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A UNIVALENCE CONDITION FOR ANALYTIC FUNCTIONS IN THE UNIT DISK

VERONICA OANA NECHITA

Abstract. In this paper we give a univalence criterion for analytic functions in the unit disk, which generalizes previously known, recent results. We use the method of Loewner chains in order to prove our main theorem.

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Key words. univalent function, univalence criteria, Loewner chain.

1. INTRODUCTION AND PRELIMINARIES

Let U_r denote the disk $\{z \in \mathbb{C} : |z| < r\}$ in the complex plane, where $0 < r \leq 1$ and consider $U = U_1$. Let $\mathcal{A}$ denote the class of analytic functions in the unit disk, which are also normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let f and F be members of $\mathcal{A}$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ [3, p.36].

A function $L(z,t), z \in U, t \geq 0$, is a subordination chain if $L(\cdot,t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z,t_1) \prec L(z,t_2)$, whenever $0 \leq t_1 \leq t_2$.

The aim of this paper is to give a new univalence criterion for analytic functions defined in the unit disk. The main tool in our development is the following result, due to Ch. Pommerenke [4], which gives a method of constructing univalence criteria.

LEMMA 1.1. *Let r be a real number such that $0 < r \leq 1$ and let $L : U_r \times [0,\infty) \to \mathbb{C}$ be a function that satisfies the following conditions:*

(i) *$L(\cdot,t)$ is analytic in U_r, for each $t \in [0,\infty)$, $L(z,t) = a_1(t)z + ...$ and locally absolutely continuous in $[0,\infty)$, locally uniformly with respect to U_r;*

(ii) *for each $t \in [0,\infty)$, $a_1(t) \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\left\{\frac{L(\cdot,t)}{a_1(t)}\right\}_{t\geq 0}$ forms a normal family in U_r;*

(iii) *there exists a function $p : U \times [0,\infty) \to \mathbb{C}$, such that $p(\cdot,t)$ is analytic in U, $\operatorname{Re} p(z,t) > 0$ for each $(z,t) \in U \times [0,\infty)$ and $\frac{\partial L(z,t)}{\partial t} = p(z,t) \cdot z \cdot \frac{\partial L(z,t)}{\partial z}$ for $z \in U_r$ and almost all $t \in [0,\infty)$.*

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Then, for each $t \in [0,\infty)$, $L(\cdot,t)$ can be analytically continued in U and gives a univalent function.

2. MAIN RESULTS

We are now able to give our main result. Let $a(t)$ a complex valued function defined on $[0,\infty)$ such that the following conditions hold:

$$a \in C^1[0,\infty), a(0) = 1, a(t) \neq 0,\ a(t) + a'(t) \neq 0, \text{ for each } t \in [0,\infty), \tag{1}$$

$$\lim_{t\to\infty} |a(t)| = \infty. \tag{2}$$

THEOREM 2.1. *Let $f \in \mathcal{A}$ and let $a : [0,\infty) \to \mathbb{C}$ be a function such that (1) and (2) hold. If $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 1$,*

$$\left|\frac{1-\alpha}{\alpha} + \frac{1-a'(0)}{2}\right| < \frac{|1+a'(0)|}{2}, \tag{3}$$

$$\begin{aligned}\max_{|z|=\mathrm{e}^{-t}} &\left|\frac{1-\alpha}{\alpha}\left[\frac{a(t)}{|z|} - \left(\frac{a(t)}{|z|} - 1\right)\frac{zf'(z)}{f(z)}\right] + \left(\frac{a(t)}{|z|} - 1\right) z\frac{\mathrm{d}}{\mathrm{d}z}\log\frac{z^2 f'(z)}{f^2(z)}\right.\\ &\left.+\frac{a(t)-a'(t)}{2a(t)}\right| \leq \frac{|a(t)+a'(t)|}{2|a(t)|},\ z \in \dot{U}, t \geq 0,\end{aligned} \tag{4}$$

then f is univalent in U.

Proof. We introduce the function $L : U \times [0,\infty) \to \mathbb{C}$,

$$L(z,t) := \left[f\left(\mathrm{e}^{-t}z\right)\right]^{1-\alpha}\left[f\left(\mathrm{e}^{-t}z\right) + \frac{\left(a(t)\mathrm{e}^{t}-1\right)\mathrm{e}^{-t}zf'\left(\mathrm{e}^{-t}z\right)}{1-\left(a(t)\mathrm{e}^{t}-1\right)\left(\frac{\mathrm{e}^{-t}zf'\left(\mathrm{e}^{-t}z\right)}{f\left(\mathrm{e}^{-t}z\right)}-1\right)}\right]^{\alpha}. \tag{5}$$

The function $f \in \mathcal{A}$ has the series expansion $f(z) = z + a_2z^2 + \ldots$. From (4) we have $f(z) \neq 0$, for each $z \in \dot{U}$, and we obtain that

$$f_1(z,t) := \frac{\mathrm{e}^{-t}zf'\left(\mathrm{e}^{-t}z\right)}{f\left(\mathrm{e}^{-t}z\right)} = 1 + ...$$

is analytic in U. Hence, the function

$$f_2(z,t) := \frac{\mathrm{e}^{-t}zf'\left(\mathrm{e}^{-t}z\right)}{f\left(\mathrm{e}^{-t}z\right)} - 1 = a_2\mathrm{e}^{-t}z + ...$$

is also analytic in U. It follows from

$$f_3(z,t) := 1 + \frac{\left(a(t)\mathrm{e}^{t}-1\right)f_1(z,t)}{1-\left(a(t)\mathrm{e}^{t}-1\right)f_2(z,t)} = a(t)\mathrm{e}^{t} + ...$$

that there is an $r \in (0,1]$ such that $f_3(z,t)$ is analytic in U_r and $f_3(z,t) \neq 0$, for each $z \in U_r$, $t \geq 0$. For the function given by

$$f_4(z,t) := [f_3(z,t)]^{\alpha} = [a(t)]^{\alpha}\mathrm{e}^{\alpha t} + ...$$

we will choose an analytic branch in U_r. We have that

$$L(z,t) = f\left(\mathrm{e}^{-t}z\right)f_4(z,t) = [a(t)]^{\alpha}\mathrm{e}^{(\alpha-1)t}z + ... \tag{6}$$

is analytic in U_r, for each $t \geq 0$. From (6) we have $L(z,t) = a_1(t) z + \ldots$, where $a_1(t) = [a(t)]^{\alpha} \mathrm{e}^{(\alpha-1)t} \neq 0$ and $|a_1(t)| = |[a(t)]^{\alpha}| \mathrm{e}^{\mathrm{Re}(\alpha-1)t}$. Because $\mathrm{Re}\,\alpha > 1$ and from (2), we can conclude that $\lim_{t\to\infty} |a_1(t)| = \infty$.

Let $p : U_r \times [0,\infty) \to \mathbb{C}$, be the function given by

$$p(z,t) = \frac{\partial L(z,t)/\partial t}{z \cdot \partial L(z,t)/\partial z}$$

and consider $w : U_r \times [0,\infty) \to \mathbb{C}$,

$$w(z,t) = \frac{1-p(z,t)}{1+p(z,t)} = \frac{z \cdot \partial L(z,t)/\partial z - \partial L(z,t)/\partial t}{z \cdot \partial L(z,t)/\partial z + \partial L(z,t)/\partial t}.$$

We determine from (5) the partial derivatives of L with respect to z and t, and by introducing the results in the previous relation, we obtain

$$w(z,t) = \left\{ \frac{1-\alpha}{\alpha} \left[a(t)\mathrm{e}^t - \left(a(t)\mathrm{e}^t - 1\right) \frac{\mathrm{e}^{-t} z f'\left(\mathrm{e}^{-t}z\right)}{f\left(\mathrm{e}^{-t}z\right)} \right] + \left(a(t)\mathrm{e}^t - 1\right) \right.$$
$$\left. \cdot \left[2 + \frac{\mathrm{e}^{-t} z f''\left(\mathrm{e}^{-t}z\right)}{f'\left(\mathrm{e}^{-t}z\right)} - 2\frac{\mathrm{e}^{-t} z f'\left(\mathrm{e}^{-t}z\right)}{f\left(\mathrm{e}^{-t}z\right)} \right] + \frac{a(t) - a'(t)}{2a(t)} \right\} \cdot \frac{2a(t)}{a(t)+a'(t)}.$$

From (4) it follows that $f(z) \cdot f'(z) \neq 0$, for each $z \in U$, and therefore we can analytically continue the function $w(\cdot,t)$ in U, and $p(\cdot,t)$ will also admit an analytic continuation in U, for each $t \in [0,\infty)$.

We have $w(z,0) = \left[\frac{1-\alpha}{\alpha} + \frac{a(0)-a'(0)}{2a(0)}\right] \cdot \frac{2a(0)}{a(0)+a'(0)}$, and hence, from (3), $|w(z,0)| < 1$ for each $z \in U$.

For fixed $t > 0$, because $w(z,t)$ is analytic in U, we obtain by the maximum principle, that $|w(z,t)| < \max_{|\zeta|=1} |w(\zeta,t)|$. We will prove that $\max_{|\zeta|=1} |w(\zeta,t)| \leq 1$.

Let $z = \mathrm{e}^{-t}\zeta$, hence $z \in U$ and $|z| = \mathrm{e}^{-t}$. We have

$$\max_{|z|=\mathrm{e}^{-t}} |w(z,t)| = \max_{|z|=\mathrm{e}^{-t}} \left\{ \frac{1-\alpha}{\alpha} \left[\frac{a(t)}{|z|} - \left(\frac{a(t)}{|z|} - 1\right) \frac{z f'(z)}{f(z)} \right] + \left(\frac{a(t)}{|z|} - 1\right) \right.$$
$$\left. \cdot \left[2 + \frac{z f''(z)}{f'(z)} - 2\frac{z f'(z)}{f(z)} \right] + \frac{a(t) - a'(t)}{2a(t)} \right\} \cdot \frac{2a(t)}{a(t)+a'(t)}.$$

From relation (4) it follows that $\max_{|\zeta|=1} |w(\zeta,t)| = \max_{|z|=\mathrm{e}^{-t}} |w(z,t)| \leq 1$.

Because $\frac{\partial L(z,t)}{\partial t} = [a(t)]^{\alpha-1} \mathrm{e}^{(\alpha-1)t} \left[\alpha a'(t) + (1-\alpha) a(t)\right] z + \ldots$, it follows that $\left|\frac{\partial L(z,t)}{\partial t}\right|$ is bounded in $[0,T]$ for each fixed $T > 0$ and for each $z \in U$. Thus, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniformly with respect to $z \in U$.

It is easy to see that there is $M > 0$ such that $\left|\frac{L(z,t)}{a_1(t)}\right| \leq M$ for all $z \in U$ and $t \in [0,\infty)$, and thus the function family $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is normal in U.

We showed that the function L given by relation (5) satisfies the conditions of Lemma 1.1, so we can conclude now that for each $t \in [0,\infty)$, $L(\cdot,t)$ has an analytic and univalent continuation in U. In particular, the function $f(z) = L(z,0)$ is univalent in U. □

If we take $a(t) = \mathrm{e}^t$ in Theorem 2.1, we obtain a result of D. Răducanu [5].

COROLLARY 2.2. *Let $f \in \mathcal{A}$ and let α be a complex number such that* $\operatorname{Re}\alpha > \frac{1}{2}$. *If*

$$\left|\frac{1-\alpha}{\alpha}\left[1-\left(1-|z|^2\right)\frac{zf'(z)}{f(z)}\right]+\left(1-|z|^2\right)z\frac{\mathrm{d}}{\mathrm{d}z}\log\frac{z^2f'(z)}{f^2(z)}\right| \le |z|^2$$

for all $z \in U$, then the function f is univalent in U.

Another consequence of Theorem 2.1 is the following result:

COROLLARY 2.3. *Let $f \in \mathcal{A}$, let $c > -1$ and $\alpha \in \mathbb{C}$ such that* $\operatorname{Re}\alpha > \max\left\{1, \frac{1+c}{2}\right\}$. *If*

$$\left|\frac{1-\alpha}{\alpha}\left[\frac{1+c|z|^2}{1+c}-\frac{\left(1-|z|^2\right)}{1+c}\frac{zf'(z)}{f(z)}\right]+\frac{\left(1-|z|^2\right)}{1+c}z\frac{\mathrm{d}}{\mathrm{d}z}\log\frac{z^2f'(z)}{f^2(z)}\right.$$
$$\left.+\frac{c|z|^4}{1+c|z|^2}\right| \le \frac{|z|^2}{1+c|z|^2}$$

for all $z \in U$, then f is univalent in U.

Proof. A simple calculation for $a(t) = \frac{\mathrm{e}^t + c\mathrm{e}^{-t}}{1+c}$ in Theorem 2.1 yields the result. □

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"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: vnechita@math.ubbcluj.ro