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AN EXTENSION OPERATOR AND LOEWNER CHAINS ON THE EUCLIDEAN UNIT BALL IN $\mathbb{C}^n$

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Abstract. We are concerned with an extension operator $\Phi_{n,\alpha}$, $\alpha \geq 0$, that provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$. In the case $\alpha = 1/(n+1)$, this operator reduces to the Pfaltzgraff-Suffridge extension operator. By using the method of Loewner chains, we prove that if $f \in S^0(B^n)$, then $\Phi_{n,\alpha}(f) \in S^0(B^{n+1})$, whenever $\alpha \in [0, 1/(n+1)]$. In particular, if $f \in S^*(B^n)$, then $\Phi_{n,\alpha}(f) \in S^*(B^{n+1})$, and if f is spirallike of type $\beta \in (-\pi/2, \pi/2)$ on B^n, then $\Phi_{n,\alpha}(f)$ is also spirallike of type β on B^{n+1}. We also prove that if f is almost starlike of order $\beta \in [0, 1)$ on B^n, then $\Phi_{n,\alpha}(f)$ is almost starlike of order β on B^{n+1}. Finally we prove that if $f \in K(B^n)$ and $1/(n+1) \leq \alpha \leq 1/n$, then the image of $F = \Phi_{n,\alpha}(f)$ contains the convex hull of the image of some egg domain contained in B^{n+1}. An extension of this result to the case of ε-starlike mappings will be also considered.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. For $n \geq 2$, let $\tilde{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ so that $z = (z_1, \tilde{z}) \in \mathbb{C}^n$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n is denoted by B^n. In the case of one complex variable, B^1 is denoted by U.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, and let I_n be the identity of $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, we denote by $H(\Omega)$ the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If $0 \in \Omega$, such a mapping f is said to be normalized if $f(0) = 0$ and $Df(0) = I_n$. A holomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B^n)$. We say that $f \in H(B^n)$ is locally biholomorphic on B^n if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in B^n$. Let $J_f(z) = \det Df(z)$.

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Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n and let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n.

A map $f \in S(B^n)$ is said to be convex if its image is a convex domain in $\mathbb{C}^n$, and starlike if the image is a starlike domain with respect to 0. We denote the classes of normalized convex and starlike mappings on B^n respectively by $K(B^n)$ and $S^*(B^n)$. In one variable we write $\mathcal{L}S_1 = \mathcal{L}S$, $S(B^1) = S$, $K(B^1) = K$ and $S^*(B^1) = S^*$.

Starlikeness has an analytic characterization due to Matsuno and Suffridge (see [21]): a locally biholomorphic map $f : B^n \to \mathbb{C}^n$ such that $f(0) = 0$ is starlike if and only if $\mathrm{Re}\, \langle [Df(z)]^{-1} f(z), z\rangle > 0,\ z \in B^n\setminus\{0\}$.

We recall that a mapping $f \in \mathcal{L}S_n$ is spirallike of type $\beta \in (-\pi/2, \pi/2)$ if $\mathrm{Re}\, [\mathrm{e}^{-\mathrm{i}\beta}\langle [Df(z)]^{-1} f(z), z\rangle] > 0,\ z \in B^n\setminus\{0\}$. We denote by $\hat{S}_\beta(B^n)$ the class of normalized spirallike mappings of type β on B^n. In the case of one variable this class is denoted by $\hat{S}_\beta$. If $\beta = 0$, we obtain that f is spirallike of type 0 if and only if f is starlike. A mapping $f \in \mathcal{L}S_n$ is almost starlike of order $\beta \in [0, 1)$ if $\mathrm{Re}\, \langle [Df(z)]^{-1} f(z), z\rangle > \beta\|z\|^2,\ z \in B^n\setminus\{0\}$. If $\beta = 0$, we obtain that f is almost starlike of order 0 if and only if f is starlike. We remark that the notion of almost starlikeness of order β was introduced by Xu and Liu in 2007 (see [22]).

We next present the notion of ε-starlikeness due to Gong and Liu (see [3]). This notion interpolates between starlikeness and convexity as ε ranges from 0 to 1.

Definition 1.1. Let $0 \in \Omega \subseteq \mathbb{C}^n$ be a domain and $f : \Omega \to \mathbb{C}^n$ be a biholomorphic mapping such that $f(0) = 0$. We say that f is ε-starlike, $0 \le \varepsilon \le 1$, if $f(\Omega)$ is starlike with respect to each point in $\varepsilon f(\Omega)$, i.e.

$$(1-\lambda)f(z) + \lambda\varepsilon f(w) \in f(\Omega),\ \lambda \in [0, 1],\ z, w \in \Omega.$$

When $\varepsilon = 0$ we obtain the family of starlike mappings on Ω, and when $\varepsilon = 1$ we obtain the family of convex mappings on Ω. The analytical characterization of ε-starlikeness was given in [4].

We next refer to the notions of subordination and Loewner chains. Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \le \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$. If g is biholomorphic on B^n, this is equivalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

Definition 1.2. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = \mathrm{e}^t I_n$ for $t \ge 0$, and $f(z, s) \prec f(z, t)$ whenever $0 \le s \le t < \infty$ and $z \in B^n$. The requirement $f(z, s) \prec f(z, t)$ is equivalent to the condition that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$ called the transition mapping associated to $f(z, t)$ such that $f(z, s) = f(v(z, s, t), t),\ z \in B^n,\ t \ge s \ge 0$.

We also note that the normalization of $f(z, t)$ implies the normalization $Dv(0, s, t) = \mathrm{e}^{s-t} I_n$ for $0 \le s \le t < \infty$.

Various results concerning Loewner chains can be found in [1], [9] and [16].

Remark 1.1. Certain subclasses of $S(B^n)$ can be characterized in terms of Loewner chains. In particular, f is starlike if and only if $f(z,t) = \mathrm{e}^t f(z)$ is a Loewner chain. Also, f is spirallike of type β if and only if $f(z,t) = \mathrm{e}^{(1-\mathrm{i}a)t} f(\mathrm{e}^{\mathrm{i}at} z)$ is a Loewner chain, where $a = \tan\beta$ and f is almost starlike of order β if and only if $f(z,t) = \mathrm{e}^{\frac{t}{1-\beta}} f(\mathrm{e}^{\frac{\beta t}{\beta-1}} z)$ is a Loewner chain.

The notion of parametric representation is related to that of a Loewner chain (see [6] and [12]; cf. [18]).

Definition 1.3. A normalized mapping $f \in H(B^n)$ has parametric representation if there exists a Loewner chain $f(z,t)$ such that $\{\mathrm{e}^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f(z) = f(z,0)$, $z \in B^n$.

Let $S^0(B^n)$ be the set of mappings which have parametric representation.

A key role in our discussion is played by the following Schwarz-type lemma for the Jacobian determinant of a holomorphic mapping from B^n into B^n[20]:

Lemma 1.1. *Let $\psi \in H(B^n)$ be such that $\psi(B^n) \subseteq B^n$. Then*

$$|J_\psi(z)| \leq \left[\frac{1 - \|\psi(z)\|^2}{1 - \|z\|^2}\right]^{\frac{n+1}{2}}, \quad z \in B^n. \tag{1}$$

This inequality is sharp and equality at a given point $z \in B^n$ holds if and only if $\psi \in Aut(B^n)$, where $Aut(B^n)$ denotes the set of holomorphic automorphisms of B^n.

For $n \geq 1$, set $z' = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

Definition 1.4. Let $\alpha \geq 0$. The extension operator $\Phi_{n,\alpha} : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ is defined by $\Phi_{n,\alpha}(f)(z) = F(z) = \Big(f(z'), z_{n+1}[J_f(z')]^\alpha\Big)$, $z = (z', z_{n+1}) \in B^{n+1}$.

We choose the branch of the power function such that $[J_f(z')]^\alpha\big|_{z'=0} = 1$. Then $F = \Phi_{n,\alpha}(f) \in \mathcal{L}S_{n+1}$ whenever $f \in \mathcal{L}S_n$. Also, if $f \in S(B^n)$ then $F \in S(B^{n+1})$. Indeed, if $F(z) = F(w)$, then $f(z') = f(w')$, which implies that $z' = w'$. Now from $z_{n+1}[J_f(z')]^\alpha = w_{n+1}[J_f(w')]^\alpha$ we obtain that $z_{n+1} = w_{n+1}$, therefore $z = w$.

If $\alpha = 1/(n+1)$, the operator $\Phi_{n,1/(n+1)}$ is denoted by Φ_n. This operator was introduced by Pfaltzgraff and Suffridge [17]. Thus the extension operator $\Phi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ is given by $\Phi_n(f)(z) = F(z) = \Big(f(z'), z_{n+1}[J_f(z')]^{\frac{1}{n+1}}\Big)$, $z = (z', z_{n+1}) \in B^{n+1}$. This operator was also investigated by Graham, Kohr and Pfaltzgraff [13]. They proved that if $f \in S^0(B^n)$, then $\Phi_n(f) \in S^0(B^{n+1})$. In particular, if $f \in S^*(B^n)$, then $\Phi_n(f) \in S^*(B^{n+1})$. If $n = 1$ and $\alpha = 1/2$, then $\Phi_{1,1/2}$ reduces to the well-known Roper-Suffridge extension operator. For $n \geq 2$, the Roper-Suffridge extension operator $\Psi_n : \mathcal{L}S \to \mathcal{L}S_n$ is defined by (see [19]) $\Psi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)})$, $z = (z_1, \tilde{z}) \in B^n$. We choose the branch of the power function such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

Roper and Suffridge proved that if f is convex on U then $\Psi_n(f)$ is also convex on B^n. Graham and Kohr proved that if f is starlike on U then so is $\Psi_n(f)$ on B^n. Graham, Kohr and Kohr [11] proved that if f has parametric representation on the unit disc, then $\Psi_n(f)$ has the same property on B^n.

Note that the operator $\Phi_{1,\alpha}$, $\alpha \in [0, \frac{1}{2}]$, was considered by Graham, Kohr and Kohr in [11]. On the other hand, Gong and Liu [3] proved that if f is an ε-starlike function on U, $\varepsilon \in [0, 1]$, and if $p \geq 1$, then $F_{1/p}$ is an ε-starlike mapping on the domain $\Omega_{n,p} = \left\{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^n |z_j|^p < 1\right\}$, where $F_{1/p} = \Phi_{n,1/p}(f)$ and $\Phi_{n,1/p}(f)(z) = \left(f(z_1), (f'(z_1))^{\frac{1}{p}}\tilde{z}\right)$, $z = (z_1, \tilde{z}) \in \Omega_{n,p}$.

Other extension operators that preserve various geometric properties have been recently considered in [2], [5], [7], [8], [10], [14], [15], [22], [23].

In this paper we prove that if $f \in S(B^n)$ can be imbedded as the first element of a Loewner chain $f(z', t)$, then $F = \Phi_{n,\alpha}(f)$ can also be imbedded as the first element of a Loewner chain $F(z, t)$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$. In particular, we obtain various consequences related to the preservation of the notions of parametric representation, starlikeness, spirallikeness of type β, and almost starlikeness of order β under $\Phi_{n,\alpha}$. Finally, we consider the preservation of ε-starlikeness under the operator $\Phi_{n,\alpha}$. In the case $\varepsilon = 1$, we obtain a partial answer to the question of whether $\Phi_{n,\alpha}$ preserves convexity.

2. LOEWNER CHAINS AND THE OPERATOR $\Phi_{N,\alpha}$

We begin this section with the following main result. In the case $\alpha = \frac{1}{n+1}$, see [13].

Theorem 2.1. *Assume $f \in S(B^n)$ can be imbedded as the first element of a Loewner chain $f(z', t)$. Then $F = \Phi_{n,\alpha}(f)$ can also be imbedded as the first element of a Loewner chain $F(z, t)$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Proof. Since $f \in S(B^n)$, we have $F \in S(B^{n+1})$. Let $v = v(z', s, t)$ be the transition mapping associated to $f(z', t)$. Then

$$f(z', s) = f(v(z', s, t), t),\ z' \in B^n,\ 0 \leq s \leq t < \infty. \tag{2}$$

Let $f_t(z') = f(z', t)$ for $z' \in B^n$ and $t \geq 0$ and let $v_{s,t}(z') = v(z', s, t)$, $z' \in B^n$, $t \geq s \geq 0$. Also, let $F : B^{n+1} \times [0, \infty) \to \mathbb{C}^{n+1}$ be given by

$$F(z, t) = (f(z', t), z_{n+1}\mathrm{e}^{t(1-n\alpha)}[J_{f_t}(z')]^{\alpha}), \tag{3}$$

for $z = (z', z_{n+1}) \in B^{n+1}$ and $t \geq 0$. We choose the branch of the power function such that $[J_{f_t}(z')]^{\alpha}\big|_{z'=0} = \mathrm{e}^{nt\alpha}$. Let us prove that $F(z, t)$ is a Loewner chain. Indeed, since $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$ and $Df(0, t) =$

$e^t I_n$, it is not difficult to see that $F(\cdot,t)$ is biholomorphic on B^{n+1}, $F(0,t)=0$ and $DF(0,t)=e^t I_{n+1}$.

Let $V_{s,t}: B^{n+1}\to\mathbb{C}^{n+1}$ be given by $V_{s,t}(z)=V(z,s,t)$, where

(4) $$V(z,s,t)=(v(z',s,t), z_{n+1}e^{(s-t)(1-n\alpha)}[J_{v_{s,t}}(z')]^\alpha),$$

for $z=(z',z_{n+1})\in B^{n+1}$ and $t\geq s\geq 0$. We choose the branch of the power function such that $[J_{v_{s,t}}(z')]^\alpha\big|_{z'=0}=e^{n\alpha(s-t)}$. Then $V_{s,t}$ is biholomorphic on B^{n+1}, $V_{s,t}(0)=0$, $DV_{s,t}(0)=e^{s-t}I_{n+1}$ and $\|V_{s,t}(z)\|<1$, $z\in B^{n+1}$. Indeed, by Lemma 1.1 and the fact that $\alpha\in[0,1/(n+1)]$, we obtain that $\|V_{s,t}(z)\|^2=\|v_{s,t}(z')\|^2+|z_{n+1}|^2e^{2(s-t)(1-n\alpha)}|J_{v_{s,t}}(z')|^{2\alpha}\leq\|v_{s,t}(z')\|^2+|z_{n+1}|^2\left[\frac{1-\|v_{s,t}(z')\|^2}{1-\|z'\|^2}\right]^{(n+1)\alpha}\leq\|v_{s,t}(z')\|^2+\frac{|z_{n+1}|^2}{1-\|z'\|^2}(1-\|v_{s,t}(z')\|^2)<\|v_{s,t}(z')\|^2+1-\|v_{s,t}(z')\|^2=1$, $z=(z',z_{n+1})\in B^{n+1}$. Hence $\|V_{s,t}(z)\|<1$ for $z\in B^{n+1}$, as claimed.

Further, taking into account (2), we can easily deduce that $F(z,s)=F(V(z,s,t),t)$ for $z\in B^{n+1}$, $t\geq s\geq 0$. Indeed,

$$\begin{aligned}F(V_{s,t}(z),t)&=(f(v_{s,t}(z'),t), z_{n+1}e^{(s-t)(1-n\alpha)}e^{t(1-n\alpha)}[J_{f_t}(v_{s,t}(z'))]^\alpha[J_{v_{s,t}}(z')]^\alpha)\\&=(f(z',s), z_{n+1}e^{s(1-n\alpha)}[J_{f_s}(z')]^\alpha)=F(z,s),\end{aligned}$$

for all $z\in B^{n+1}$ and $t\geq s\geq 0$. We have used (2) and the fact that $J_{f_s}(z')=J_{f_t}(v_{s,t}(z'))J_{v_{s,t}}(z')$, $z'\in B^n$, $t\geq s\geq 0$. This completes the proof. □

Taking into account Theorem 2.1, we next prove that the operator $\Phi_{n,\alpha}$ preserves the notions of parametric representation, starlikeness, spirallikeness of type β, and almost starlikeness of order β. Note that Corollaries 2.1 and 2.2 have been obtained in [13] in the case $\alpha=\frac{1}{n+1}$.

Corollary 2.1. *Assume $f\in S^0(B^n)$. Then $F=\Phi_{n,\alpha}(f)\in S^0(B^{n+1})$, for $\alpha\in\left[0,\frac{1}{n+1}\right]$.*

Proof. Since $f\in S^0(B^n)$, there exists a Loewner chain $f(z',t)$ such that $f(z',0)=f(z')$, $z'\in B^n$ and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family. Then

(5) $$\frac{r}{(1+r)^2}\leq\|e^{-t}f(z',t)\|\leq\frac{r}{(1-r)^2},\ \|z'\|=r<1,\ t\geq 0.$$

Applying the Cauchy integral formula for vector valued holomorphic functions, it is easy to see that for each $r\in(0,1)$ there is $K=K(r)\geq 0$ such that $e^{-t}\|Df(z',t)\|\leq K(r)$, $\|z'\|\leq r$, $t\geq 0$. Moreover, since $|J_{f_t}(z')|\leq\|Df_t(z')\|^n$, $z'\in B^n$, we deduce that there is some $K^*=K^*(r)\geq 0$ such that

(6) $$|J_{f_t}(z')|^\alpha\leq e^{nt\alpha}K^*(r),\ \|z'\|\leq r,\ t\geq 0.$$

Let $F: B^{n+1}\times[0,\infty)\to\mathbb{C}^{n+1}$ be the Loewner chain given by (3). Taking into account (5) and (6) we now easily deduce that for each $r\in(0,1)$ there is some $L=L(r)\geq 0$ such that $e^{-t}\|F(z,t)\|\leq L(r)$, $\|z\|\leq r$, $t\geq 0$. Consequently,

$\{e^{-t}F(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family on B^{n+1}, and thus is normal. Hence $F = F(\cdot,0) \in S^0(B^{n+1})$. This completes the proof. $\square$

COROLLARY 2.2. *Assume $f \in S^*(B^n)$. Then $F = \Phi_{n,\alpha}(f) \in S^*(B^{n+1})$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Proof. The fact that f is starlike on B^n is equivalent to the statement that $f(z',t) = e^t f(z')$ is a Loewner chain. With this choice of $f(z',t)$, we deduce that $F(z,t)$ given by (3) is a Loewner chain. On the other hand, we have $F(z,t) = (e^t f(z'), z_{n+1}e^{t(1-n\alpha)}e^{nt\alpha}[J_f(z')]^\alpha) = e^t(f(z'), z_{n+1}[J_f(z')]^\alpha) = e^t F(z)$, $z \in B^{n+1}$, $t \geq 0$. Thus $F = F(\cdot,0) \in S^*(B^{n+1})$, as claimed. $\square$

COROLLARY 2.3. *Assume $f \in \hat{S}_\beta(B^n)$, where $\beta \in (-\pi/2, \pi/2)$. Then $F = \Phi_{n,\alpha}(f) \in \hat{S}_\beta(B^{n+1})$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Proof. The fact that f is spirallike of type β on B^n is equivalent to the statement that $f(z',t) = e^{(1-ia)t}f(e^{iat}z')$ is a Loewner chain, where $a = \tan\beta$. With this choice of $f(z',t)$, we deduce that $F(z,t)$ given by (3) is a Loewner chain. On the other hand, we have

$$\begin{aligned} F(z,t) &= (e^{(1-ia)t}f(e^{iat}z'), z_{n+1}e^{t(1-n\alpha)}e^{nt\alpha}[J_f(e^{iat}z')]^\alpha) \\ &= (e^{(1-ia)t}f(e^{iat}z'), z_{n+1}e^{t}[J_f(e^{iat}z')]^\alpha) \\ &= e^{(1-ia)t}(f(e^{iat}z'), z_{n+1}e^{iat}[J_f(e^{iat}z')]^\alpha) = e^{(1-ia)t}F(e^{iat}z), \end{aligned}$$

for $z \in B^{n+1}$ and $t \geq 0$. Thus $F = F(\cdot,0) \in \hat{S}_\beta(B^{n+1})$, as claimed. This completes the proof. $\square$

The following result yields that the operator $\Phi_{n,\alpha}$ preserves the notion of almost starlikeness of order $\beta \in [0,1)$. In the case $n = 1$, see [22].

COROLLARY 2.4. *Assume f is an almost starlike mapping of order β on B^n, where $\beta \in [0,1)$. Then $F = \Phi_{n,\alpha}(f)$ is almost starlike mapping of order β on B^{n+1}, where $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Proof. The fact that f is almost starlike mapping of order β on B^n is equivalent to the statement that $f(z',t) = e^{\frac{t}{1-\beta}}f(e^{\frac{\beta t}{\beta-1}}z')$ is a Loewner chain. With this choice of $f(z',t)$, we deduce that $F(z,t)$ given by (3) is a Loewner chain. On the other hand, we have

$$\begin{aligned} F(z,t) &= (e^{\frac{t}{1-\beta}}f(e^{-\frac{\beta t}{1-\beta}}z'), z_{n+1}e^{t(1-n\alpha)}e^{tn\alpha}[J_f(e^{-\frac{\beta t}{1-\beta}}z')]^\alpha) \\ &= (e^{\frac{t}{1-\beta}}f(e^{-\frac{\beta t}{1-\beta}}z'), z_{n+1}e^{t}[J_f(e^{-\frac{\beta t}{1-\beta}}z')]^\alpha) \\ &= e^{\frac{t}{1-\beta}}(f(e^{-\frac{\beta t}{1-\beta}}z'), z_{n+1}e^{-\frac{\beta t}{1-\beta}}[J_f(e^{-\frac{\beta t}{1-\beta}}z')]^\alpha) = e^{\frac{t}{1-\beta}}F(e^{-\frac{\beta t}{1-\beta}}z) \end{aligned}$$

for $z \in B^{n+1}$ and $t \geq 0$. Thus $F = F(\cdot,0)$ is almost starlike mapping of order β on B^{n+1}. This completes the proof. $\square$

3. ε-STARLIKENESS AND THE OPERATOR $\Phi_{N,\alpha}$

We next discuss the case of ε-starlike mappings associated with the operator $\Phi_{n,\alpha}$, for $\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$. For $a \in (0,1]$, let $\Omega_{a,n,\alpha} = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : |z_{n+1}|^2 < a^{2n\alpha}(1 - \|z'\|^2)^{(n+1)\alpha}\}$. Then $\Omega_{a,n,\alpha} \subseteq B^{n+1}$. Indeed, from $a \in (0,1]$ and $\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, we obtain that $|z_{n+1}|^2 < 1 - \|z'\|^2$, i.e. $\Omega_{a,n,\alpha} \subseteq B^{n+1}$. For $a = 1$ and $\alpha = \frac{1}{n+1}$, we obtain $\Omega_{1,n,\frac{1}{n+1}} = B^{n+1}$. We are now able to prove the main result of this section, which when $\varepsilon = 1$ gives a partial answer to the question of whether $\Phi_{n,\alpha}$ preserves convexity.

THEOREM 3.1. *Let $\varepsilon \in [0,1]$ and $f : B^n \to \mathbb{C}^n$ be a normalized ε-starlike mapping. Also let $F = \Phi_{n,\alpha}(f)$, for $\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, and let $a_1, a_2 > 0$ be such that $a_1 + a_2 \leq 1$. Then $(1-\lambda)F(z) + \lambda\varepsilon F(w) \in F(\Omega_{a_1+a_2,n,\alpha})$, $z \in \Omega_{a_1,n,\alpha}$, $w \in \Omega_{a_2,n,\alpha}$, $\lambda \in [0,1]$.*

Proof. Since f is biholomorphic on B^n, it follows that $F = \Phi_{n,\alpha}(f)$ is also biholomorphic on B^{n+1}. Fix $\lambda \in [0,1]$ and let $z \in \Omega_{a_1,n,\alpha}$, $w \in \Omega_{a_2,n,\alpha}$. We want to find a point $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,\alpha}$ such that $(1-\lambda)F(z) + \lambda\varepsilon F(w) = F(u)$, i.e. $f(u') = (1-\lambda)f(z') + \lambda\varepsilon f(w')$ and $u_{n+1}[J_f(u')]^\alpha = (1-\lambda)z_{n+1}[J_f(z')]^\alpha + \lambda\varepsilon w_{n+1}[J_f(w')]^\alpha$. If $\lambda = 0$, let $u = z$. If $\lambda = 1$, then using the fact that f is ε-starlike and the equality $\varepsilon F(w) = F(u)$, we easily deduce that $u = (u', u_{n+1}) \in \Omega_{a_2,n,\alpha} \subseteq \Omega_{a_1+a_2,n,\alpha}$. Hence, it suffices to assume that $\lambda \in (0,1)$. Since f is ε-starlike, we obtain that $u' = f^{-1}((1-\lambda)f(z') + \lambda\varepsilon f(w'))$. Then $u' = u'(z', w')$ can be viewed as a mapping from $B^n \times B^n$ into B^n. Let $u_{n+1} = (1-\lambda)z_{n+1}\left[\frac{J_f(z')}{J_f(u')}\right]^\alpha + \lambda\varepsilon w_{n+1}\left[\frac{J_f(w')}{J_f(u')}\right]^\alpha$. We prove that $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,\alpha}$. It is obvious that $\frac{\partial u'}{\partial z'} = (1-\lambda)[Df(u')]^{-1}Df(z')$ and $\frac{\partial u'}{\partial w'} = \lambda\varepsilon[Df(u')]^{-1}Df(w')$. Hence $u_{n+1} = (1-\lambda)^{1-n\alpha}z_{n+1}[J_{u'_{z'}}]^\alpha + (\lambda\varepsilon)^{1-n\alpha}w_{n+1}[J_{u'_{w'}}]^\alpha$. Using Lemma 1.1 in the previous equation, we obtain

$$\begin{aligned}
|u_{n+1}| \leq &(1-\lambda)^{1-n\alpha}|z_{n+1}|\left[\frac{1-\|u'(z',w')\|^2}{1-\|z'\|^2}\right]^{\frac{(n+1)\alpha}{2}} \\
&+ (\lambda\varepsilon)^{1-n\alpha}|w_{n+1}|\left[\frac{1-\|u'(z',w')\|^2}{1-\|w'\|^2}\right]^{\frac{(n+1)\alpha}{2}} \\
=&(1-\|u'\|^2)^{\frac{(n+1)\alpha}{2}}\left\{(1-\lambda)^{1-n\alpha}\left[\frac{|z_{n+1}|^{\frac{2}{(n+1)\alpha}}}{1-\|z'\|^2}\right]^{\frac{(n+1)\alpha}{2}}\right. \\
&\left.+ (\lambda\varepsilon)^{1-n\alpha}\left[\frac{|w_{n+1}|^{\frac{2}{(n+1)\alpha}}}{1-\|w'\|^2}\right]^{\frac{(n+1)\alpha}{2}}\right\}.
\end{aligned}$$

We have two cases:

First case. If $\varepsilon = 0$ (i.e. f is starlike), then we obtain that

$$|u_{n+1}| \leq (1 - \|u'\|^2)^{\frac{(n+1)\alpha}{2}} (1-\lambda)^{1-n\alpha} \frac{|z_{n+1}|}{(1-\|z'\|^2)^{\frac{(n+1)\alpha}{2}}} < a_1^{n\alpha}(1-\|u'\|^2)^{\frac{(n+1)\alpha}{2}}.$$

Here we have used the fact that $z = (z', z_{n+1}) \in \Omega_{a_1,n,\alpha}$. Hence $|u_{n+1}|^2 < a_1^{2n\alpha}(1 - \|u'\|^2)^{(n+1)\alpha}$, i.e. $u = (u', u_{n+1}) \in \Omega_{a_1,n,\alpha}$. On the other hand, since $\Omega_{a_1,n,\alpha} \subseteq \Omega_{a_1+a_2,n,\alpha}$, we deduce that $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,\alpha}$, as desired.

Second case. For $\varepsilon \in (0, 1]$, using Hölder's inequality we obtain

$$\begin{aligned} |u_{n+1}| \leq & (1 - \|u'\|^2)^{\frac{(n+1)\alpha}{2}} (1-\lambda+\lambda\varepsilon)^{1-n\alpha} \left\{ \left[\frac{|z_{n+1}|^{\frac{2}{(n+1)\alpha}}}{1-\|z'\|^2} \right]^{\frac{n+1}{2n}} \right. \\ & + \left. \left[\frac{|w_{n+1}|^{\frac{2}{(n+1)\alpha}}}{1-\|w'\|^2} \right]^{\frac{n+1}{2n}} \right\}^{n\alpha} < (1-\|u'\|^2)^{\frac{(n+1)\alpha}{2}} (a_1+a_2)^{n\alpha}. \end{aligned}$$

Therefore, we have proved that $|u_{n+1}|^2 < (a_1 + a_2)^{2n\alpha}(1 - \|u'\|^2)^{(n+1)\alpha}$, i.e. $u = (u', u_{n+1}) \in \Omega_{a_1+a_2,n,\alpha}$. This completes the proof. □

Taking $\varepsilon = 1$ in Theorem 3.1, we obtain the following convexity result for the operator $\Phi_{n,\alpha}$. In the case $\alpha = \frac{1}{n+1}$, see [13].

Corollary 3.1. *If $f \in K(B^n)$ and $F = \Phi_{n,\alpha}(f)$, then $(1 - \lambda)F(z) + \lambda F(w) \in F(\Omega_{a_1+a_2,n,\alpha})$, $z \in \Omega_{a_1,n,\alpha}$, $w \in \Omega_{a_2,n,\alpha}$, $\lambda \in [0, 1]$, where $a_1, a_2 > 0$, $a_1 + a_2 \leq 1$.*

Taking $\alpha = \frac{1}{n+1}$ in Theorem 3.1, we obtain the following result regarding ε-starlikeness for the Pfaltzgraff-Suffridge extension operator Φ_n:

Corollary 3.2. *Let $\varepsilon \in [0, 1]$ and $f : B^n \to \mathbb{C}^n$ be a normalized ε-starlike mapping. Also let $F = \Phi_n(f)$ and $a_1, a_2 > 0$ such that $a_1 + a_2 \leq 1$. Then $(1 - \lambda)F(z) + \lambda\varepsilon F(w) \in F(\Omega_{a_1+a_2,n,1/(n+1)})$, for all $z \in \Omega_{a_1,n,1/(n+1)}$, $w \in \Omega_{a_2,n,1/(n+1)}$ and $\lambda \in [0, 1]$.*

Taking $a_1 = a_2 = \frac{1}{2}$ in Corollary 3.2 and using the fact that $\Omega_{1,n,1/(n+1)} = B^{n+1}$, we obtain the following corollary. In the case $\varepsilon = 1$, see [13].

Corollary 3.3. *If f is a normalized ε-starlike mapping on B^n, $\varepsilon \in [0, 1]$, and $F = \Phi_n(f)$, then $(1 - \lambda)F(z) + \lambda\varepsilon F(w) \in F(B^{n+1})$, $z, w \in \Omega_{1/2,n,1/(n+1)}$, $\lambda \in [0, 1]$.*

REFERENCES

[1] Curt, P., *Special Chapters of Geometric Function Theory of Several Complex Variables* (in Romanian), Editura Albastră, Cluj-Napoca, 2001.

[2] Elin, M., *Extension operators via semigroups*, J. Math. Anal. Appl., **377** (2011), 239–250.

[3] Gong, S. and Liu, T., *On the Roper-Suffridge extension operator*, J. Anal. Math., **88** (2002), 397–404.

[4] Gong, S. and Liu, T., *Criterion for the family of ε starlike mappings*, J. Math. Anal. Appl., **274** (2002), no. 2, 696–704.

[5] Gong, S. and Liu, T., *The generalized Roper-Suffridge extension operator*, J. Math. Anal. Appl., **284** (2003), 425–434.

[6] Graham, I., Hamada, H. and Kohr, G., *Parametric representation of univalent mappings in several complex variables*, Canad. J. Math., **54** (2002), 324–351.

[7] Graham, I. and Kohr, G., *Univalent mappings associated with the Roper-Suffridge extension operator*, J. Anal. Math., **81** (2000), 331–342.

[8] Graham, I. and Kohr, G., *An Extension Theorem and Subclasses of Univalent Mappings in Several Complex Variables*, Complex Var. Eliptic Equ., **47** (2002), 59–72.

[9] Graham, I. and Kohr, G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker Inc., New York, 2003.

[10] Graham, I. and Kohr, G., *The Roper-Suffridge extension operator and classes of biholomorphic mappings*, Sci. China Math., **49** (2006), 1539–1552.

[11] Graham, I., Kohr, G. and Kohr, M., *Loewner chains and the Roper-Suffridge Extension Operator*, J. Math. Anal. Appl., **247** (2000), 448–465.

[12] Graham, I., Kohr, G. and Kohr, M., *Loewner chains and parametric representation in several complex variables*, J. Math. Anal. Appl., **281** (2003), 425–438.

[13] Graham, I., Kohr, G. and Pfaltzgraff, J., *Parametric representation and linear functionals associated with extension operators for biholomorphic mappings*, Rev. Roumaine Math. Pures Appl., **52** (2007), 47–68.

[14] Liu, T.S. and Xu, Q.H., *Loewner chains associated with the generalized Roper-Suffridge extension operator*, J. Math. Anal. Appl., **322** (2006), 107–120.

[15] Muir, J.R. Jr., *A class of Loewner chain preserving extension operators*, J. Math. Anal. Appl., **337** (2008), 862–879.

[16] Pfaltzgraff, J.A., *Subordination chains and univalence of holomorphic mappings in* $\mathbb{C}^n$, Math. Ann., **210** (1974), 55–68.

[17] Pfaltzgraff, J.A. and Suffridge, T.J., *An extension theorem and linear invariant families generated by starlike maps*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, **53** (1999), 193–207.

[18] Poreda, T., *On the univalent holomorphic maps of the unit polydisc of $\mathbb{C}^n$ which have the parametric representation, I-the geometrical properties*, Ann. Univ. Mariae Curie-Sklodowska Sect A **41** (1987), 105–113.

[19] Roper, K. and Suffridge, T.J., *Convex mappings on the unit ball of $\mathbb{C}^n$*, J. Anal. Math., **65** (1995), 333–347.

[20] Rudin, W., *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, New York, 1980.

[21] Suffridge, T.J., *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Lecture Notes in Math., **599**, 146–159.

[22] Xu, Q.H. and Liu, T.S., *Löwner chains and a subclass of biholomorphic mappings*, J. Math. Appl., **334** (2007), 1096-1105.

[23] Zhu, Y. and Liu, M.S., *Loewner chains associated with the generalized Roper–Suffridge extension operator on some domains*, J. Math. Anal. Appl., **337** (2008), 949–961.

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