MATHEMATICA, Tome 54 (77), N° 2, 2012, pp. 103–116

AN INTERMEDIATE NEWTON ITERATIVE SCHEME AND GENERALIZED ZABREJKO-NGUEN AND KANTOROVICH EXISTENCE THEOREMS FOR NONLINEAR EQUATIONS

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Abstract. We revisit a one-step intermediate Newton iterative scheme that was used by Uko and Velásquez in [17] for the constructive solution of nonlinear equations of the type $f(u) + g(u) = 0$. By utilizing weaker hypotheses of the Zabrejko-Nguen kind and a modified majorizing sequence we perform a semilocal convergence analysis which yields finer error bounds and more precise information on the location of the solution that the ones obtained in [17]. We also give two generalizations of the well-known Kantorovich theorem on the solvability of nonlinear equations and the convergence of Newton's method. Illustrative examples are provided in the paper.

MSC 2000. 65H99, 49M15.

Key words. Nonlinear equations, Newton's method, intermediate Newton method, Zabrejko-Nguen conditions, iterative solution, majorant method, majorizing sequence, Lipschitz condition, center-Lipschitz condition.

1. INTRODUCTION

Let X and Y be Banach spaces, let $u_0 \in X$, let D be a non-empty closed ball $B\left[u_0, T\right]$ in X and let $F\colon D \longmapsto Y$ be a continuous function – assumed Fréchet differentiable on the open ball $D_0 = B(u_0, T)$ – fixed all through this paper. We are interested in the solvability of the equation

$$F(u) = 0. \tag{1}$$

In the sequel, we will assume that the function F has a splitting $F = f + g$, where $f, g\colon D \longmapsto Y$ are continuous functions that are Fréchet differentiable on D_0 and that therefore we can reformulate equation (1) in the form

$$f(u) + g(u) = 0. \tag{2}$$

In [17] it was shown that if f and g satisfy Zabrejko-Nguen type conditions then the iterates obtained from the equation

$$u_{m+1} = u_m - \left[f'(u_m) + g'(u_0)\right]^{-1}\left[f(u_m) + g(u_m)\right], \quad m = 0, 1, \ldots \tag{3}$$

converge to a solution of problem (2).

When $g = 0$, the scheme (3) becomes the classical Newton scheme for the equation $f(u) = 0$, and when $f = 0$, it becomes the modified Newton scheme for the equation $g(u) = 0$. Therefore this scheme provides an interesting unified setting for the study of both Newton's method and the modified Newton's method.

The iterative scheme (3) can also be viewed as an intermediate scheme between the method of Newton, defined by the iterations

$$u_{m+1} = u_m - F'(u_m)^{-1}F(u_m), \quad m = 0, 1, \ldots \tag{4}$$

and the modified Newton scheme, defined by the iterations

$$u_{m+1} = u_m - F'(u_0)^{-1}F(u_m), \quad m = 0, 1, \ldots.$$

It is well-known that although Newton's method usually requires fewer iterations than the modified Newton method to achieve a specified level of accuracy, the later is less expensive to implement than Newton's method. This led several authors (cf. [1, 6, 17]) to propose intermediate Newton methods which converge faster than the modified method and are cheaper to implement than Newton's method. The iterative scheme(3) is an interesting intermediate Newton scheme of this kind and is particularly useful in situations in which the Jacobian derivative of f is relatively easy to compute.

Convergence results on some intermediate Newton schemes have been given by Argyros [6, 7, 8], Uko & Velázquez [17] and Appel, De Pacale, Evkuta & Zabrejko [1].

The fundamental result on the solvability of problem (2) and the convergence of the Newton iterates in (4) was obtained by Kantorovich [12] and some improvements of this result were later obtained by several authors, including Ortega & Rheinboldt [14], Ostrowski [15] and Zabrejko & Nguen[19]. A fundamental improvement of the Kantorovich Theorem was obtained recently by Argyros [7] who used a combination of Lipschitz and center-Lipschitz conditions in place of the Lipschitz conditions used by Kantorovich.

In the sequel we will use the majorant method to obtain some results on the solvability of problem (2) and the convergence of the intermediate Newton scheme (3) under Zabrejko-Nguen type conditions and Kantorovich-type conditions. In Section 2 we will define the majorizing sequences that will be used and study their properties. Section 3 contains results on the solvability of problem (2) and the convergence of the intermediate scheme (3). Of particular interest are Theorem 1 which generalizes the Nguen-Zabrejko theorem in [19] and Theorems 2 and 3 which generalize the basic Kantorovich theorem in [12]. The later Theorems also generalize the improved Kantorovich-type result that was proved by Argyros in [7]. Two illustrative results are given in Section 3 of problems whose solvability can be deduced from Theorems 2 and 3 but not from the results in [12] or [7].

The Kantorovich Theorem is a fundamental tool in nonlinear analysis for proving the existence and uniqueness of solutions of nonlinear equations arising in various fields (cf. [2, 3, 4, 5, 12]). The generalized Kantorovich Theorems that we present in Section 3 are extensions of the Kantorovich and Argyros results and should ultimately lead to an enlargement of the class of nonlinear problems that can be solved with the Kantorovich technique and/or a weakening of the solvability conditions for some of the previously solved problems.

2. MAJORANT SEQUENCES

In this section we define the majorant sequences that we will use and give their main properties.

PROPOSITION 1. *Let $a \geq 0$, let $\alpha(t)$, $\alpha_0(t)$ and $\beta_0(t)$ be non-negative non-decreasing functions defined on an interval $[0, T]$. For any $0 \leq t \leq T$, let*

$$\kappa(t) = \int_0^t \alpha(s)\,\mathrm{d}s, \qquad \kappa_0(t) = \int_0^t \alpha_0(s)\,\mathrm{d}s, \quad \sigma(t) = a + \int_0^t \kappa(s)\,\mathrm{d}s - t,$$

$$\sigma_0(t) = a + \int_0^t \kappa_0(s)\,\mathrm{d}s - t, \quad \pi_0(t) = \int_0^t \beta_0(s)\,\mathrm{d}s, \quad \tau_0(t) = \int_0^t \pi_0(s)\,\mathrm{d}s.$$

Let $\mu(t) \equiv \sigma(t) + \tau_0(t)$, $t_0 = 0$, $t_1 = a$ and for $m = 1, 2, \ldots,$ let

$$t_{m+1} = t_m - \frac{\mu(t_m) - \mu(t_{m-1}) - (t_m - t_{m-1})\sigma'(t_{m-1}))}{\sigma_0'(t_m)}. \tag{5}$$

Suppose that $\kappa_0(T) < 1$ and $t_m \leq T$ for all m. Then the sequence $\{t_m\}$ is well defined and convergent and, for $m = 1, 2, \ldots,$ we have

$$t_{m-1} \leq t_m \leq t_* \equiv \lim_{n\to\infty} t_m \leq T. \tag{6}$$

Proof. Since $\sigma_0'(t_m) = \kappa_0(t_m) - 1 \leq \kappa_0(T) - 1 < 0$ for all m and $\sigma(t)$ is convex, we see that if $t_{m-1} \leq t_m$, then $\mu(t_m) - \mu(t_{m-1}) - (t_m - t_{m-1})\sigma'(t_{m-1}) = \tau_0(t_m) - \tau_0(t_{m-1}) + \sigma(t_m) - \sigma(t_{m-1}) - (t_m - t_{m-1})\sigma'(t_{m-1}) \geq 0$. Therefore, it follows from an easy induction argument that the sequence $\{t_m\}$ is well defined, monotone increasing and bounded above by T and as such it converges to its unique least upper bound t_*. □

REMARK 1. According to Proposition 1 in [17] the condition $\kappa_0(T) < 1$ can be replaced by the requirement that the function $\mu(t) = \sigma(t) + \tau_0(t)$ have a unique zero t_* in $[0, T]$.

In the next series of results we will consider specific choices of $\alpha(t)$, $\alpha_0(t)$ and $\beta(t)$ for which the hypotheses of Proposition 1 hold.

PROPOSITION 2. *Let $a \geq 0$, $L_0 \geq 0$, $0 \leq M_0 \leq M$, $0 \leq \theta < 2$, and $T = 2a/(2 - \theta)$. Suppose that*

$$\frac{M\theta}{2} + (2L_0 + \theta M_0)\left(\frac{\theta}{2}\right) \leq M, \tag{7}$$

$$(M + 2L_0 + M_0\theta)a \leq \theta. \tag{8}$$

Then the sequence $t_0 = 0$, $t_1 = a$,

$$t_{m+1} = t_m + \frac{M(t_m - t_{m-1})^2 + L_0(t_m^2 - t_{m-1}^2)}{2(1 - M_0 t_m)}, \quad m = 1, 2, \ldots \tag{9}$$

is well defined and converges to a real number t_* that satisfies condition (6) *and the inequality*

$$(M_0 + L_0)t_* \leq 1. \tag{10}$$

Proof. If we set $\alpha(t) \equiv Mt$, $\alpha_0(t) \equiv M_0 t$ and $\beta_0(t) \equiv L_0 t$ then the sequence in (5) reduces to the form (9). Therefore it suffices to verify the hypotheses of Proposition 1.

If $a = 0$ or $\theta = 0$ or $M^2 + L_0^2 = 0$ then $t_m = t_1$ for all $m \geq 1$ and the conclusions of the Proposition hold trivially. So we assume that $a > 0$ and $M^2 + L_0^2 > 0$ and $\theta > 0$. We show by induction that the inequalities

$$t_{m-1} \leq t_m \leq T, \tag{11}$$

$$M(t_m - t_{m-1}) + 2L_0 t_m + M_0 \theta t_m \leq \theta, \tag{12}$$

$$M_0 t_m < 1 \tag{13}$$

hold for all $m \geq 1$. It follows from (8) that they hold when $m = 1$. Suppose, by induction, that $k \geq 1$ and that these inequalities hold for all $m \leq k$. Then it follows from (12), (9) and (13) that the inequalities $M(t_m - t_{m-1}) + L_0(t_m + t_{m-1}) + M_0\theta t_m \leq M(t_m - t_{m-1}) + 2L_0 t_m + M_0 \theta t_m \leq \theta$, and $t_{m+1} - t_m = (t_m - t_{m-1})\frac{M(t_m - t_{m-1}) + L_0(t_m + t_{m-1})}{2(1 - M_0 t_m)} \leq \frac{\theta}{2}(t_m - t_{m-1})$ hold for all $m \leq k$. Hence

$$0 \leq t_{k+1} - t_k \leq \frac{\theta}{2}(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\theta}{2}\right)^k a, \tag{14}$$

$$\begin{aligned} t_{k+1} &= t_1 + (t_2 - t_1) + \cdots + (t_{k+1} - t_k) \\ &\leq \left[1 + \cdots + \left(\frac{\theta}{2}\right)^k\right] a = \frac{a\left[1 - \left(\frac{\theta}{2}\right)^{k-1}\right]}{1 - \frac{\theta}{2}} < \frac{a}{1 - \frac{\theta}{2}} = T. \end{aligned} \tag{15}$$

Therefore (11) holds when we replace m with $k+1$. It follows from (15), (8) and (7) that

$$\begin{aligned} &M(t_{k+1} - t_k) + 2L_0 t_{k+1} + M_0 \theta t_{k+1} \\ \leq &Ma\left(\frac{\theta}{2}\right)^k + \frac{(2L_0 + M_0\theta)a\left[1 - \left(\frac{\theta}{2}\right)^{k+1}\right]}{1 - \frac{\theta}{2}} \\ = &T\left[(2L_0 + \theta M_0) + \left(\frac{\theta}{2}\right)^k (M - M\left(\frac{\theta}{2}\right) - (2L_0 + \theta M_0)\left(\frac{\theta}{2}\right))\right] \\ \leq &T\left[(2L_0 + \theta M_0) + M - M\left(\frac{\theta}{2}\right) - (2L_0 + \theta M_0)\left(\frac{\theta}{2}\right))\right] \\ = &T(M + 2L_0 + M_0\theta)\left(1 - \left(\frac{\theta}{2}\right)\right) = (M + 2L_0 + M_0\theta)a \leq \theta. \end{aligned}$$

This shows that the inequality (12) holds when $m = k+1$ and implies that

$$t_{k+1} \leq \frac{Mt_k + \theta}{M + 2L_0 + M_0\theta}. \tag{16}$$

If $M_0 = 0$ then (13) holds trivially when we replace m with $k+1$. If $M_0 > 0$ then it follows from (16) and the induction hypotheses that $t_{k+1} <$

$\frac{(M/M_0)+\theta}{M+2L_0+M_0\theta} \le \frac{(M/M_0)+\theta}{M+M_0\theta} = \frac{1}{M_0}$. Therefore (13) also holds when we replace m with $k+1$. We conclude, by induction, that (11)–(13) hold for all m.

It follows from (9) and a straightforward induction argument that the sequence $\{t_m\}$ is monotone increasing and bounded above by T and as such it converges to its unique least upper bound t_*. To conclude the proof we rewrite (12) in the form $M(t_m - t_{m-1}) + 2L_0 t_m \le \theta(1 - M_0 t_m)$ and let m tend to infinity. We obtain the inequality $2L_0 t_* \le \theta(1 - M_0 t_*) \le 2(1 - M_0 t_*)$ which implies (10). □

A setback of this result is the fact that θ has to be found by trial and error and there is no guarantee that the choice we make leads to minimal conditions on M, M_0, L_0 and a that ensure the convergence of the sequence $\{t_m\}$. The next result resolves this problem by identifying the best possible value of θ provided that M_0 and M are not allowed to vanish at the same time.

Proposition 3. *Let* $0 \le L_0$, $0 \le \gamma$, $0 \le M_0 \le M \le \gamma M_0$ *and set*

$$\theta_*(M, M_0, L_0) \equiv \begin{cases} \frac{4M}{M+2L_0+\sqrt{[2L_0+M]^2+8MM_0}}, & \text{if } M + L_0 > 0 \\ 0, & \text{if } M + L_0 = 0. \end{cases} \tag{17}$$

Then

$$0 \le \theta_*(M, M_0, L_0) < 2. \tag{18}$$

Moreover, there exists $\theta \in [0, 2)$ *satisfying conditions* (7)–(8) *if and only if*

$$[2L_0 + M + M_0\theta_*(M, M_0, L_0)]\, a \le \theta_*(M, M_0, L_0). \tag{19}$$

Proof. If $M + L_0 = 0$ then condition (18) holds trivially. If $M + L_0 > 0$ then $\theta_*(M, M_0, L_0) \le \frac{4M}{2M+4L_0} \le 2$. Since M and L_0 cannot both vanish at the same time, one of these inequalities is strict, which shows (18) also holds in this case.

If $M + L_0 = 0$ then $M = L_0 = M_0 = 0$. Therefore conditions (7)–(8) and (19) are redundant and so the equivalence of these conditions holds trivially. Therefore, without loss of generality we may assume that $M + L_0 > 0$. In this case it is easy to verify – by solving a quadratic inequality – that $\theta_*(M, M_0, L_0)$ is by definition the largest value of θ for which the inequality (7) holds.

Suppose that (19) holds. Then, if we set $\theta = \theta_*(M, M_0, L_0)$ we see that condition (7) holds as an equality and that (8) reduces to condition (19).

Suppose now that conditions (7) and (8) hold for some $\theta \in [0, 2)$. Then condition (8) implies that $M_0 a \le \frac{M_0\theta}{M+M_0\theta+2L_0} < 1$. It follows that condition (8) can be expressed in the equivalent form $0 \le (M + 2L_0)a/(1 - M_0 a) \le \theta$. Since $\theta_*(M, M_0, L_0)$ is the largest value of θ for which the inequality (7) holds we conclude that $0 \le (M+2L_0)a/(1-M_0 a) \le \theta \le \theta_*(M, M_0, L_0)$. This shows that (19) holds, and that completes the proof of the Proposition. □

The next result gives an alternative set of hypotheses that guarantee the convergence of the scalar majorant sequence (9).

PROPOSITION 4. *Suppose* $a \geq 0$, $0 \leq M_0 \leq M$, $0 \leq L_0 \leq M$, $0 \leq \theta < 2$, *and* $T = 2a/(2-\theta)$. *If*

$$\frac{M\theta}{2} + (L_0 + \theta M_0)\left(\frac{\theta}{2}\right) \leq M, \tag{20}$$

$$(M + L_0 + M_0\theta)a \leq \theta, \tag{21}$$

then the sequence $\{t_m\}$ *defined in* (9) *is well defined and converges to a real number* t_* *that satisfies conditions* (6) *and* (10).

Proof. As in the proof of Proposition 2 we may assume without loss of generality that $a > 0$ and and $M^2 + L_0^2 > 0$ and $\theta > 0$. We show by induction that the inequalities

$$t_{m-1} \leq t_m \leq T, \tag{22}$$

$$M(t_m - t_{m-1}) + L_0(t_m + t_{m-1}) + M_0\theta t_m \leq \theta, \tag{23}$$

$$M_0 t_m < 1 \tag{24}$$

hold for all $m \geq 1$. It follows from (21) that they hold when $m = 1$. Suppose, by induction, that $k \geq 1$ and that these inequalities hold for all $m \leq k$. Then it follows from (9) and (24) that the inequalities $t_{m+1} - t_m \leq \frac{\theta}{2}(t_m - t_{m-1})$ hold for all $m \leq k$. By reasoning as in the proof of Proposition 2 we see that (14)–(15) hold and hence that (22) holds when we replace k with $k+1$.

On using (15), (8) and (7) we see that

$$\begin{aligned}
&M(t_{k+1} - t_k) + L_0(t_{k+1} + t_k) + M_0\theta t_{k+1} \\
\leq& Ma\left(\frac{\theta}{2}\right)^k + \frac{a\left[2L_0 + M_0\theta - (L_0 + M_0\theta)\left(\frac{\theta}{2}\right)^{k+1} - L_0\left(\frac{\theta}{2}\right)^k\right]}{1 - \frac{\theta}{2}} \\
=& T\left[(2L_0 + \theta M_0) + \left(\frac{\theta}{2}\right)^k\left(M - M\left(\frac{\theta}{2}\right) - L_0 - (L_0 + \theta M_0)\left(\frac{\theta}{2}\right)\right)\right] \\
\leq& T\left[(2L_0 + \theta M_0) + \left(M - M\left(\frac{\theta}{2}\right) - L_0 - (L_0 + \theta M_0)\left(\frac{\theta}{2}\right)\right)\right] \\
=& T(M + L_0 + M_0\theta)\left(1 - \left(\frac{\theta}{2}\right)\right) = (M + L_0 + M_0\theta)a \leq \theta.
\end{aligned}$$

This shows that the inequality (23) holds when $m = k+1$ and implies that

$$t_{k+1} \leq \frac{(M - L_0)t_k + \theta}{M + L_0 + M_0\theta}. \tag{25}$$

If $M_0 = 0$ then (24) holds trivially when we replace m with $k+1$. If $M_0 > 0$ then it follows from (25) and the induction hypotheses that $t_{k+1} \leq \frac{(M+L_0)t_k+\theta}{M+L_0+M_0\theta} < \frac{(M+L_0)/M_0+\theta}{M+L_0+M_0\theta} = \frac{1}{M_0}$. Therefore (24) also holds when we replace m with $k+1$.

We conclude, by induction, that (22)–(24) hold for all m.

The conclusion now follows as in the proof of Proposition 2. □

The next result identifies the best possible value of θ that can be used in Proposition 4 whenever M and M_0 are not allowed to vanish at the same time.

PROPOSITION 5. *Let* $0 \le \gamma$, $0 \le M_0 \le M \le \gamma M_0$, $0 \le L_0 \le M$ *and let*

$$\Theta_*(M, M_0, L_0) \equiv \begin{cases} \frac{4(M-L_0)}{M+L_0+\sqrt{[L_0+M]^2+8M_0(M-L_0)}}, & \text{if } M+L_0>0 \\ 0, & \text{if } M+L_0=0. \end{cases} \tag{26}$$

Then

$$0 \le \Theta_*(M, M_0, L_0) < 2. \tag{27}$$

Moreover, there exists $\theta \in [0, 2)$ *satisfying conditions* (20)–(21) *if and only if*

$$[L_0 + M + M_0\Theta_*(M, M_0, L_0)]\, a \le \Theta_*(M, M_0, L_0). \tag{28}$$

Proof. The proof is similar to the demonstration of the analogous result in Proposition 3. □

3. EXISTENCE AND CONVERGENCE RESULTS

In this section we will use the scalar majorant sequences described in the previous section to derive Zabrejko-Nguen and Kantorovich type existence results for problem (2) and convergence results for the intermediate Newton scheme (3).

The following result will be used in the sequel. The proof can be found in [19, Proposition 1].

LEMMA 1. *Let* v *be a function defined on the closed ball* $B\,[u_0, T]$ *in* X *with values in* Y. *Suppose that there exists a non-decreasing function* $\theta(t)$ *defined on the closed interval* $[0, T]$ *such that, for all* $0 \le t \le T$, *we have*

$$\|v(x) - v(y)\| \le \theta(t)\|x - y\| \quad \forall x, y \in B(u_0, t).$$

Then, whenever $0 \le t \le s \le T$, $x \in B\,[u_0, t]$ *and* $y \in B\,[x, s-t]$ *we have*

$$\|v(x) - v(y)\| \le \int_t^s \theta(s)\,\mathrm{d}s.$$

We next prove the convergence of the intermediate Newton-type scheme (3) under weaker Zabrejko-Nguen-type hypotheses than the ones used in [17].

THEOREM 1. *Suppose that* $J_0 = f'(u_0) + g'(u_0)$ *is invertible and that there exists* $a \ge 0$ *such that*

$$\|J_0^{-1}\,[f(u_0) + g(u_0)]\,\| \le a. \tag{29}$$

Suppose further that there exist functions $\alpha(t)$, $\alpha_0(t)$ *and* $\beta_0(t)$ *satisfying the hypotheses of Proposition 1 and that whenever* $0 \le t \le T$ *and* $x, y \in B(u_0, t)$

we have

$$\|J_0^{-1}\left[f'(x)-f'(y)\right]\| \le \alpha(t)\|x-y\|, \tag{30}$$

$$\|J_0^{-1}\left[f'(x)-f'(u_0)\right]\| \le \kappa_0(t) \tag{31}$$

$$\|J_0^{-1}\left[g'(x)-g'(u_0)\right]\| \le \pi_0(t). \tag{32}$$

Let t_* *and* $\{t_m\}$ *be defined as in Proposition* 1. *Suppose that* $t_* \le T$ *and that*

$$\kappa_0(T)+\pi_0(T) < 1. \tag{33}$$

Then the intermediate Newton iterates in (3) *are well defined and converge to a unique solution* u *of equation* (2) *in* $B\,[u_0, T]$, *with error estimates*

$$\|u_m - u_{m-1}\| \le t_m - t_{m-1}, \tag{34}$$

$$\|u_m - u_0\| \le t_m, \tag{35}$$

$$\|u - u_m\| \le t_* - t_m. \tag{36}$$

Proof. Let $\sigma(t)$, $\tau_0(t)$ and $\mu(t)$ be defined as in Proposition 1.

If $a=0$, then $u=u_0$ solves equation (2) and, since $u_m=u_0$ and $t_m=t_0$ for all m, the estimates (34) – (36) hold trivially. In the rest of the proof we assume $a>0$. Since $\|u_1-u_0\| = a \le t_1 - t_0$, we see that (34)–(35) hold when $m=1$.

Suppose now, by induction, that $m \ge 1$ and that the u_m are well defined and satisfy (34)–(35). Then, on letting $J_m \equiv f'(u_m)+g'(u_0) = J_0(I+A)$ or, equivalently, $A = J_0^{-1}\left[f'(u_m)-f'(u_0)\right]$, and applying Lemma 1, we see that $\|A\| \le \kappa_0(t_m) \le \kappa_0(T) < 1$. Therefore it follows from the Banach Lemma on invertible operators [12] that $(I+A)^{-1}$ exists, and $\|(I+A)^{-1}\| \le 1/\left[1-\kappa_0(t_m)\right] = -1/\sigma_0'(t_m)$. Therefore J_m is invertible and $\|J_m^{-1}J_0\| = \|(I+A)^{-1}\| \le -1/\sigma_0'(t_m)$. Now

$$\begin{aligned}
&\|J_0^{-1}\left[f(u_m)+g(u_m)\right]\| \\
\le&\|J_0^{-1}\left[f(u_m)-f(u_{m-1})-f'(u_{m-1})(u_m-u_{m-1})\right]\| \\
+&\|J_0^{-1}\left[g(u_m)-g(u_{m-1})-g'(u_0)(u_m-u_{m-1})\right]\| \\
\le&\|\int_0^1 J_0^{-1}\left[f'(u_m+s(u_m-u_{m-1}))-f'(u_{m-1})\right](u_m-u_{m-1})\,\mathrm{d}s\| \\
+&\|\int_0^1 J_0^{-1}\left[g'(u_m+s(u_m-u_{m-1}))-g'(u_0)\right](u_m-u_{m-1})\,\mathrm{d}s\|.
\end{aligned}$$

Therefore an application of (30)–(32), Lemma 1 and the induction hypotheses shows that

$$
\begin{aligned}
\|J_0^{-1}\left[f(u_m)+g(u_m)\right]\| &\le \int_0^1 \left[\int_{t_{m-1}}^{t_m+s(t_m-t_{m-1})} \alpha(w)\,dw\right](t_m-t_{m-1})\,\mathrm{d}s \\
&+\int_0^1 \pi_0(t_m+s(t_m-t_{m-1}))(t_m-t_{m-1})\,\mathrm{d}s \\
&=\int_0^1 \left[\kappa(t_m+s(t_m-t_{m-1}))-\kappa(t_{m-1})\right](t_m-t_{m-1})\,\mathrm{d}s+\int_{t_{m-1}}^{t_m}\pi_0(s)\,\mathrm{d}s \\
&=\int_{t_{m-1}}^{t_m}\kappa(s)\,\mathrm{d}s-\kappa(t_{m-1})(t_m-t_{m-1})+\tau_0(t_m)-\tau_0(t_{m-1}) \\
&=(t_m-t_{m-1})(1-\kappa(t_{m-1}))+\sigma(t_m)-\sigma(t_{m-1})+\tau_0(t_m)-\tau_0(t_{m-1}) \\
&=\mu(t_m)-\mu(t_{m-1})-(t_m-t_{m-1})\sigma'(t_{m-1})).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|u_{m+1}-u_m\| &= \|J_m^{-1}\left[f(u_m)+g(u_m)\right]\| \le \|J_m^{-1}J_0\|\|J_0^{-1}\left[f(u_m)+g(u_m)\right]\| \\
&\le -\frac{\mu(t_m)-\mu(t_{m-1})-(t_m-t_{m-1})\sigma'(t_{m-1})}{\sigma_0'(t_m)} \\
&= t_{m+1}-t_m, \\
\|u_{m+1}-u_0\| &\le \|u_{m+1}-u_m\|+\|u_m-u_0\| \le t_{m+1}-t_m+t_m=t_{m+1}.
\end{aligned}
$$

It follows that (34) and (35) also hold when m is replaced with $m+1$ and hence, by induction, that they hold for all positive integral values of m. This implies that $\|u_{m+q}-u_m\| \le \sum_{k=m+1}^{m+q}\|u_k-u_{k-1}\| \le \sum_{k=m+1}^{m+q}(t_k-t_{k-1}) = t_{m+q}-t_m$. Since $\{t_m\}$ is a Cauchy sequence, it follows that $\{u_m\}$ is also a Cauchy sequence converging to some $u \in B\,[u_0,T]$. On letting q tend to infinity we see that (36) holds. It follows from (3) that $\left[f'(u_m)+g'(u_0)\right](u_{m+1}-u_m)+f(u_m)+g(u_m)=0$ and on letting m tend to infinity we see that u solves equation (2).

To prove uniqueness, we suppose that v is another solution of equation (2) in $B\,[u_0,T]$. Then, on setting $F=f+g$ and making use of (31) and (32), we see that $\|u-v\| = \|J_0^{-1}\left[F(u)-F(v)-F'(u_0)(u-v)\right]\|$, hence

$$
\begin{aligned}
\|u-v\| \le \|&\int_0^1 J_0^{-1}\left[F'(su+(1-s)v)-F'(u_0)\right](u-v)\,\mathrm{d}s\| \\
&\le \int_0^1 \kappa_0(s\|u-u_0\|+(1-s)\|v-u_0\|)\|u-v\|\,\mathrm{d}s \\
&\quad+\int_0^1 \pi_0(s\|u-u_0\|+(1-s)\|v-u_0\|)\|u-v\|\,\mathrm{d}s \\
&\le \int_0^1 (\kappa_0(T)+\pi_0(T))\|u-v\|\,\mathrm{d}s = (\kappa_0(T)+\pi_0(T))\|u-v\|.
\end{aligned}
$$

Therefore it follows immediately from condition (33) that $u=v$. $\square$

REMARK 2. If we introduce the following stronger condition that was used in [17], $\|J_0^{-1}[g'(x) - g'(y)]\| \leq \beta(t)\|x - y\|$, for all $x, y \in B[u_0, t]$, then in order to compare the iterates $\{t_m\}$ with the analogous ones used in [17], we set $\pi(t) = \int_0^t \beta(s)\,\mathrm{d}s$, $\tau(t) = \int_0^t \pi(s)\,\mathrm{d}s$ and $\lambda(t) \equiv \sigma(t) + \tau(t)$. It is evident that

$$(37) \qquad \alpha_0(t) \leq \alpha(t) \text{ and } \beta_0(t) \leq \beta(t), \quad \forall t \in [0, T]\,.$$

Consider the iterations defined by $s_0 = 0$, $s_1 = a$ and for $m = 1, 2, \ldots$, let

$$(38) \qquad s_{m+1} = s_m - \frac{\lambda(s_m) - \lambda(s_{m-1}) - (t_m - t_{m-1})\sigma'(t_{m-1})}{\sigma'(t_m)}.$$

Suppose that $s_* = \lim_{m\to\infty} s_m$ exists.

If $\alpha_0(t) = \alpha(t)$ and $\beta_0(t) = \beta(t)$ for all $t \in [0, T]$ then $s_m = t_m$ for all m and $s_* = t_*$ and in this case our results coincide with those of [17]. Examples have been given in [6, 7] with $\alpha(t) \gg \alpha_0(t)$ and $\beta(t) \gg \beta_0(t)$. Therefore one or more of the inequalities in (37) can be strict and whenever this happens, a straightforward induction argument shows that whenever $m \geq 2$ we have $t_m < s_m$, $t_{m+1} - t_m < s_{m+1} - s_m$, $t_* - t_m \leq s_* - s_m$, $t_* \leq s_*$. In this case Theorem 1 constitutes an improvement over the results of [17] because the error estimates are more precise, and more precise information is given on the solution of equation (2). It is also interesting to observe that these improvements are obtained under weaker hypotheses and at the same computational cost as the results of [17] because the evaluation of α and β requires the evaluation of α_0 and β_0.

We now present two generalizations of Argyros' extension [7] of the Kantorovich theorem on the solvability of nonlinear equations and the convergence of Newton's method. In the sequel, t_* and majorant sequence $\{t_m\}$ will be defined as in Proposition 2.

THEOREM 2. *Suppose that $J_0 = f'(u_0) + g'(u_0)$ is invertible and there exist $a \geq 0$, $L_0 \geq 0$, $0 \leq \theta < 2$ and $0 \leq M_0 \leq M$ such that* (7), (8) *and* (29) *hold and*

$$(39) \qquad \|J_0^{-1}\left[f'(x) - f'(y)\right]\| \leq M\|x - y\|, \quad \forall x, y \in D_0,$$

$$(40) \qquad \|J_0^{-1}\left[f'(x) - f'(u_0)\right]\| \leq M_0\|x - u_0\|, \quad \forall x \in D_0,$$

$$(41) \qquad \|J_0^{-1}\left[g'(x) - g'(u_0)\right]\| \leq L_0\|x - u_0\|, \quad \forall x \in D_0.$$

Then the intermediate Newton iterates in (3) *are well defined and converge to a solution u of equation* (2) *in $B[u_0, t_*]$ and the error estimates* (34)–(36) *hold. If $(M_0 + L_0)(t_* + T) < 2$ then this solution is unique in $B[u_0, T]$.*

Proof. The existence result follows from Theorem 1 and Proposition 2. To prove uniqueness, we suppose that v is another solution of equation (2) in $B[u_0, t_*]$. Then, on setting $F = f + g$ and using the Lipschitz conditions (40) and (41), we see that

$$\begin{aligned}
\|u-v\| &= \|J_0^{-1}\left[F(u)-F(v)-F'(u_0)(u-v)\right]\| \\
&\le \|\int_0^1 J_0^{-1}\left[F'(su+(1-s)v)-F'(u_0)\right](u-v)\,\mathrm{d}s\| \\
&\le \int_0^1 (M_0+L_0)\left[s\|u-u_0\|+(1-s)\|v-u_0\|\right]\|u-v\|\,\mathrm{d}s \\
&= \frac{1}{2}(M_0+L_0)\left[\|u-u_0\|+\|v-u_0\|\right]\|u-v\| \\
&\le \frac{1}{2}(M_0+L_0)(T+t_*)\|u-v\|.
\end{aligned}$$

Then it follows immediately from the hypotheses that $u=v$. □

THEOREM 3. *Suppose that $J_0=f'(u_0)+g'(u_0)$ is invertible and there exist $a\ge 0$, $0\le\theta<2$, $0\le M_0\le M$ and $0\le L_0\le M$ such that* (20), (21), (29), (39), (40) *and* (41) *hold. Then the intermediate Newton iterates in* (3) *are well defined and converge to solution u of equation* (2) *in $B\,[u_0,t_*]$ and the error estimates* (34)–(36) *hold. If $(M_0+L_0)(t_*+T)<2$ then this solution is unique in $B\,[u_0,T]$.*

Proof. This result follows immediately from Theorem 1 and Proposition 4. □

REMARK 3. If we set $L_0=0$ and $g=0$ in Theorem 2 or 3 we recover Argyros' generalization [7] of the Kantorovich Theorem, and if further set $\theta=1$ and $M=M_0$, we recover the Kantorovich theorem [12].

REMARK 4. If we set $M_0=0$ in condition (40) then $f'(x)=f'(u_0)$ for all $x\in D_0$, which implies that $f'(x)=f'(y)$ for all $x,y\in D_0$. Evidently it makes sense to set $M=0$ also, in this case. Therefore, in the study of the solvability of equation (2), there is no loss of generality in assuming that whenever $M_0=0$ then $M=0$. The simplest way of making sure that this happens is by imposing a condition of the form $0\le M_0\le M\le\gamma M_0$ – as was done in Propositions 3 and 5 – where γ is any positive constant. This condition will also be employed in the next two results.

REMARK 5. A setback of Theorems 2 and 3 is the fact that θ has to be found by trial and error and there is no guarantee that the choice we make leads to minimal conditions on M, M_0, L_0 and a that ensure the solvability of problem (2). The two next results resolve this problem by identifying the best possible values of θ.

THEOREM 4. *Suppose that $J_0=f'(u_0)+g'(u_0)$ is invertible and there exist $a\ge 0$, $\gamma\ge 0$, $L_0\ge 0$ and $0\le M_0\le M\le\gamma M_0$ such that* (29) *and* (39)–(41) *hold. Suppose further that*

$$[2L_0+M+M_0\theta_*(M,M_0,L_0)]\,a\le\theta_*(M,M_0,L_0) \tag{42}$$

where $\theta_*(M, M_0, L_0)$ *is defined in* (17)*. Then the intermediate Newton iterates in* (3) *are well defined and converge to a solution* u *of equation* (2) *in* $B[u_0, t_*]$ *and the error estimates* (34)–(36) *hold. If* $(M_0 + L_0)(t_* + T) < 2$ *then this solution is unique in* $B[u_0, T]$.

Proof. This result follows immediately from Theorem 2 and Proposition 3. □

THEOREM 5. *Suppose that* $J_0 = f'(u_0) + g'(u_0)$ *is invertible and there exist* $a \geq 0$, $\gamma \geq 0$, $0 \leq L_0 \leq M_0$ *and* $0 \leq M_0 \leq M$ *such that* (29), (39), (40) *and* (41) *hold. Suppose further that*

$$[L_0 + M + M_0\Theta_*(M, M_0, L_0)]\, a \leq \Theta_*(M, M_0, L_0), \tag{43}$$

where $\Theta_*(M, M_0, L_0)$ *is defined in* (26)*. Then the intermediate Newton iterates in* (3) *are well defined and converge to a solution* u *of equation* (2) *in* $B[u_0, t_*]$ *and the error estimates* (34)–(36) *hold. If* $(M_0 + L_0)(t_* + T) < 2$ *then this solution is unique in* $B[u_0, T]$.

Proof. This result follows immediately from Theorem 3 and Proposition 5. □

REMARK 6. Theorems 4 and 5 are generalizations of the well-known Kantorovich theorem [12] on the solvability of nonlinear equations and the convergence of Newton's method. If the conditions (29) and (39)–(41) hold with $g = 0$, $f = F$ and $L_0 = 0$ then it follows from the Kantorovich theorem that if

$$2Ma \leq 1, \tag{44}$$

then the Newton iterates in (4) converge to a solution of equation (2). In this case it was shown in [7] that the same conclusion holds under the weaker condition

$$(M + M_0)a \leq 1. \tag{45}$$

In fact, this condition is precisely condition (8) of Theorem 2 with $L_0 = 0$ and $\theta = 1$. However, in this case the Newton scheme (4) coincides with the intermediate Newton scheme (3) and it follows from Theorem 5 that the convergence of the Newton iterates to a solution of equation (2) holds under the condition

$$[M + M_0\Theta_*(M, M_0, 0)]\, a \leq \Theta_*(M, M_0, 0), \tag{46}$$

which is weaker than (45).

EXAMPLE 1. Let $X = Y = \mathbb{R}$, $D = [8/9, 10/9]$, $g = 0$, $f(x) = x^3 + x/3 - 0.47118$ and $u_0 = 1$. Then it is easy to verify that $M = 2$, $M_0 = 1.9$, $L_0 = 0$, $\theta_*(M, M_0, L_0) = 1.0171451$ and $a = 0.258646$. In this case condition (46) holds, but the Kantorovich condition (44) and the Argyros condition (45) do not hold.

Example 2. Let $X = Y = \mathbb{R}$, $u_0 = 1$, $T = 1 - c$, $D = [c, 2 - c]$ for some $c \in [0, 1/2)$, $g = 0$, and $f(x) = x^3 - c$. We see that $a = (1 - c)/3$, $M_0 = 3 - c$, $L_0 = 0$ and $M = 2(2 - c)$. The Kantorovich condition (44) does not hold since $2Ma = 4(1 - c)(2 - c)/3 > 1$ whenever $c \in [0, 1/2)$. In [7] it was shown that Newton's method converges for this problem whenever $c \in [0.4505, 1/2)$. We can get a slight improvement of this result by using our generalized Kantorovich Theorems. For instance, it follows from (46) and a brief calculation that the slightly weaker condition $c \in [0.450339002, 1/2)$ is sufficient for the convergence of Newton's method for this problem.

Example 3. Let $X = Y = C\,[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with norm $\|x\| = \max_{0 \le s \le 1} |x(s)|$. Let $c \in [0, 1]$ be a given parameter. Consider the cubic integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s, t) u(t)\,\mathrm{d}t + y(s) - c. \tag{47}$$

Here the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$; the parameter λ is a real number called the “albedo” for scattering; $y(s)$ is a given continuous function defined on $[0, 1]$ and $x(s)$ is the unknown function sought in $C\,[0, 1]$. Equations of the form (47) arise in the theory of radiactive transfer, neutron transport, and the kinetic theory of gasses [2, 3, 4, 5, 9].

For simplicity, we choose $u_0(s) = y(s) = 1$ and $q(s, t) = s/(s + t)$, for all $s \in [0, 1]$ and $t \in [0, t]$ with $s + t \neq 0$. If we let $D = B\,[u_0, 1 - c]$, $g = 0$ and define the operator f on D by

$$f(x)(s) = x^3(s) + \lambda x(s) \int_0^1 q(s, t) x(t)\,\mathrm{d}t + y(s) - c, \tag{48}$$

for all $s \in [0, 1]$, then every zero of f satisfies equation (47). We have the estimate

$$\max_{0 \le s \le 1} \left| \int_0^1 s/(s + t)\,\mathrm{d}t \right| = \ln 2. \tag{49}$$

Therefore if we set $b = \|f'(u_0)^{-1}\|$, then it follows from (48) and (49) that conditions (39), (40) and (41) hold with $L_0 = 0$, $a = b(|\lambda| \ln 2 + 1 - c)$, $M = 2b\,[|\lambda| \ln 2 + 3(2 - c)]$ and $M_0 = b\,[2|\lambda| \ln 2 + 3(3 - c)]$. It follows from Theorem 2 that if condition (46) holds, then problem (47) has a unique solution near u_0. This condition is weaker than the conditions employed in [2, 3, 4] for equations of this type.

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Received November 24, 2011
Accepted October 22, 2012

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