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SUBCLASSES OF STARLIKE FUNCTIONS INVOLVING A CERTAIN INTEGRAL OPERATOR

GANGADHARAN MURUGUSUNDARAMOORTHY

Abstract. Making use of the generalized integral operator, we define a new subclass of uniformly convex functions and a corresponding subclass of starlike functions with negative coefficients and obtain coefficient estimates, extreme points, the radii of close to convexity, starlikeness and convexity . In particular, we obtain integral means inequalities for the function $f(z)$ belongs to the class $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$ in the unit disc.

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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C},\ |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in U. By $\mathcal{K}(\beta)$, and $\mathcal{S}^*(\beta)$ respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \quad \text{and} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad z \in U$$

for $0 \leqq \beta < 1$. In particular, $\mathcal{K} = \mathcal{K}(0)$ and $\mathcal{S}^* = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions. Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqq 0, \tag{2}$$

that are analytic in the open unit disk U. This class was introduced and studied by Silverman [13]. Analogous to the subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of $\mathcal{S}$ respectively, the subclasses of $\mathcal{T}$ denoted by $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$, $0 \leqq \beta < 1$, were also investigated by Silverman in [13]. For functions $f \in A$ given by (1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

(3) $$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

In [4] Catas extended the multiplier transformation and defined the operator $\mathcal{I}^m(\lambda, \mu)f(z)$ on A by the following series:

(4) $$I^m(\lambda, \mu)f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+\mu+\lambda(n-1)}{1+\mu} \right)^m a_n z^n,$$

$\lambda \geq 0; \mu \geq 0; m \in N_0 = N \cup \{0\}; N = \{1, 2, 3, ...\}; z \in U$). Note that $I^0(1;0)f(z) = f(z)$ and $I^1(1;0)f(z) = zf'(z)$.

Now, we define the integral operator $\mathcal{J}^m(\lambda;\mu)f(z)$ for $f(z) \in A$ and $z \in U$, as follows:

$$\mathcal{J}^0(\lambda;\mu)f(z) = f(z)$$

$$\mathcal{J}^1(\lambda;\mu)f(z) = \left(\frac{1+\mu}{\lambda} \right) z^{1-\left(\frac{1+\mu}{\lambda}\right)} \int_0^z t^{\left(\frac{1+\mu}{\lambda}\right)-2} f(t) \mathrm{d}t$$

$$\mathcal{J}^2(\lambda;\mu)f(z) = \left(\frac{1+\mu}{\lambda} \right) z^{1-\left(\frac{1+\mu}{\lambda}\right)} \int_0^z t^{\left(\frac{1+\mu}{\lambda}\right)-2} \mathcal{J}^1(\lambda;\mu)f(t) \mathrm{d}t$$

and in general

(5) $$\begin{aligned} \mathcal{J}^m(\lambda;\mu)f(z) &= \left(\frac{1+\mu}{\lambda} \right) z^{1-\left(\frac{1+\mu}{\lambda}\right)} \int_0^z t^{\left(\frac{1+\mu}{\lambda}\right)-2} \mathcal{J}^{m-1}(\lambda;\mu)f(t) \mathrm{d}t \\ &= \underbrace{\mathcal{J}^1(\lambda;\mu)\left(\frac{z}{1-z}\right) * \mathcal{J}^1(\lambda;\mu)\left(\frac{z}{1-z}\right) * ... * \mathcal{J}^1(\lambda;\mu)\left(\frac{z}{1-z}\right)}_{m \text{ times}} * f(z). \end{aligned}$$

We note that if $f(z) \in A$ then from (1) and (5) we have

(6) $$\begin{aligned} \mathcal{J}^m(\lambda;\mu)f(z) &= z + \sum_{n=2}^{\infty} \left[\frac{1+\mu}{1+\mu+\lambda(n-1)} \right]^m a_n z^n \\ &= z + \sum_{n=2}^{\infty} \Psi_n^{\lambda}(m,\mu) a_n z^n, \end{aligned}$$

where

(7) $$\Psi_n^{\lambda}(m,\mu) = \left[\frac{1+\mu}{1+\mu+\lambda(n-1)} \right]^m$$

and (throughout this paper unless otherwise mentioned) the parameters μ, λ are constrained as $\lambda \geq 0; \mu \geq 0; m \in N_0 = N \cup \{0\}; N = \{1, 2, 3, ...\}; z \in U$. For various choices of m, λ and μ we note the following:

1) $\mathcal{J}^0(\lambda;\mu)f(z) := f(z)$;
2) $\mathcal{J}^1(1;0)f(z) := \int_0^z \frac{f(t)}{t} \mathrm{d}t := \mathcal{L}f(z)$(see Alexander [1]);

3) $\mathcal{J}^1(1;\nu)f(z) := \frac{1+\nu}{z^\nu}\int_0^z t^{\nu-1}f(t)\mathrm{d}t := z + \sum\limits_{n=2}^{\infty}\left(\frac{1+\nu}{n+\nu}\right)a_n z^n = \mathcal{B}_\nu f(z)$ (see Bernardi [3]);

4) $\mathcal{J}^1(1;1)f(z) := \frac{2}{z}\int_0^z f(t)\mathrm{d}t := z + \sum\limits_{n=2}^{\infty}\left(\frac{2}{n+1}\right)a_n z^n = \mathcal{I}f(z)$ (see [10];

5) $\mathcal{J}^m(1;1)f(z) := z + \sum\limits_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^m a_n z^n := \mathcal{I}^m f(z)$ (see Flett [5] and Jung et al. [8]).

Motivated by earlier works of [6, 7, 10, 11, 14]) we define an unified class of analytic function based on the operator $\mathcal{J}^m(\lambda;\mu)$. For $0 \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$, $\alpha \geqq 0$, and for all $z \in \Delta$, we let the class $\mathcal{UCT}(\alpha,\beta,\gamma,\ \lambda,\ m)$, consists of functions $f \in \mathcal{T}$ is said to be in the class satisfying the condition

$$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha\left|\frac{zF'(z)}{F(z)} - 1\right| + \beta, \tag{8}$$

with

$$F(z) := (1-\gamma)\mathcal{J}^m(\lambda;\mu)f(z) + \gamma z[\mathcal{J}^m(\lambda;\mu)f(z)]', \tag{9}$$

where $\mathcal{J}^m(\lambda;\mu)f(z)$ is given by (6). The family $\mathcal{UCT}(\alpha,\ \beta,\gamma,\ \lambda,\ m)$ unifies various well-known classes of analytic univalent functions, as follows.

Example 1. If $m = 0$, then

$$\mathcal{S}(\alpha,\beta,\gamma) := \left\{f \in \mathcal{A} : \operatorname{Re}\ \left\{\frac{zF'(z)}{F(z)} - \alpha\right\} > \beta\left|\frac{zF'(z)}{F(z)} - 1\right|,\ \ z \in U\right\}.$$

Further $\mathcal{TS}(\alpha,\beta,\gamma) = \mathcal{S}(\alpha,\beta,\gamma)\cap\mathcal{T}$, where $\mathcal{T}$ is given by (2).

Example 2. If $m = 1$, $\lambda = 1$, $\mu = 0$, then

$$\mathcal{L}(\alpha,\beta,\gamma) := \left\{f \in \mathcal{A} : \operatorname{Re}\ \left\{\frac{z(\mathcal{L}F(z))'}{\mathcal{L}F(z)} - \alpha\right\} > \beta\left|\frac{z(\mathcal{L}F(z))'}{\mathcal{L}F(z)} - 1\right|,\ \ z \in U\right\},$$

where $\mathcal{L}F(z)$ is defined by $\mathcal{L}F(z) := z + \sum\limits_{n=2}^{\infty}\left(\frac{1+n\gamma-\gamma}{n}\right)a_n z^n$. Also $\mathcal{TL}(\alpha,\beta,\gamma) = \mathcal{L}(\alpha,\beta,\gamma)\cap\mathcal{T}$, where $\mathcal{T}$ is given by (2).

Example 3. If $m = 1$, $\lambda = 1$, $\mu = \nu$ ($\nu > -1$), then

$$\mathcal{B}_\nu(\alpha,\beta,\gamma) = \left\{f \in \mathcal{A} : \operatorname{Re}\ \left\{\frac{z(\mathcal{B}_\nu F(z))'}{\mathcal{B}_\nu F(z)} - \alpha\right\} > \beta\left|\frac{z(\mathcal{B}_\nu F(z))'}{\mathcal{B}_\nu F(z)} - 1\right|,\ \ z \in U\right\},$$

where $\mathcal{B}_\nu F(z)$ is given by $\mathcal{B}_\nu F(z) := z + \sum\limits_{n=2}^{\infty}(1+n\gamma-\gamma)\left(\frac{1+\nu}{n+\nu}\right)a_n z^n$. Further, $TB_\nu(\alpha,\beta,\gamma) = B_\nu(\alpha,\beta,\gamma)\cap T$, where T is given by (2).

Example 4. If $\lambda = 1$, $\mu = 1$, then

$$\mathcal{I}(\alpha,\beta,\gamma) \quad := \left\{f \in \mathcal{A} : \operatorname{Re}\ \left\{\frac{z(\mathcal{I}^m F(z))'}{\mathcal{I}^m F(z)} - \alpha\right\} > \beta\left|\frac{z(\mathcal{I}^m F(z))'}{\mathcal{I}^m F(z)} - 1\right|,\ \ z \in U\right\},$$

where $\mathcal{I}^m f(z)$ is defined by $\mathcal{I}^m F(z) := z + \sum\limits_{n=2}^{\infty}(1+n\gamma-\gamma)\left(\frac{2}{n+1}\right)^m a_n z^n$.

In the present paper, we obtain a characterization, coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{UCT}(\alpha,\ \beta,\gamma,\ \lambda,\ k)$.

2. CHARACTERIZATION AND COEFFICIENT ESTIMATES

THEOREM 5. *Let* $f \in \mathcal{T}$. *Then* $f \in \mathcal{UCT}(\alpha,\ \beta,\gamma,\ \lambda,\ m)$, $0 \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$ *and* $\alpha \geqq 0$,

$$\sum_{n=2}^{\infty} [n(\alpha+1)-(\alpha+\beta)]\,(\gamma(n-1)+1)\,\Psi_n^{\lambda}(m,\mu)|a_n| \leqq 1-\beta. \tag{10}$$

This result is sharp for the function

$$f(z) = z - \frac{1-\beta}{[n(\alpha+1)-(\alpha+\beta)][\gamma(n-1)+1]\Psi_n^{\lambda}(m,\mu)}\, z^n,\ n \geqq 2. \tag{11}$$

Proof. We employ the technique from [2]. We have $f \in \mathcal{UCT}(\alpha,\ \beta,\ \gamma,\ \lambda,\ m)$, if and only if the condition (8) is satisfied, which is equivalent to

$$\operatorname{Re}\left\{\frac{zF'(z)(1+k\mathrm{e}^{\mathrm{i}\theta})-F(z)k\mathrm{e}^{\mathrm{i}\theta}}{F(z)}\right\} > \beta, \quad -\pi \leqq \theta < \pi. \tag{12}$$

Now, letting $G(z) = zF'(z)(1+k\mathrm{e}^{\mathrm{i}\theta}) - F(z)k\mathrm{e}^{\mathrm{i}\theta}$, (12) is equivalent to

$$|G(z)+(1-\beta)F(z)| > |G(z)-(1+\beta)F(z)|,\ 0 \leqq \beta < 1,$$

where $F(z)$ is as defined in (9). Now a simple computation gives

$$\begin{aligned}
&|G(z)+(1-\beta)F(z)|\\
&\geqq |(2-\beta)z| - \left|\sum_{n=2}^{\infty}\{n+1-\beta\}\{\gamma(n-1)+1\}\Psi_n^{\lambda}(m,\mu)a_n z^n\right|\\
&-\left|k\mathrm{e}^{\mathrm{i}\theta}\sum_{n=2}^{\infty}(n-1)\{\gamma(n-1)+1\}\Psi_n^{\lambda}(m,\mu)a_n z^n\right|\\
&\geqq (2-\beta)|z| - \sum_{n=2}^{\infty}\{n+1-\beta\}\{\gamma(n-1)+1\}\Psi_n^{\lambda}(m,\mu)a_n|z|^n\\
&-k\sum_{n=2}^{\infty}(n-1)\{\gamma(n-1)+1\}\Psi_n^{\lambda}(m,\mu)a_n|z|^n\\
&\geqq (2-\beta)|z| - \sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)+1\Big)\Big(\gamma(n-1)+1\Big)\Psi_n^{\lambda}(m,\mu)a_n|z|^n
\end{aligned}$$

and similarly,

$$
\begin{aligned}
&|G(z)-(1+\beta)F(z)| \\
&\leqq \beta|z|+\sum_{n=2}^{\infty}\Big((n(\alpha+1)-(\alpha+\beta)-1)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n^{\lambda}(m,\mu)a_n|z|^n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|G(z)+(1-\beta)F(z)|-|G(z)-(1+\beta)F(z)|\geqq 2(1-\beta)|z| \\
&-2\sum_{n=2}^{\infty}\Big((n(\alpha+1)-(\alpha+\beta))\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n^{\lambda}(m,\mu)a_n|z|^n\geqq 0,
\end{aligned}
$$

which is equivalent to the result (10).

On the other hand, for all $-\pi\leqq\theta<\pi$, $\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}(1+k\mathrm{e}^{\mathrm{i}\theta})-k\mathrm{e}^{\mathrm{i}\theta}\right\}>\beta$. Now choosing the values of z on the positive real axis, where $0\leqq|z|=r<1$, and using $\operatorname{Re}\{-\mathrm{e}^{\mathrm{i}\theta}\}\geqq-|\mathrm{e}^{\mathrm{i}\theta}|=-1$, the above inequality can be written as

$$
\operatorname{Re}\left\{\frac{(1-\beta)-\displaystyle\sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n^{\lambda}(m,\mu)a_nr^{n-1}}{1-\displaystyle\sum_{n=2}^{\infty}\Big(\gamma(n-1)+1\Big)\,\Psi_n^{\lambda}(m,\mu)a_nr^{n-1}}\right\}\geqq 0.
$$

Setting $r\to 1^-$, we get the desired result. □

By taking $\alpha=0,\gamma=1,$ and $m=0$ in Theorem 5, we get the following interesting result given in [13].

Corollary 6 ([13]). *If $f\in\mathcal{T}$, then we have $f\in\mathcal{C}(\beta)$ if and only if $\sum_{n=2}^{\infty}n(n-\beta)a_n\leqq 1-\beta$.*

Indeed, since $f\in\mathcal{UCT}(\alpha,\,\beta,\gamma,\ \lambda,\,m)$, (10), we have

$$
\sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n^{\lambda}(m,\mu)a_n\leqq 1-\beta.
$$

Hence for all $n\geqq 2$, we have

$$
a_n\leqq\frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)},
$$

whenever $0\leqq\gamma\leqq 1,\ 0\leqq\beta<1$ and $\alpha\geqq 0$. Hence we state this important observation as a separate theorem.

Theorem 7. *If $f\in\mathcal{UCT}(\alpha,\,\beta,\gamma,\ \lambda,\,m)$, then*

$$
a_n\leqq\frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)},\ n\geqq 2, \tag{13}
$$

where $0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ *and* $\alpha \geqq 0$. *Equality in* (13) *holds for the function*

$$f(z) = z - \frac{1-\beta}{\big(\, n(\alpha+1) - (\alpha+\beta)\big) \big(\, \gamma(n-1)+1\big)\, \Psi_n^{\lambda}(m,\mu)}. \tag{14}$$

3. DISTORTION AND COVERING THEOREMS

Theorem 8. *If* $f \in \mathcal{UCT}(\alpha,\, \beta, \gamma,\ \lambda,\, m)$, *then* $f \in \mathcal{T}^*(\delta)$, *where*

$$\delta = 1 - \frac{1-\beta}{\big(\, 2(\alpha+1) - (\alpha+\beta)\big) \big(\, \gamma+1\big)\, \Psi_2^{\lambda}(m,\mu) - (1-\beta)}.$$

This result is sharp with the extremal function being

$$f(z) = z - \frac{1-\beta}{\big(\, 2(\alpha+1) - (\alpha+\beta)\big) \big(\, \gamma+1\big)\, \Psi_2^{\lambda}(m,\mu)} z^2.$$

Proof. It is sufficient to show that (10) implies $\displaystyle\sum_{n=2}^{\infty}(n-\delta)a_n \leqq 1-\delta$ [13], that is,

$$\frac{n-\delta}{1-\delta} \leqq \frac{\big(\, n(\alpha+1) - (\alpha+\beta)\big) \big(\, \gamma(n-1)+1\big)\, \Psi_n^{\lambda}(m,\mu)}{1-\beta},\ \ n \geqq 2. \tag{15}$$

Since, for $n \geqq 2$, (15) is equivalent to

$$\delta \ \leqq\ 1 - \frac{(n-1)(1-\beta)}{\big(\, n(\alpha+1) - (\alpha+\beta)\big) \big(\, \gamma(n-1)+1\big)\, \Psi_n^{\lambda}(m,\mu) - (1-\beta)} = \Phi(n),$$

and $\Phi(n) \leqq \Phi(2)$, (15) holds true for any $0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ and $\alpha \geqq 0$. This completes the proof of the Theorem 8. □

As in the previous cases we note this result has many special cases. If we take $\alpha = 0, \gamma = 1,$ and $m = 0$ in Theorem 8, then we have the following result of Silverman [13].

Corollary 9 ([13]). *If* $f \in \mathcal{C}(\beta)$, *then* $f \in \mathcal{T}^*\left(\dfrac{2}{3-\beta}\right)$. *The result is sharp for the extremal function* $f(z) = z - \frac{1-\beta}{2(2-\beta)} z^2$.

Remark. Since distortion theorem and covering theorem are available for the class $\mathcal{T}^*(\beta)$ [13], we can also obtain the corresponding results for the class $\mathcal{UCT}(\alpha,\, \beta, \gamma,\ \lambda,\, m)$, from the respective results of $\mathcal{T}^*(\beta)$ by using Theorem 8, and we state them without proof.

Theorem 10. *Let* $\Psi_n^{\lambda}(m,\mu)$ *be defined as in* (7). *Then, for every* $f \in \mathcal{UCT}(\alpha,\, \beta, \gamma,\ \lambda,\, m)$, *with* $z = r\mathrm{e}^{\mathrm{i}\theta} \in \Delta$, *we have*

$$r - B(\alpha,\, \beta,\, \gamma,\, \lambda) r^2 \leqq |f(z)| \leqq r + B(\alpha,\, \beta,\, \gamma,\, \lambda) r^2, \tag{16}$$

where $B(\alpha,\, \beta,\, \gamma,\, \lambda) := \frac{1-\beta}{\big(\, 2(\alpha+1) - (\alpha+\beta)\big) \big(\, \gamma+1\big) \Psi_2^{\lambda}(m,\mu)}$.

THEOREM 11. *If* $f \in \mathcal{UCT}(\alpha,\, \beta, \gamma,\; \lambda,\, m)$, *then for* $|z| = r < 1$

$$1 - B(\alpha,\, \beta,\, \gamma,\, \lambda)r \leqq |f'(z)| \leqq 1 + B(\alpha,\, \beta,\, \gamma,\, \lambda)r\,, \tag{17}$$

where $B(\alpha,\, \beta,\, \gamma,\, \lambda)$ *as in Theorem* 10.

Note that in Theorem 10 and Theorem 11 equality holds for the function

$$f(z) = z - \frac{1-\beta}{\big(\,2(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma+1\big)\,\Psi_2^{\lambda}(m,\mu)} z^2.$$

4. EXTREME POINTS OF THE CLASS $\mathcal{UCT}(\alpha,\, \beta, \gamma,\; \lambda,\, K)$,

THEOREM 12. *Let* $f_1(z) = z$,

$$f_n(z) = z - \frac{1-\beta}{\big(\,n(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)} z^n, \quad n \geqq 2$$

and $\Psi_n^{\lambda}(m,\mu)$ *be as defined in* (7)*. Then* $f \in \mathcal{UCT}(\alpha,\, \beta, \gamma,\; \lambda,\, m)$*, if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \quad \omega_n \geqq 0, \quad \sum_{n=1}^{\infty} \omega_n = 1. \tag{18}$$

Proof. Suppose $f(z)$ can be written as in (18). Then

$$f(z) = z - \sum_{n=2}^{\infty} \omega_n \left\{ \frac{1-\beta}{\big(\,n(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)} \right\} z^n.$$

Now we have

$$\sum_{n=2}^{\infty} \omega_n \frac{\big(\,n(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)(1-\beta)}{(1-\beta)\big(\,n(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)}$$

$$= \sum_{n=2}^{\infty} \omega_n = 1 - \omega_1 \leqq 1.$$

Thus $f \in \mathcal{UCT}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. Conversely, let us have $f \in \mathcal{UCT}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. Then by using (13), we may write

$$\omega_n = \frac{\big(\,n(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)}{1-\beta} a_n, \quad n \geqq 2,$$

and $\omega_1 = 1 - \sum\limits_{n=2}^{\infty} \omega_n$. Then $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$, with $f_n(z)$ is as in the Theorem. □

COROLLARY 13. *The extreme points of* $f \in \mathcal{UCT}(\alpha,\, \beta, \gamma,\; \lambda,\, m)$*, are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\beta}{\big(\,n(\alpha+1) - (\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)} z^n, \quad n \geqq 2.$$

Remark. As in previous theorems, we can deduce known results for various other classes and we omit details.

THEOREM 14. *The class $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$ is a convex set.*

Proof. Let the function

$$f_j(z) = \sum_{n=2}^{\infty} a_{n,\,j} z^n, \quad a_{n,\,j} \geqq 0, \quad j = 1, 2\,, \tag{19}$$

be the class $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$. It sufficient to show that the function defined by $g(z) = \omega f_1(z) + (1-\omega) f_2(z)$, $0 \leqq \mu \leqq 1$, is in the class $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$. Since $g(z) = z - \sum_{n=2}^{\infty} [\omega a_{n,1} + (1-\omega) a_{n,2}] z^n$, an easy computation with the aid of Theorem 5 gives

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta) \Big) \Big(\gamma(n-1) + 1 \Big) \, \Psi_n^{\lambda}(m,\mu) [\omega a_{n,1} + (1-\omega) a_{n,2}]$$

$$+ (1-\omega) \sum_{n=2}^{\infty} (\, n(\alpha+1) - (\alpha+\beta))(\, \gamma(n-1) + 1) \Psi_n^{\lambda}(m,\mu)$$

$$\leqq \omega(1-\beta) + (1-\omega)(1-\beta) \leqq 1-\beta,$$

which implies that $g \in \mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$. Hence $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$ is convex. □

5. MODIFIED HADAMARD PRODUCTS

For functions of the form (19), we define the modified Hadamard product:

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,\,1}\, a_{n,\,2} z^n. \tag{20}$$

THEOREM 15. *If $f_j(z) \in \mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$, $j = 1,\ 2$, then*

$$(f_1 * f_2)(z) \in \mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m,\ \xi),$$

where

$$\xi = \frac{(2-\beta) \big(2(\alpha+1) - (\alpha+\beta) \big) \big(\gamma + 1 \big) \, \Psi_2^{\lambda}(m,\mu) - 2(1-\beta)^2}{(2-\beta) \big(2(\alpha+1) - (\alpha+\beta) \big) \big(\gamma + 1 \big) \, \Psi_2^{\lambda}(m,\mu) - (1-\beta)^2},$$

with $\Psi_n^{\lambda}(m,\mu)$ be defined as in (7).

Proof. Since $f_j(z) \in \mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$, $j = 1,\ 2$, we have

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta) \Big) \Big(\gamma(n-1) + 1 \Big) \, \Psi_n^{\lambda}(m,\mu) a_{n,\,j} \leqq 1-\beta. \tag{21}$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \frac{\big(\, n(\alpha+1) - (\alpha+\beta) \big) \big(\, \gamma(n-1) + 1 \big) \, \Psi_n^{\lambda}(m,\mu) a_{n,\,j}}{1-\beta} \sqrt{a_{n,\,1}\, a_{n,\,2}} \leqq 1. \tag{22}$$

Note that we need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{\big(n(k+1)-(k+\xi)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)a_{n,j}}{1-\xi} a_{n,1}\,a_{n,2} \leqq 1. \tag{23}$$

Therefore, in view of (22) and (23), whenever $\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}\,a_{n,2}} \leqq \frac{n-\beta}{1-\beta}$ $(n \geqq 2)$ holds, then (23) is satisfied. We have, from (22),

$$\sqrt{a_{n,1}\,a_{n,2}} \leqq \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)},\ n \geqq 2. \tag{24}$$

Thus, for $n \geq 2$, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left[\frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)}\right] \leqq \frac{n-\beta}{1-\beta},$$

or, if

$$\xi \leqq \frac{(n-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)-n(1-\beta)^2}{(n-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)-(1-\beta)^2},$$

then (22) is satisfied. Note that the right hand side of the above expression is an increasing function on n. Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = z - \frac{1-\beta}{(2-\beta)\big(2(\alpha+1)-(\alpha+\beta)\big)\big(\gamma+1\big)\,\Psi_2^{\lambda}(m,\mu)} z^2,$$

we see that the result is sharp. $\square$

6. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

THEOREM 16. *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{UCT}(\alpha,\,\beta,\gamma,\ \lambda,\,m)$. Then $f(z)$ is close-to-convex of order ρ, $0 \leqq \rho < 1$ in $|z| < r_1(\alpha,\,\beta,\gamma,\rho)$, where*

$$r_1(\alpha,\,\beta,\gamma,\rho) = \inf_{n\geq 2}\left[\frac{(1-\rho)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)}{n(1-\beta)}\right]^{\frac{1}{n-1}},$$

with $\Psi_n^{\lambda}(m,\mu)$ be defined as in (7). *This result is sharp for the function $f(z)$ given by* (11).

Proof. It is sufficient to show that $|f'(z)-1| \leqq 1-\rho$, $0 \leqq \rho < 1$, for $|z| < r_1(\alpha,\,\beta,\gamma,\rho)$, or equivalently

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right) a_n|z|^{n-1} \leqq 1. \tag{25}$$

By Theorem 5, (25) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leqq \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\,\Psi_n^{\lambda}(m,\mu)}{1-\beta}$$

or, if

$$|z| \leqq \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n^\lambda(m,\mu)}{n(1-\beta)} \right]^{\frac{1}{n-1}}, \tag{26}$$

for $n \geqq 2$. The theorem follows easily from (26). □

THEOREM 17. *Let $f(z)$ defined by* (2) *be in the class $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \lambda,\ m)$. Then $f(z)$ is starlike of order ρ, $0 \leqq \rho < 1$ in $|z| < r_2(\alpha,\ \beta, \gamma, \rho)$, where*

$$r_2(\alpha,\ \beta,\gamma,\rho) = \inf_{n\geq 2} \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n^\lambda(m,\mu)}{(n-\rho)(1-\beta)} \right]^{\frac{1}{n-1}},$$

with $\Psi_n^\lambda(m,\mu)$ be defined as in (7)*. This result is sharp for the function $f(z)$ given by* (11)*.*

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1-\rho$, or equivalently

$$\sum_{n=2}^{\infty} \left(\frac{n-\rho}{1-\rho}\right) a_n |z|^{n-1} \leqq 1, \tag{27}$$

for $0 \leqq \rho < 1$, and $|z| < r_2(\alpha,\ \beta,\gamma,\rho)$. Proceeding as in Theorem 16, with the use of Theorem 5, we get the required result. Hence, by Theorem 5, (27) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1} \leqq \frac{\{n(\alpha+1)-(\alpha+\beta)\}\{\gamma(n-1)+1\}\Psi_n^\lambda(m,\mu)}{1-\beta}$$

or, if

$$|z| \leqq \left[\frac{\{n(\alpha+1)-(\alpha+\beta)\}\{\gamma(n-1)+1\}\Psi_n^\lambda(m,\mu)}{(n-\rho)(1-\beta)} \right]^{\frac{1}{n-1}},\ n \geqq 2. \tag{28}$$

The theorem follows easily from (28). □

THEOREM 18. *Let $f(z)$ defined by* (2) *be in the class $\mathcal{UCT}(\alpha,\ \beta,\gamma,\lambda,m)$. Then $f(z)$ is convex of order ρ, $0 \leqq \rho < 1$ in $|z| < r_3(\alpha,\ \beta,\gamma,\rho)$, where*

$$r_3(\alpha,\ \beta,\gamma,\rho) = \inf_{n\geq 2} \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n^\lambda(m,\mu)}{n(n-\rho)(1-\beta)} \right]^{\frac{1}{n-1}},$$

with $\Psi_n^\lambda(m,\mu)$ be defined as in (7)*. This result is sharp for the function $f(z)$ given by* (11)*.*

Proof. It is sufficient to show that $\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1-\rho$, or equivalently

$$\sum_{n=2}^{\infty} \left(\frac{n(n-\rho)}{1-\rho}\right) a_n |z|^{n-1} \leqq 1, \tag{29}$$

for $0 \leqq \rho < 1$ and $|z| < r_3(\alpha,\ \beta,\gamma,\rho)$. Proceeding as in Theorem 16, we get the required result. □

7. INTEGRAL MEANS

In order to find the integral means inequality and to verify the Silverman Conjuncture[12] for $f \in \mathcal{UCT}(\alpha, \beta, \gamma, \lambda, m)$ we need the following subordination result due to Littlewood [9] .

LEMMA 19 ([9]). *If the functions $f(z)$ and $g(z)$ are analytic in U with $g(z) \prec f(z)$, then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta, \quad \eta > 0, \quad z = re^{i\theta} \text{ and } 0 < r < 1. \tag{30}$$

Applying Theorem 5 with extremal function given by (11) and Lemma 19, we prove the following theorem.

THEOREM 20. *Let $\eta > 0$. If $f(z) \in \mathcal{UCT}(\alpha, \beta, \gamma, \lambda, m)$, and the sequence $\{\Phi(\alpha, \beta, \gamma, n)\}_{n=2}^{\infty}$ is non-decreasing, then for $z = re^{i\theta}$ and $0 < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\eta} d\theta, \tag{31}$$

where$f_2(z) = z - \frac{1-\beta}{\big(2(\alpha+1)-(\alpha+\beta)\big)\big(\gamma+1\big)\Psi_2^{\lambda}(m,\mu)} z^2$, *or* $f_2(z) = z - \frac{1-\beta}{\Phi(\alpha,\beta,\gamma,2)} z^2$, *and* $\Phi(\alpha, \beta, \gamma, n) = [n(\alpha + 1) - (\alpha + \beta)] (\gamma(n - 1) + 1) \Psi_n^{\lambda}(m, \mu)$.

Proof. Let $f(z)$ of the form (2) and $f_2(z) = z - \frac{1-\beta}{\Phi(\alpha,\beta,\gamma,2)} z^2$. Then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\beta}{\Phi(\alpha, \beta, \gamma, 2)} z\right|^{\eta} d\theta.$$

By Lemma 19, it suffices to show that $1 - \sum\limits_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\beta}{\Phi(\alpha,\beta,\gamma,2)} z$. Setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\beta}{\Phi(\alpha, \beta, \gamma, 2)} w(z). \tag{32}$$

From (32) and (10), we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\Phi(\alpha, \beta, \gamma, n)}{1-\beta} a_n z^{n-1}\right| \leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\alpha, \beta, \gamma, n)}{1-\beta} a_n \leq |z| < 1.$$

This completes the proof of the Theorem 20. □

Concluding remarks. By suitably specializing the various parameters involved in our theorems, we can state the corresponding results for the new subclasses defined in our examples and also for many relatively more familiar function classes. Further, by choosing $\alpha = 0$, $m = 0$ and $\gamma = 0$, $\gamma = 1$ the results obtained for the class $\mathcal{UCT}(\alpha,\ \beta, \gamma,\ \ \lambda,\ m)$ yield the results of Silverman [13], and for $m = 0$, $\gamma = 0$, $\gamma = 1$, the results in [14].

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VIT University
School of Advanced Sciences
Vellore – 632014, India
E-mail: `gmsmoorthy@yahoo.com`