functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see, for example, [2], [6], [9], [10], [21]–[24], [27], [29], [37]–[46], and [49]).

Given a mapping $f : A \to B$ consider the quartic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) + 24f(y). \tag{1}$$

In 1999 J. M. Rassias [37] proved the stability of (1). Recently, Chang and Sahoo [8] solved (1). On the other hand, it is easy to see that the solution f of (1) is even, thus the above equation can be written as

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y).$$

Prak and Bae proved in [36] that a function f between two real vector spaces X and Y is a solution of (1) if and only if there exists a unique symmetric multi–additive function $M : X \times X \times X \times X \to Y$ such that $f(x) = M(x, x, x, x)$ for all x. We refer for more information on such problems to [1], [8] and [15]. Recently, M. Bavand Savadkouhi, M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour gave in [3] approximate ternary Jordan derivations on Banach ternary algebras. For more details we refer to [7], [10]–[12], [17]–[20], [25], [26], [35], and [47].

In this paper we prove the generalized Ulam stability of 4^{th} order ternary derivations on Fréchet algebras, associated with the following quartic functional equation $f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$.

2. MAIN RESULTS

DEFINITION 1. Let $\mathcal{A}$ be a Banach ternary algebra over $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{X}$ be a ternary Banach $\mathcal{A}$-module. A quartic mapping $D:(\mathcal{A},[\ ]_{\mathcal{A}})\to(\mathcal{X},[\ ]_{\mathcal{X}})$ is called a 4^{th} *order ternary derivation* if

$$D([x,y,z])=[D(x),y^4,z^4]+[x^4,D(y),z^4]+[x^4,y^4,D(z)],\ \text{for all } x,\ y,\ z\in\mathcal{A}.$$

Now we investigate the generalized Ulam stability of 4^{th} order ternary derivations on Fréchet algebras.

THEOREM 1. *Let A and B be two Fréchet algebras with metrics d_1 and d_2, respectively, and let $f:A\to B$ be a mapping with the property that there exists a function $\psi:A\times A\times A\to[0,\infty)$ such that the following relations hold for every $x,\ y,\ z\in A$*

$$\sum_{j=0}^{\infty}\frac{1}{2^{4j}}\psi(2^jx,2^jy,2^jz)<\infty, \tag{2}$$

$$\begin{aligned}&d_2(f(2x+y)+f(2x-y)+6f(y),4f(x+y)+4f(x-y)+24f(x))\\&\le\psi(x,y,0),\end{aligned} \tag{3}$$

$$\begin{aligned}&d_2(f([x,y,z]),[f(x),y^4,z^4]+[x^4,f(y),z^4]+[x^4,y^4,f(z)])\\&\le\psi(x,y,z).\end{aligned} \tag{4}$$

Then there exists a unique 4^{th} order ternary derivation $D:A\to B$ such that

$$d_2(f(x),D(x))\le\frac{1}{8}\sum_{j=0}^{\infty}\frac{\psi(2^jx,0,0)}{2^{4j}},\ \textit{for all } x\in A. \tag{5}$$

Proof. Putting $x=y=0$ in (3), we get $f(0)=0$, and putting $y=0$ in (3), we obtain

$$d_2(2f(2x),32f(x))\le\psi(x,0,0),\ \text{for all } x\in A. \tag{6}$$

Multiplying both sides of (6) by $\frac{1}{32}$, we get

$$d_2\left(\frac{f(2x)}{2^4},f(x)\right)\le\frac{\psi(x,0,0)}{32},\ \text{for all } x\in A. \tag{7}$$

One can use induction to show that, for all non-negative integers n,

$$d_2\left(\frac{f(2^nx)}{2^{4n}},f(x)\right)\le\frac{1}{32}\sum_{j=0}^{n-1}\frac{\psi(2^jx,0,0)}{2^{4j}},\ \text{for all } x\in A. \tag{8}$$

Hence we get that for all non-negative integers n and m with $n \geq m$

$$d_2\left(\frac{f(2^{n+m}x)}{2^{4(n+m)}}, \frac{f(2^m x)}{2^{4m}}\right) \leq \frac{1}{32}\sum_{j=m}^{n+m-1}\frac{\psi(2^j x, 0, 0)}{2^{4j}}, \text{ for all } x \in A.$$

It follows from (2) that $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ is a Cauchy sequence, so, since B is complete, this sequence is convergent. Define now the mapping $D : A \to B$ by

$$D(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^{4n}}, \text{ for all } x \in A. \tag{9}$$

If we replace in (3) the elements x and y, respectively, by $2^n x$ and $2^n y$, and multiply both sides of (3) by $\frac{1}{2^{4n}}$, we get, for all non-negative integers n,

$$\begin{aligned} &d_2(D(2x+y) + D(2x-y) + 6D(y), 4D(x+y) + 4D(x-y) + 24D(x)) \\ &= \lim_{n\to\infty}\frac{1}{2^{4n}}d_2(f(2^n(2x+y)) + f(2^n(2x-y)) + 6f(2^n y), 4f(2^n(x+y)) \\ &+ 4f(2^n(x-y)) + 24f(2^n x)) \leq \lim_{n\to\infty}\frac{\psi(2^n x, 2^n y, 0)}{2^{4n}}, \text{ for all } x, y \in A. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $D(2x+y) + D(2x-y) + 6D(y) = 4D(x+y) + 4D(x-y) + 24D(x)$, for all x, $y \in A$. Moreover, (8) and (9) yield that $d_2(f(x), D(x)) \leq \frac{1}{32}\sum_{j=0}^{\infty}\frac{\psi(2^j x,0,0)}{2^{4j}}$, for all $x \in A$. It follows from (4) that, for, all non-negative integers n,

$$\begin{aligned} &d_2(D([x,y,z]), [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)]) \\ &= \lim_{n\to\infty}\frac{1}{2^{12n}}d_2(f(2^{3n}[x,y,z]), [f(2^n x), (2^n y)^4, (2^n z)^4] \\ &+ [(2^n x)^4, f(2^n y), (2^n z)^4] + [(2^n x)^4, (2^n y)^4, f(2^n z)]) \leq \lim_{n\to\infty}\frac{\psi(2^n x, 2^n y, 2^n z)}{2^{12n}}, \end{aligned}$$

for all $x \in A$. Taking the limit as $n \to \infty$, we obtain

$$D([x,y,z]) = [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^3, D(z)],$$

for all $x \in A$. Consider now $D' : A \to B$ to be another 4^{th} order ternary derivation satisfying (5). Then we have, for all $x \in A$, $d_2(D(x), D'(x)) = \frac{1}{2^{4n}}d_2(D(2^n x), D'(2^n x)) \leq \frac{1}{2^{4n}}(d_2(D(2^n x), f(2^n x)) + d_2(f(2^n x), D'(2^n x)))$, so $d_2(D(x), D'(x)) \leq \frac{1}{16}\sum_{j=n}^{\infty}\frac{\psi(2^j x,0,0)}{2^{4j}}$. By taking the limit as $n \to \infty$, we conclude that $D(x) = D'(x)$ for all $x \in A$. Hence D is the unique 4^{th} order ternary derivation satisfying (5). □

THEOREM 2. *Let A and B be two Fréchet algebras with metrics d_1 and d_2, respectively, and let $f : A \to B$ be a mapping with the property that there exists a function $\psi : A \times A \times A \to [0, \infty)$ satisfying, for all x, y, $z \in A$, the inequalities* (3), (4) *and*

$$\sum_{j=1}^{\infty} 2^{12j}\psi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty. \tag{10}$$

Then there exists a unique 4^{th} *order ternary derivation* $D : A \to B$ *such that*

$$d_2(f(x), D(x)) \leq \frac{1}{32}\sum_{j=1}^{\infty} 2^{4j}\psi\left(\frac{x}{2^j}, 0, 0\right), \text{ for all } x \in A. \tag{11}$$

Proof. If we replace x by $\frac{x}{2}$ in (6), and multiply both sides of (6) by $\frac{1}{2}$, we get

$$d_2\left(2^4 f(\frac{x}{2}), f(x)\right) \leq \frac{1}{2}\psi(\frac{x}{2}, 0, 0), \text{ for all } x \in A. \tag{12}$$

Using induction, one obtains, for all non-negative integers n, that
(13)

$$d_2\left(2^{4n} f(\frac{x}{2^n}), f(x)\right) \leq \frac{1}{2}\sum_{j=0}^{n-1} 2^{4j}\psi\left(\frac{x}{2^{j+1}}, 0, 0\right) = \frac{1}{32}\sum_{j=1}^{n} 2^{4j}\psi\left(\frac{x}{2^j}, 0, 0\right),$$

for all $x \in A$, hence, for all non-negative integers n and m with $n \geq m$, $d_2\left(2^{4(n+m)} f\left(\frac{x}{2^{n+m}}\right), 2^{4m} f\left(\frac{x}{2^m}\right)\right) \leq \frac{1}{32}\sum_{j=1+m}^{n+m} 2^{4j}\psi\left(\frac{x}{2^j}, 0, 0\right)$, for all $x \in A$. It follows from (10) that $\left\{2^{4n} f(\frac{x}{2^n})\right\}$ is a Cauchy sequence, so, since B is complete, this sequence is convergent. Define now the mapping $D : A \to B$ by

$$D(x) := \lim_{n\to\infty} 2^{4n} f\left(\frac{x}{2^n}\right), \text{ for all } x \in A. \tag{14}$$

Replacing in (3) the elements x and y, respectively, by $\frac{x}{2^n}$ and $\frac{y}{2^n}$, and multiplying both sides of (3) by 2^{4n}, we get, for all non-negative integers n, that

$$\begin{aligned} &d_2(D(2x+y) + D(2x-y) + 6D(y), 4D(x+y) + 4D(x-y) + 24D(x)) \\ &= \lim_{n\to\infty} 2^{4n} d_2\left(f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) + 6f\left(\frac{y}{2^n}\right), 4f\left(\frac{x+y}{2^n}\right)\right. \\ &\left. + 4f\left(\frac{x-y}{2^n}\right) + 24f\left(\frac{x}{2^n}\right)\right) \leq \lim_{n\to\infty} 2^{4n}\psi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right), \text{ for all } x, y \in A. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $D(2x+y) + D(2x-y) = 4D(x+y) + 4D(x-y) + 24D(x) - 6D(y)$, for all $x, y \in A$. Moreover, it follows from (13) and (14) that $d_2(f(x), D(x)) \leq \frac{1}{32}\sum_{j=1}^{\infty} 2^{4j}\psi\left(\frac{x}{2^j}, 0, 0\right)$, for all $x \in A$. Relation (4) yields for all non-negative integers n that

$$\begin{aligned} &d_2(D([x,y,z]), [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)]) \\ &= \lim_{n\to\infty} 2^{12n} d_2\left(f\left(\frac{[x,y,z]}{2^{4n}}\right), \left[f\left(\frac{x}{2^n}\right), \left(\frac{y}{2^n}\right)^4, \left(\frac{z}{2^n}\right)^4\right]\right. \\ &\left. + \left[\left(\frac{x}{2^n}\right)^4, f\left(\frac{y}{2^n}\right), \left(\frac{z}{2^n}\right)^4\right] + \left[\left(\frac{x}{2^n}\right)^4, \left(\frac{y}{2^n}\right)^4, f\left(\frac{z}{2^n}\right)\right]\right) \\ &\leq \lim_{n\to\infty} 2^{12n}\psi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), \text{ for all } x, y, z \in A. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$D([x,y,z]) = [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)], \text{ for all } x, y, z \in A.$$

Consider now $D' : A \to B$ to be another 4^{th} order ternary derivation satisfying (11). Then we have, for all $x \in A$, that

$$\begin{aligned} d_2(D(x), D'(x)) &= 2^{4n} d_2\left(D\left(\frac{x}{2^n}\right), D'\left(\frac{x}{2^n}\right)\right) \\ &\leq 2^{4n}\left(d_2\left(D\left(\frac{x}{2^n}\right), f\left(\frac{x}{2^n}\right)\right) + d_2\left(f\left(\frac{x}{2^n}\right), D'\left(\frac{x}{2^n}\right)\right)\right) \\ &\leq \frac{1}{16}\sum_{j=1+n}^{\infty} 2^{4j}\psi\left(\frac{x}{2^j}, 0, 0\right). \end{aligned}$$

Taking limit as $n \to \infty$, we conclude that $D(x) = D'(x)$, for all $x \in A$. This proves that the the mapping $D : A \to B$ is the unique 4^{th} order ternary derivation satisfying (11). □

Using Theorems 1 and 2, we show now the following Ulam stability of 4^{th} order ternary derivations on Fréchet algebras.

COROLLARY 1. *Let A and B be two Fréchet algebras with metrics d_1 and d_2, respectively, let $p \geq 0$ be such that $p \neq 4$, and let ϵ, θ be nonnegative real numbers. If $f : A \to B$ is a mapping such that the following relations hold for all $x, y, z \in A$*

$$\begin{aligned} &d_2(f(2x+y) + f(2x-y) + 6f(y), 4f(x+y) + 4f(x-y) + 24f(x)) \\ &\leq \epsilon(d_1(x,0)^p + d_1(y,0)^p), \end{aligned} \tag{15}$$

$$\begin{aligned} &d_2(f([x,y,z]), [f(x), y^4, z^4] + [x^4, f(y), z^4] + [x^4, y^4, f(z)]) \\ &\leq \epsilon(d_1(x,0)^p + d_1(y,0)^p + d_1(z,0)^p), \end{aligned} \tag{16}$$

then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$d_2(D(x), f(x)) \leq \frac{\theta d_1(x,0)^p}{2(2^4 - 2^p)}, \text{ for all } x \in X, \tag{17}$$

when $p < 4$, and

$$d_2(D(x), f(x)) \leq \frac{\theta d_1(x,0)^p}{2(2^p - 2^4)}, \text{ for all } x \in X, \tag{18}$$

when $p > 4$.

Proof. Let $\psi(x,y,z) := \epsilon(d_1(x,0)^p + d_1(y,0)^p + d_1(z,0)^p)$, for all $x, y, z \in A$. The statement follows now from Theorems 1 and 2. □

Using Theorem 1, we prove the following generalized Ulam stability of 4^{th} order ternary derivations in ternary Banach algebras.

Corollary 2. *Let A and B be two ternary Banach algebras. Suppose that $f : A \to B$ is a mapping such that there exists a function $\psi : A\times A\times A \to [0, \infty)$ satisfying for all $x, y, z \in A$ the relations* (2),

$$\|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) + 24f(x) - 6f(y)\| \leq \psi(x, y, 0), \tag{19}$$

$$\|f([x, y, z]) - [f(x), y^4, z^4] - [x^4, f(y), z^4] - [x^4, y^4, f(z)]\| \leq \psi(x, y, z). \tag{20}$$

Then there exists a unique 4^{th} *order ternary derivation* $D : A \to B$ *such that* $\|f(x) - D(x)\| \leq \frac{1}{32} \sum_{j=0}^{\infty} \frac{1}{2^{4j}} \psi(2^j x, 0, 0)$, *for all* $x \in A$.

Proof. Put $d_1(x, y) := \|x - y\|$, for all $x, y \in A$, and apply Theorem 1. □

Corollary 3. *Let A and B be two ternary Banach algebras. Suppose that $f : A \to B$ is a mapping such that there exists a function $\psi : A\times A\times A \to [0, \infty)$ satisfying* (10), (19) *and* (20) *for all $x, y, z \in A$. Then there exists a unique* 4^{th} *order ternary derivation $D : A \to B$ such that*

$$\|f(x) - D(x)\| \leq \frac{1}{32} \sum_{j=1}^{\infty} 2^{4j} \psi\left(\frac{x}{2^j}, 0, 0\right), \text{ for all } x \in A.$$

Proof. Put $d_1(x, y) := \|x - y\|$, for all $x, y \in A$, and apply Theorem 2. □

Corollary 4. *Let A and B be two ternary Banach algebras, let $p \geq 0$ be such that $p \neq 4$, and let ϵ, θ be nonnegative real numbers. If $f : A \to B$ is a mapping such that*

$$\|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$

and

$$\|f([x, y, z])-[f(x), y^4, z^4]-[x^4, f(y), z^4]-[x^4, y^4, f(z)]\| \leq \epsilon(\|x\|^p+\|y\|^p+\|z\|^p),$$

hold for all $x, y, z \in A$, then there exists a unique 4^{th} *order ternary derivation $D : A \to B$ such that*

$$\|D(x) - f(x)\| \leq \frac{\epsilon}{2(2^4 - 2^p)} \|x\|^p, \text{ for all } x \in X,$$

when $p < 4$, and

$$\|D(x) - f(x)\| \leq \frac{\theta}{2(2^p - 2^4)} \|x\|^p, \text{ for all } x \in X,$$

when $p > 4$.

Proof. Consider $d_1(x, y) = \|x - y\|$ and $d_2(x, y) = \|x - y\|$, for all $x, y \in A$, and apply Theorems 1 and 2. □

The following corollary is about the Ulam stability of 4^{th} order ternary derivations in ternary Banach algebras.

Corollary 5. *Let A and B be two ternary Banach algebras, let ϵ be a nonnegative real number, and let $f : A \to B$ be a mapping such that*

$$\|f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\| \leq \epsilon,$$

$$\|f([x,y,z])-[f(x),y^4,z^4]-[x^4,f(y),z^4]-[x^4,y^4,f(z)]\| \leq \epsilon,$$

for all $x, y, z \in A$. Then there exists a unique 4^{th} order ternary derivation $D : A \to B$ such that $\|D(x)-f(x)\| \leq \frac{\epsilon}{30}$, for all $x \in X$.

Proof. The statement follows from Corollary 4 if one takes there $p := 0$. □

The next result reveals the JMRassias stability for 4^{th} order ternary derivations associated to the mixed type product-sum function $(x,y) \to \epsilon(\|x\|^r\|y\|^s + \|x\|^p + \|y\|^p)$.

Corollary 6. *Let r, s, p be real numbers with $p \neq 4$, $r+s \neq 4$, $\epsilon > 0$. Let A and B be two ternary Banach algebras and let $f : A \to B$ be a mapping such that the relations $\|f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\| \leq \epsilon(\|x\|^r\|y\|^s+\|x\|^p+\|y\|^p)$ and $\|f([x,y,z])-[f(x),y^4,z^4]-[x^4,f(y),z^4]-[x^4,y^4,f(z)]\| \leq \epsilon(\|x\|^r\|y\|^s+\|x\|^p+\|y\|^p)$ hold for all $x, y, z \in A$. Then there exists a unique 4^{th} order ternary derivation $D : A \to B$ such that $\|D(x)-f(x)\| \leq \frac{\epsilon}{2|2^4-2^p|}\|x\|^p$, for all $x \in A$.*

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