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LOEWNER CHAINS AND GENERALIZED ALMOST STARLIKE MAPPINGS

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Abstract. In this paper, we introduce the notion of generalized almost starlikeness on the unit disc as well as on the unit ball B^n in $\mathbb{C}^n$, and we prove that this notion can be characterized in terms of Loewner chains. Finally, we use the theory of Loewner chains to deduce that certain classes of generalized Roper-Suffridge extension operators preserve generalized almost starlikeness.

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Key words. Loewner chain, Roper-Suffridge extension operator, spirallike function, starlike function, subordination,subordination chain, biholomorphic mapping.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, ..., z_n)'$ with respect to the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the norm $\|z\| = \sqrt{\langle z, z\rangle}$. The symbol $'$ means the transpose of vectors and matrices.

Let $B_r^n = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B^n = B_1^n$. In the case of one complex variable B_r^n is denoted by U_r and U_1 by U.If G is an open set in $\mathbb{C}^n$, let $H(G)$ be the set of holomorphic maps from G into $\mathbb{C}^n$.

By $L(\mathbb{C}^n, \mathbb{C}^m)$ we denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I denote the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

A holomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exist and is holomorphic on the open set $f(B^n)$. A mapping $f \in H(B^n)$ is said to be locally biholomorphic if the Frechet derivative $Df(z)$ has a bounded inverse for each $z \in B^n$. If $f \in H(B^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. Let $S(B^n)$ be the set of all normalized biholomorphic mappings on B^n. We denote the classes of normalized convex and starlike mappings on B^n respectively by $K(B^n)$ and $S^*(B^n)$. In one variable we write $S(U) = S, K(U) = K$ and $S^*(U) = S^*$.

For $n \geq 2$, let $\widetilde{z} = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$, so that $z = (z_1, \widetilde{z}) \in \mathbb{C}^n$.

Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$)if there is a Schwarz mapping v (i.e., $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|, z \in B^n$) such that $f(z) = g(v(z)), z \in B^n$. If g is biholomorphic on B^n, this is equivalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

Definition 1.1. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if it satisfies the following conditions:

(i) $f(\cdot,t)$ is holomorphic and univalent on B^n, $f(0,t)=0$ and $Df(0,t)=\mathrm{e}^tI$ for each $t\geq 0$;

(ii) $f(\cdot,s)\prec f(\cdot,t)$ whenever $0\leq s\leq t<\infty$ and $z\in B^n$.

The subordination condition (ii) implies that there is a unique univalent Schwarz mapping $v=v(z,s,t)=\mathrm{e}^{(s-t)}z+\cdots$, called the transition mapping associated to $f(z,t)$, such that $f(z,s)=f(v(z,s,t),t), 0\leq s\leq t<\infty, z\in B^n$.

A key role in our discussion is played by the Caratheodory set (see [22]):

$$\mathcal{M}=\{h\in H(B^n): h(0)=0, Dh(0)=I, \Re\langle h(z),z\rangle>0, z\in B^n\}.$$

The following result was obtained by Pfaltzgraff ([22]).

Lemma 1.2. *Let $f: B^n\times[0,\infty)\to\mathbb{C}^n$ be a mapping such that*

(a) *$f(\cdot,t)\in H(B^n)$ for each $t\geq 0$;*

(b) *$f(z,t)$ is a locally absolutely continuous function of $t\in[0,\infty)$ locally uniformly with respect to $z\in B^n$.*

Let $h: B^n\times[0,\infty)\to\mathbb{C}^n$ satisfy the following conditions:

(i) *$h(\cdot,t)\in\mathcal{M}, \quad t\geq 0$;*

(ii) *for each $z\in B^n, h(z,t)$ is a measurable function of $t\in[0,\infty)$.*

Suppose that

$$\frac{\partial f}{\partial t}(z,t)=Df(z,t)h(z,t), \quad a.e. \quad t\geq 0$$

and for all $z\in B^n$, and suppose there exists a nonnegative sequence $\{t_m\}$, increasing to ∞ such that $\lim\limits_{m\to\infty}\mathrm{e}^{-t_m}f(z,t_m)=G(z)$ locally uniformly on B^n. Then $f(z,t)$ is a Loewner chain on B^n.

The converse of Lemma 1.3 is due to Graham, Kohr and Kohr [10].

Lemma 1.3. *Let $f(z,t)$ be a Loewner chain. Then there is a mapping $h=h(z,t)$ such that $h(\cdot,t)\in\mathcal{M}$ for each $t\geq 0, h(z,t)$ is measurable in t for each $z\in B^n$, and for a.e. $t\geq 0$,*

$$\frac{\partial f}{\partial t}(z,t)=Df(z,t)h(z,t), \forall z\in B^n. \tag{1.1}$$

We next recall the definition of spirallikeness with respect to a linear operator A on the unit ball B^n(see [27]). Clearly, if $A=0$, we obtain the usual notion of starlikeness. Also, in the case of one complex variable, if $A=\mathrm{e}^{-\mathrm{i}\alpha}$ in Definition 1.5., where $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$, we deduce the usual notion of spirallikeness of type α in the unit disc.

Definition 1.4. Let $A\in L(\mathbb{C}^n,\mathbb{C}^n)$ be such that $\Re\langle A(z),z\rangle>0$ for $z\in\mathbb{C}^n\setminus\{0\}$. Also let $f: B^n\to\mathbb{C}^n$ be a holomorphic mapping on B^n such that $f(0)=0$. We say that f is *spirallike* with respect to A if f is biholomorphic on B^n and $e^{-tA}f(z)\in f(B^n)$ for $z\in B^n$ and $t\in[0,\infty)$, where $\mathrm{e}^{-tA}=\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k!}A^k$.

A mapping $g\in H(B^n)$ is called *spirallike of type* α, $\alpha\in(\frac{-\pi}{2},\frac{\pi}{2})$, if g is spirallike with respect to $A=\mathrm{e}^{-\mathrm{i}\alpha}I$.

REMARK 1.5. Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$be such that $\Re\langle A(z), z\rangle > 0$ for $z \in \mathbb{C}^n \setminus \{0\}$. T. Suffridge [27] proved that a locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ with $f(0) = 0$, is spirallike with respect to A if and only if

$$\Re\langle [Df(z)]^{-1} Af(z), z\rangle > 0, \quad z \in B^n \setminus \{0\}.$$

The following result [11] (see [24] for $n = 1$), provides a necessary and sufficient condition for a holomorphic mapping to be spirallike of type α.

LEMMA 1.6. *Assume f is a normalized locally biholomorphic mapping on $B^n, \alpha \in (\frac{-\pi}{2}, \frac{\pi}{2}), a = \tan\alpha$. Then f is a spirallike mapping of type α if and only if*

$$F(z,t) = \mathrm{e}^{(1-\mathrm{i}a)t} f(\mathrm{e}^{\mathrm{i}at} z), \quad z \in B^n, \quad t \ge 0,$$

is a Loewner chain. In particular, f is a starlike mapping if and only if $F(z,t) = \mathrm{e}^t f(z)$is a Loewner chain.

The following definition was introduced by Feng [1] in the case of $\alpha \in [0, 1)$ and on the unit ball in a complex Banach space X. For our purpose, we present this notion only on the Euclidean setting.

DEFINITION 1.7. Suppose $0 \le \alpha < 1$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be almost starlike of order α if

$$\Re\langle [Df(z)]^{-1} f(z), z\rangle > \alpha\|z\|^2, \quad z \in B^n \setminus \{0\}.$$

It is clear that if f is almost starlike of order α, then f is also starlike. Q-H. Xu and T-S. Liu [29] proved the following characterization of almost starlikeness of order α in terms of Loewner chains.

LEMMA 1.8. *Suppose f is a normalized locally biholomorphic mapping in B^n and $0 \le \alpha < 1$. Then f is almost starlike of order α if and only if*

$$F(z,t) = \mathrm{e}^{\frac{1}{1-\alpha}t} f(\mathrm{e}^{\frac{\alpha}{\alpha-1}t} z), \quad z \in B^n, t \ge 0,$$

is a Loewner chain. In particular, f is a starlike mapping (i.e., $\alpha = 0$) if and only if $F(z,t) = \mathrm{e}^t f(z)$ is a Loewner chain.

In this paper, we shall introduce the notion of generalized almost starlikeness and we shall use the method of Loewner chains to give an analytical characterization of this notion. Finally, we shall prove that the Roper-Suffridge extension operators preserve generalized almost starlikeness .

2. PRELIMINARIES CONCERNING THE GENERALIZED ROPER-SUFFRIDGE EXTENSION OPERATORS

In this section we give a brief description of the well known Roper-Suffridge extension operator. In 1995, Roper and Suffridge [26] introduced the operator below which gives a way of extending a (locally) univalent function f on the unit disc to a (locally) univalent mapping of B^n into $\mathbb{C}^n$,

$$F(z) = \Phi_n(f)(z) = (f(z_1), \widetilde{z}\sqrt{f'(z_1)}), \quad z = (z_1, \widetilde{z})' \in B^n. \tag{2.1}$$

The branch of the square root is choosen such that $\sqrt{f'(0)} = 1$.

The Roper-Suffridge extension operator, given by (2.1) preserves convexity, as proved by K. Roper and T. Suffridge [26]. Also the operator Φ_n preserves starlikeness (see [6]).

We next mention the following results related to the preservation of the univalence by certain generalization of the Roper-Suffridge extension operator.

Lemma 2.1. [9] *Suppose that* $f \in S$ *and* $\beta \in [0, \frac{1}{2}]$. *Then* $\Phi_{n,\beta}(f)$ *can be embedded in a Loewner chain,where*

$$\Phi_{n,\beta}(f)(z) = (f(z_1), (f'(z_1)^{\beta})\widetilde{z})', \quad z = (z_1, \widetilde{z})' \in B^n.$$

The branch of the power function is chosen such that $(f'(z_1))^{\beta}|_{z_1=0} = 1$.

Lemma 2.2. [5] *Assume that* $f \in S$ *and* $\alpha \in [0, 1], \beta \in [0, \frac{1}{2}]$, *with* $\alpha + \beta \leq 1$. *Then*

$$\Phi_{n,\alpha,\beta}(f)(z) = (f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta}\widetilde{z})', \quad z = (z_1, \widetilde{z})' \in B^n,$$

and $z_1 \in U, \widetilde{z} = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$. *The branches of the power functions is chosen such that* $\left(\frac{f(z_1)}{z_1}\right)^{\alpha}|_{z_1=0} = 1$ *and* $(f'(z_1))^{\beta}|_{z_1=0} = 1$.

Other generalizations of the Roper-Suffridge extension operator were considered by S. Gong and T. Liu [2], [3], M. S. Liu and Y. C. Zhu [28], Liu and Xu [11], etc.

3. MAIN RESULTS

In this section we introduce the notion of generalized almost starlikeness and prove a characterization of this notion in terms of Loewner chains.

Definition 3.1. Let $a : [0, \infty) \to \mathbb{C}$ be of class C^{∞} with $\Re a(t) \leq 0, t \in [0, \infty)$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *generalized almost starlike with respect to a* (generalized almost starlike) if

$$\Re[(1 - a'(t))\mathrm{e}^{-a(t)}\langle[Df(\mathrm{e}^{a(t)}z)]^{-1}f(\mathrm{e}^{a(t)}z), z\rangle] \geq -\Re a'(t)\|z\|^2, \tag{3.1}$$

$z \in B^n, t \in [0, \infty)$.

It is easy to see that in the case of one variable, the above relation becomes

$$\Re[(1 - a'(t))\frac{f(\mathrm{e}^{a(t)}z)}{\mathrm{e}^{a(t)}zf'(\mathrm{e}^{a(t)}z)}] \geq -\Re a'(t), z \in U, t \geq 0. \tag{3.2}$$

Remark 3.2. Note that if $a(t) = \lambda t, t \in [0, \infty)$, in Definition 3.1, where $\lambda \in \mathbb{C}, \Re\lambda \leq 0$, one obtains the notion of almost starlikeness of complex order λ. This notion has been recently introduced by V. Nechita and M. Balaeti [21]. On the other hand, if $a(t) = \frac{\alpha t}{\alpha - 1}, t \in [0, \infty)$, where $\alpha \in [0, 1)$, we obtain the notion of almost starlikeness of order α due to Feng [1]. Finally, if $a(t) \equiv it\tan\alpha, \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, in Definition 3.1, we obtain the notion of spirallikeness of type α.

The following result provides a necessary and sufficient condition of generalized almost starlikeness in terms of Loewner chains.

THEOREM 3.3. *Let $f : B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping and let $a : [0, \infty) \to \mathbb{C}$ be a function of class C^∞. Assume that $\{e^{-a(t)} f(e^{a(t)} z)\}_{t\geq 0}$ is locally uniformly bounded on B^n. Then f is generalized almost starlike mapping if and only if*

$$g(z,t) = e^{t-a(t)} f(e^{a(t)} z), \quad z \in B^n,\ t \in [0,\infty),$$

is a Loewner chain. In particular, f is a starlike mapping (i.e. $a(t) = 0$) if and only if $g(z,t) = e^t f(z)$ is a Loewner chain.

Proof. First assume that f is an generalized almost starlike mapping. By Definition 3.1 , we have

(3.3)
$$\Re[(1 - a'(t))e^{-a(t)} \langle [Df(e^{a(t)} z)]^{-1} f(e^{a(t)} z), z\rangle] \geq -\Re a'(t)\|z\|^2, z \in B^n, t \geq 0.$$

It is easy to see that $g(\cdot,t) \in \mathcal{H}(B^n), g(0,t) = 0$ and $Dg(0,t) = e^t I$ for $t \geq 0$. We note that $g(z,t)$ is a C^∞ mapping on $B^n \times [0,\infty)$. Also, $\{e^{-t} g(\cdot,t)\}_{t\geq 0}$ is locally uniformly bounded on B^n, since $\{e^{-a(t)} f(e^{a(t)} z)\}_{t\geq 0}$ is a locally uniformly bounded family on B^n.

Obviously, we deduce that

$$\frac{\partial g}{\partial t}(z,t) = e^{t-a(t)}[(1 - a'(t))f(e^{a(t)} z) + Df(e^{a(t)} z) \cdot e^{a(t)} \cdot a'(t) \cdot z]$$

and $Dg(z,t) = e^t Df(e^{a(t)} z)$. Then

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t), \quad z \in B^n, t \geq 0,$$

where

$$h(z,t) = (1 - a'(t))e^{-a(t)}[Df(e^{a(t)} z)]^{-1} f(e^{a(t)} z) + a'(t)z. \tag{3.4}$$

It is clear that $h(z,t)$ is a measurable function of $t \in [0,\infty)$ for each $z \in B^n$, $h(0,t) = 0$, and $Dh(0,t) = I$. Using (3.3) it is easy to see that

$$\Re\langle h(z,t), z\rangle > 0, \quad z \in B^n \setminus \{0\},\ t \geq 0.$$

Hence, $h(\cdot,t) \in \mathcal{M}$. Since the conditions of Lemma 1.3 are satisfied, we conclude that $g(z,t)$ is a Loewner chain, as desired.

Conversely, assume that $g(z,t)$ is a Loewner chain. Since $g(z,t)$ is of class C^∞ on $B^n \times [0,\infty)$, the mapping $h(z,t) \in \mathcal{M}$ given by Lemma 1.4 is also of class C^∞ on $B^n \times [0,\infty)$, and (1.1) holds for all $z \in B^n$ and $t \geq 0$. It obvious that $h(z,t)$ is given by (3.4).Using the fact that $\Re\langle h(z,t), z\rangle > 0$ for $z \neq 0$ and $t \geq 0$, and the relation (3.4)we obtain that

$$\Re[(1 - a'(t))e^{-a(t)} \langle [Df(e^{a(t)} z)]^{-1} f(e^{a(t)} z), z\rangle] \geq -\Re a'(t)\|z\|^2, z \in B^n, t \geq 0.$$

Thus f is an generalized almost starlike mapping, as desired. This completes the proof. $\square$

We next obtain various results related to the preservation of the notion of generalized almost starlikeness by the Roper-Suffridge extension operators.These results also provide concrete examples of generalized almost starlike mappings on the unit ball in $\mathbb{C}^n$.

THEOREM 3.4. *Assume that f is a generalized almost starlike function on U and $g_\beta(z) = \Phi_{n,\beta}(f)(z)$ is defined as in Lemma 2.1, where $\beta \in [0, 1/2]$. Then g_β is a generalized almost starlike mapping on B^n.*

Proof. Let $g_\beta(z,t)$ be defined by

$$g_\beta(z,t) = (g(z_1,t), \mathrm{e}^{(1-\beta)t}(g'(z_1,t))^\beta \widetilde{z}),$$

$g(z_1,t) = \mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)} z_1)$ is a Loewner chain on U. We prove that if $G(z,t) = \mathrm{e}^{t-a(t)} g_\beta(\mathrm{e}^{a(t)} z)$ is a Loewner chain, then it follows that $g_\beta(z) = G(z,0)$ is generalized almost starlike on B^n. We know that if $f(z_1,t)$ is a Loewner chain on U, then

$$g(z,t) = (g(z_1,t), \mathrm{e}^{(1-\beta)t} \widetilde{z}(f'(z_1,t))^\beta)$$

is a Loewner chain on B^n for $\beta \in [0, \frac{1}{2}]$ (see [9]). In particular, let $f(z_1,t) = g(z_1,t) = \mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)} z_1)$ an Loewner chain on U. It follows that

$$g_\beta(z,t) = (g(z_1,t), \mathrm{e}^{(1-\beta)t} \widetilde{z}(g'(z_1,t))^\beta)$$

is a Loewner chain. But

$$\begin{aligned} g_\beta(z,t) &= (\mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)} z_1), \mathrm{e}^{(1-\beta)t} \widetilde{z}(\mathrm{e}^t f'(\mathrm{e}^{a(t)} z_1))^\beta) \\ &= (\mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)} z_1), \mathrm{e}^t \widetilde{z}(f'(\mathrm{e}^{a(t)} z_1))^\beta). \end{aligned} \tag{3.5}$$

But

$$\begin{aligned} G(z,t) :=& \mathrm{e}^{t-a(t)} g_\beta(\mathrm{e}^{a(t)} z) = \mathrm{e}^{t-a(t)} (f(\mathrm{e}^{a(t)} z_1), \mathrm{e}^{a(t)} \widetilde{z}(f'(\mathrm{e}^{a(t)} z_1))^\beta) \\ =& (\mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)} z_1), \mathrm{e}^t \widetilde{z}(f'(\mathrm{e}^{a(t)} z_1))^\beta) = g(z,t) \quad \text{(from (3.5)}. \end{aligned}$$

It follows that $G(z,t)$ is a Loewner chain on B^n and g_β is generalized almost starlike on B^n. □

We finish this section with the following preservation result by the operator $\Phi_{n,\alpha,\beta}$ given by Lemma 2.3. We omit the proof of this result, since it suffices to use arguments similar to those in the proof of the above result.

THEOREM 3.5. *Assume that f is an generalized almost starlike function on U, and $g_{\alpha,\beta}(z) = \Phi_{n,\alpha,\beta}(f)(z)$ is defined as in Lemma 2.3. Then $g_{\alpha,\beta}$ is a generalized almost starlike mapping on B^n.*

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