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APPROXIMATION BY COMPLEX q-LORENTZ POLYNOMIALS, $q > 1$

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Abstract. In this paper, for $q > 1$ we obtain quantitative estimate in the Voronovskaja's theorem and the exact orders in simultaneous approximation by the complex q-Lorentz polynomials of degree $n \in \mathbb{N}$, attached to analytic functions in compact disks of the complex plane. The geometric progression order of approximation q^{-n} is attained, which essentially improves the approximation order $1/n$ for the case $q = 1$, obtained in the very recent paper [2]. Also, some approximation properties of the iterates of these complex q-polynomials are studied.

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1. INTRODUCTION

In the recent book [1] (see also the papers cited there in References), estimates for the convergence in Voronovskaja's theorem and the exact approximation orders in simultaneous approximation for several important classes of complex Bernstein-type operators attached to an analytic function f in compact disks of the complex plane were obtained.

The goal of the present paper is to extend these type of results to the complex q-Lorentz polynomials, $q > 1$. The complex Lorentz polynomials attached to any analytic function f in a domain containing the origin were introduced in [3, p. 43, formula (2)], under the name of degenerate Bernstein polynomials by the formula

$$L_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k f^{(k)}(0), n \in \mathbb{N}, z \in \mathbb{C}. \tag{1}$$

In the same book [3, p. 121-124], some qualitative approximation results were studied. In the recent paper [2], exact quantitative estimates of order $\frac{1}{n}$ in approximation by $L_n(f)(z)$ and by its iterates in compact disks of the complex plane were obtained.

In this paper, by introducing the complex q-Lorentz polynomials, $q \geq 1$, given by

$$L_{n,q}(f)(z) = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q \left(\frac{z}{[n]_q}\right)^k D_q^k(f)(0), n \in \mathbb{N},$$

for $q > 1$ we consider their approximation properties in compact disks. Here $[n]_q = \frac{q^n-1}{q-1}$ if $q \neq 1$, $[n]_q = n$ if $q = 1$, $[n]_q! = [1]_q[2]_q \cdot \ldots \cdot [n]_q$, $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, $D_q^k(f)(z) = D_q[D_q^{k-1}(f)](z)$, $D_q^0(f)(z) = f(z)$, $D_q(f)(z) = \frac{f(qz)-f(z)}{(q-1)z}$.

Note that because $D_q(e_k)(z) = [k]_q z^{k-1}$, where $e_k(z) = z^k$, if f is analytic in a disk $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$, that is we have $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{D}_R$, then $D_q(f)(z) = \sum_{k=1}^{\infty} c_k [k]_q z^{k-1}$, $D_q^2(f)(z) = \sum_{k=2}^{\infty} c_k [k]_q [k-1]_q z^{k-2}$ and so on. This immediately implies that $D_q^k(f)(0) = c_k [k]_q!$, for all $k = 0, 1, \ldots,$. Also, since $[n]_1 = n$ and $D_1(f)(z) = f'(z)$, it is immediate that $L_{n,1}(f)(z)$ become the complex original Lorentz polynomials $L_n(f)(z)$ given above by (1) and already studied in [3] and [2].

The plan of the present paper goes as follows. Section 2 deals with upper estimates in simultaneous approximation by these q-polynomials, in Section 3 we obtain a Voronovskaja result with a quantitative estimate and in Section 4 one obtain exact estimates in simultaneous approximation for these q-operators. Section 5 presents an approximation result for the iterates of the complex polynomials $L_{n,q}(f)(z)$. The quantitative estimates are obtained in compact disks centered at origin and are of exact order $\frac{1}{[n]_q}$, which by the inequalities $(q-1)\frac{1}{q^n} \leq \frac{1}{[n]_q} \leq q\frac{1}{q^n}$, implies the exact order of approximation q^{-n}, with $q > 1$. This essentially improves the exact order $1/n$ obtained for $L_{n,1}(f)(z) := L_n(f)(z)$ by the very recent paper [2].

2. UPPER APPROXIMATION ESTIMATES

The main result of this section is the following.

THEOREM 1. *Let $R > q > 1$. Denoting $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$, suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) *Let $1 \leq r < \frac{r_1}{q} < \frac{R}{q}$ be arbitrary fixed. For all $|z| \leq r$ and $n \in \mathbb{N}$, we have the upper estimate $|L_{n,q}(f)(z) - f(z)| \leq \frac{M_{r_1,q}(f)}{[n]_q}$, where $M_{r_1,q}(f) = \frac{q+1}{(q-1)^2} \cdot \sum_{k=0}^{\infty} |c_k|(k+1) r_1^k < \infty$.*

(ii) *Let $1 \leq r < r^* < \frac{r_1}{q} < \frac{R}{q}$ be arbitrary fixed. For the simultaneous approximation by complex Lorentz polynomials, for all $|z| \leq r$, $p \in \mathbb{N}$ and $n \in \mathbb{N}$, we have $|L_{n,q}^{(p)}(f)(z) - f^{(p)}(z)| \leq \frac{p! r^* M_{r_1,q}(f)}{[n]_q (r^*-r)^{p+1}}$, where $M_{r_1,q}(f)$ is given as at the above point* (i).

Proof. (i) Denoting $e_j(z) = z^j$, firstly we easily get that $L_{n,q}(e_0)(z) = 1$, $L_{n,q}(e_1)(z) = e_1(z)$. Then, since for all $j, n \in \mathbb{N}$, $2 \leq j \leq n$, we have

$$L_{n,q}(e_j)(z) = q^{j(j-1)/2} \binom{n}{j}_q [j]_q! \cdot \frac{z^j}{[n]_q^j},$$

taking into account the relationship (7) in [4, p. 236], we get

$$L_{n,q}(e_j)(z) = z^j \left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right) ... \left(1 - \frac{[j-1]_q}{[n]_q}\right).$$

Also, note that for $j \geq n+1$ we easily get $L_{n,q}(e_j)(z) = 0$. Since an easy computation shows that $L_{n,q}(f)(z) = \sum_{j=0}^{\infty} c_j L_{n,q}(e_j)(z)$, for all $|z| \leq r$, we immediately obtain

$$\begin{aligned}
&|L_{n,q}(f)(z) - f(z)| \\
&\leq \sum_{j=0}^{n} |c_j| \cdot |L_{n,q}(e_j)(z) - e_j(z)| + \sum_{j=n+1}^{\infty} |c_j| \cdot |L_{n,q}(e_j)(z) - e_j(z)| \\
&\leq \sum_{j=2}^{n} |c_j| r^j \left| \left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right) ... \left(1 - \frac{[j-1]_q}{[n]_q}\right) - 1 \right| + \sum_{j=n+1}^{\infty} |c_j| r^j,
\end{aligned}$$

for all $|z| \leq r$. Taking into account the inequality proved in [4, p. 247],

$$1 - \left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{n}\right) ... \left(1 - \frac{[j-1]_q}{n}\right) \leq \frac{(j-1)[j-1]_q}{[n]_q},$$

we obtain

$$\begin{aligned}
&\sum_{j=2}^{n} |c_j| r^j \left| \left(1 - \frac{[1]_q}{[n]_q}\right)\left(1 - \frac{[2]_q}{[n]_q}\right) ... \left(1 - \frac{[j-1]_q}{[n]_q}\right) - 1 \right| \\
&\leq \frac{1}{[n]_q} \sum_{j=2}^{\infty} |c_j| (j-1)[j-1]_q r^j \leq \frac{1}{[n]_q} \sum_{j=2}^{\infty} |c_j| \cdot \frac{j q^j}{q-1} \cdot r^j \\
&\leq \frac{1}{[n]_q} \cdot \frac{1}{q-1} \sum_{j=2}^{\infty} |c_j| (j+1)(rq)^j \leq \frac{1}{[n]_q} \cdot \frac{1}{q-1} \sum_{j=2}^{\infty} |c_j| (j+1) r_1^j,
\end{aligned}$$

where by hypothesis on f we have $\sum_{j=0}^{\infty} |c_j| (j+1) r_1^j < \infty$.

On the other hand, the analyticity of f implies $c_j = \frac{f^{(k)}(0)}{j!}$ and by the Cauchy's estimates of the coefficients c_j in the disk $|z| \leq r_1$, we have $|c_j| \leq \frac{K_{r_1}}{r_1^j}$, for all $j \geq 0$, where

$$K_{r_1} = \max\{|f(z)|; |z| \leq r_1\} \leq \sum_{j=0}^{\infty} |c_j| r_1^j \leq \sum_{j=0}^{\infty} |c_j| (j+1) r_1^j := R_{r_1}(f) < \infty.$$

Therefore we get

$$\begin{aligned}
&\sum_{j=n+1}^{\infty} |c_j| r^j \leq R_{r_1}(f) \left[\frac{r}{r_1}\right]^{n+1} \sum_{j=0}^{\infty} \left(\frac{r}{r_1}\right)^j = R_{r_1}(f) \left[\frac{r}{r_1}\right]^{n+1} \cdot \frac{r_1}{r_1 - r} \\
&= R_{r_1}(f) \cdot \frac{r}{r_1 - r} \cdot \left[\frac{r}{r_1}\right]^n \leq \frac{R_{r_1}(f)}{q-1} \cdot \left[\frac{r}{r_1}\right]^n \leq \frac{R_{r_1}(f)}{q-1} \cdot \frac{1}{q^n} \leq \frac{2R_{r_1}(f)}{(q-1)^2} \cdot \frac{1}{[n]_q}.
\end{aligned}$$

Collecting the estimates, finally we obtain

$$|L_{n,q}(f)(z)-f(z)| \leq \frac{1}{[n]_q}\cdot\frac{R_{r_1}(f)}{q-1}\left(1+\frac{2}{q-1}\right)=\frac{1}{[n]_q}\cdot\frac{q+1}{(q-1)^2}\cdot\sum_{j=0}^{\infty}|c_j|(j+1)r_1^j,$$

for all $n \in \mathbb{N}$ and $|z| \leq r$.

(ii) Denoting by γ the circle of radius $r^* > r$ and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v-z| \geq r^* - r$, by the Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$|L_{n,q}^{(p)}(f)(z) - f^{(p)}(z)| = \frac{p!}{2\pi}\left|\int_\gamma \frac{L_{n,q}(f)(v)-f(v)}{(v-z)^{p+1}}\mathrm{d}v\right|$$

$$\leq \frac{M_{r_1,q}(f)}{[n]_q}\frac{p!}{2\pi}\cdot\frac{2\pi r^*}{(r^*-r)^{p+1}} = \frac{M_{r_1,q}(f)}{[n]_q}\cdot\frac{p!r^*}{(r^*-r)^{p+1}},$$

which proves (ii) and the theorem. □

3. QUANTITATIVE VORONOVSKAJA-TYPE THEOREM

The following Voronovskaja-type result hold.

THEOREM 2. *For $R > q^4 > 1$ let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{D}_R$, and let $1 \leq r < \frac{r_1}{q^3} < \frac{R}{q^4}$ be arbitrary fixed. For all $n \in \mathbb{N}, |z| \leq r$ we have $\left|L_{n,q}(f)(z) - f(z) + \frac{P_q(f)(z)}{[n]_q}\right| \leq \frac{Q_{r_1,q}(f)}{[n]_q^2}$, where $P_q(f)(z) = \sum_{k=2}^{\infty} c_k \frac{[k]_q-k}{q-1} z^k = \sum_{k=2}^{\infty} c_k([1]_q + ... + [k-1]_q)z^k$, and $Q_{r_1,q}(f) = \frac{q^2-2q+2}{(q-1)^3}\cdot\sum_{k=0}^{\infty}|c_k|(k+1)(k+2)^2(r_1 q)^k < \infty$.*

Proof. We have

$$\left|L_{n,q}(f)(z) - f(z) + \frac{P_q(f)(z)}{[n]_q}\right|$$

$$= \left|\sum_{k=0}^{\infty} c_k\left[L_{n,q}(e_k)(z) - e_k(z) + \frac{[k]_q - k}{(q-1)[n]_q}e_k(z)\right]\right|$$

$$\leq \left|\sum_{k=0}^{n} c_k\left[L_{n,q}(e_k)(z) - e_k(z) + \frac{[k]_q - k}{(q-1)[n]_q}e_k(z)\right]\right|$$

$$+ \left|\sum_{k=n+1}^{\infty} c_k z^k\left(\frac{[k]-k}{(q-1)[n]_q} - 1\right)\right|$$

$$\leq \left|\sum_{k=0}^{n} c_k\left[L_{n,q}(e_k)(z) - e_k(z) + \frac{[k]_q - k}{(q-1)[n]_q}e_k(z)\right]\right|$$

$$+ \sum_{k=n+1}^{\infty}|c_k|r^k\left(\frac{[k]_q-k}{(q-1)[n]_q} - 1\right),$$

for all $|z| \le r$ and $n \in \mathbb{N}$.

In what follows, firstly we will prove by mathematical induction with respect to k that

$$0 \le E_{n,k,q}(z) \le \frac{r_1^2}{[n]_q^2}(k-1)(k-2)^2[k-2]_q, \tag{2}$$

for all $2 \le k \le n$ (here $n \in \mathbb{N}$ is arbitrary fixed) and $|z| \le r$, where

$$\begin{aligned} E_{n,k,q}(z) &= L_{n,q}(e_k)(z) - e_k(z) + \frac{[k]_q - k}{(q-1)[n]_q} e_k(z) \\ &= L_{n,q}(e_k)(z) - e_k(z) + \frac{1}{[n]_q}\left([1]_q + ... + [k-1]_q\right) e_k(z). \end{aligned}$$

Note that the relationship $\frac{[k]_q - k}{(q-1)[n]_q} e_k(z) = \frac{1}{[n]_q}\left([1]_q + ... + [k-1]_q\right) e_k(z)$, $k \ge 2$, easily follows by mathematical induction.

On the other hand, by the formula for $L_{n,q}(e_k)$ in the proof of Theorem 1 (i), simple calculation leads to $E_{n,2,q}(z) = 0$, for all $n \in \mathbb{N}$ and to the recurrence formulas

$$L_{n,q}(e_{j+1})(z) = -\frac{z^2}{[n]_q} D_q\left[L_{n,q}(e_j)\right](z) + zL_{n,q}(e_j)(z),\ j \ge 1, n \in \mathbb{N}, |z| \le r,$$

$$E_{n,k,q}(z) = -\frac{z^2}{[n]_q} D_q(L_{n,q}(e_{k-1})(z) - z^{k-1}) + zE_{n,k-1,q}(z),\ n \ge k \ge 3, |z| \le r.$$

Passing to absolute value above with $|z| \le r$ and $3 \le k \le n$ and applying the mean value theorem in complex analysis, with the general notation $\|f\|_r = \max\{|f(z)|; |z| \le r\}$, one obtains

$$\begin{aligned} |E_{n,k,q}(z)| &\le \frac{r^2}{[n]_q} \|(L_{n,q}(e_{k-1})(z) - z^{k-1})'\|_{qr} + r \cdot |E_{n,k-1,q}(z)| \\ &\le r \cdot |E_{n,k-1,q}(z)| + \frac{r^2}{[n]_q} \cdot \frac{k-1}{qr} \|L_{n,q}(e_{k-1})(z) - z^{k-1}\|_{qr} \\ &\le r \cdot |E_{n,k-1,q}(z)| + \frac{r^2}{[n]_q} \cdot \frac{k-1}{qr} \cdot (qr)^{k-1} \cdot \frac{(k-2)[k-2]_q}{[n]_q}, \end{aligned}$$

where above we used the estimate which easily follows from the proof of Theorem 1 (i): $|L_{n,q}(e_k)(z) - z^k| \le r^k \frac{(k-1)[k-1]_q}{[n]_q}$, $|z| \le r, k \ge 2$. Therefore, for all $|z| \le r$, $3 \le k \le n$, we get

$$\begin{aligned} |E_{n,k,q}(z)| &\le r \cdot |E_{n,k-1,q}(z)| + \frac{r^2}{[n]_q} \cdot \frac{k-1}{qr} \cdot (qr)^{k-1} \cdot \frac{(k-2)[k-2]_q}{[n]_q} \\ &\le r_1 \cdot |E_{n,k-1,q}(z)| + \frac{r_1^2}{[n]_q} \cdot (k-1) r_1^{k-2} \cdot \frac{(k-2)[k-2]_q}{[n]_q} \\ &= r_1 \cdot |E_{n,k-1,q}(z)| + \frac{(k-1)(k-2)[k-2]_q}{[n]_q^2} \cdot r_1^k. \end{aligned}$$

Taking $k = 3, 4, ...,$ step by step we easily obtain the estimate

$$|E_{n,k,q}(z)| \leq \frac{r_1^k}{[n]_q^2}(\sum_{j=3}^{k}(j-1)(j-2)[j-2]_q) \leq \frac{r_1^k}{[n]_q^2}(k-1)(k-2)^2[k-2]_q$$
$$\leq \frac{(r_1 q)^k}{(q-1)[n]_q^2}(k-1)(k-2)^2,$$

for all $|z| \leq r$ and $3 \leq k$, because $[k-2]_q \leq \frac{q^k}{q-1}$. In conclusion, (2) is valid, which implies

$$\left|\sum_{k=0}^{n} c_k \left[L_{n,q}(e_k)(z) - e_k(z) + \frac{[k]_q - k}{(q-1)[n]_q}e_k(z)\right]\right| \leq \sum_{k=0}^{n}|c_k| \cdot |E_{n,k,q}(z)|$$
$$\leq \frac{1}{(q-1)[n]_q^2}\sum_{k=3}^{n}|c_k|(k-1)(k-2)^2(r_1 q)^k$$
$$\leq \frac{1}{(q-1)[n]_q^2}\sum_{k=0}^{\infty}|c_k|(k+1)(k+2)^2(r_1 q)^k.$$

On the other hand, since $\frac{[k]_q - k}{(q-1)[n]_q} - 1 \geq 0$ for all $k \geq n+1$, reasoning as in the proof of Theorem 1 (i), and keeping the notation for $R_{r_1}(f)$ there, we get

$$\sum_{k=n+1}^{\infty}|c_k| r^k \left(\frac{[k]_q - k}{(q-1)[n]_q} - 1\right) \leq \sum_{k=n+1}^{\infty}|c_k| r^k \cdot \frac{[k]_q}{(q-1)[n]_q}$$
$$\leq \frac{R_{r_1}(f)}{(q-1)[n]_q}\sum_{k=n+1}^{\infty}\frac{1}{r_1^k} \cdot r^k \cdot q^k$$
$$= \frac{R_{r_1}(f)}{(q-1)[n]_q}\sum_{k=n+1}^{\infty}\left[\left(\frac{r}{r_1}\right)^{1/3}\right]^k \cdot \left[\left(\frac{r}{r_1}\right)^{1/3}\right]^{2k} q^k$$
$$\leq \frac{R_{r_1}(f)}{(q-1)[n]_q} \cdot \left(\frac{r}{r_1}\right)^{(n+1)/3}\sum_{k=0}^{\infty}\left[\left(\frac{r}{r_1}\right)^{1/3}\right]^k$$
$$= \frac{R_{r_1}(f)}{(q-1)[n]_q} \cdot \left[\left(\frac{r}{r_1}\right)^{1/3}\right]^n \cdot \frac{r^{1/3}}{r_1^{1/3} - r^{1/3}} \leq \frac{R_{r_1}(f)}{(q-1)^3[n]_q^2}$$
$$\leq \frac{1}{(q-1)^3[n]_q^2}\sum_{k=0}^{\infty}|c_k|(k+1)(k+2)^2(r_1 q)^k,$$

where we used the inequalities, $[k]_q \leq kq^k$, $\frac{1}{q^n} \leq \frac{1}{(q-1)[n]_q}$, $\frac{r^{1/3}}{r_1^{1/3} - r^{1/3}} \leq \frac{1}{q-1}$, and where for $\rho = \left(\frac{r}{r_1}\right)^{1/3} \leq \frac{1}{q}$, we used the obvious inequality $\rho^{2k} \cdot q^k \leq 1$.

Collecting now all the estimates and taking into account that $\frac{1}{q-1}+\frac{1}{(q-1)^3}=\frac{q^2-2q+2}{(q-1)^3}$, we arrive at the desired estimate. □

4. EXACT APPROXIMATION ESTIMATES

The first main result of this section is the following.

THEOREM 3. *Let $R>q^4>1$, $f:\mathbb{D}_R\to\mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{D}_R$, and $1\le r<\frac{r_1}{q^3}<\frac{R}{q^4}$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then for all $n\in\mathbb{N}$ and $|z|\le r$ we have $\|L_{n,q}(f)-f\|_r\ge\frac{C_{r,r_1,q}(f)}{[n]_q}$, where the constant $C_{r,r_1,q}(f)$ depends only on f, r and r_1. Here $\|f\|_r$ denotes* $\max_{|z|\le r}\{|f(z)|\}$.

Proof. For $P_q(f)(z)$ defined in the statement of Theorem 2, all $|z|\le r$ and $n\in\mathbb{N}$ we have

$$L_{n,q}(f)(z)-f(z)$$
$$=\frac{1}{[n]_q}\left\{-P_q(f)(z)+\frac{1}{[n]_q}\left[[n]_q^2\left(L_{n,q}(f)(z)-f(z)+\frac{P_q(f)(z)}{[n]_q}\right)\right]\right\}.$$

In what follows we will apply to this identity the following obvious property:

$$\|F+G\|_r\ge|\,\|F\|_r-\|G\|_r\,|\ge\|F\|_r-\|G\|_r.$$

It follows

$$\|L_{n,q}(f)-f\|_r\ge\frac{1}{[n]_q}\left\{\|P_q(f)\|_r-\frac{1}{[n]_q}\left[[n]_q^2\left\|L_{n,q}(f)-f+\frac{P_q(f)}{[n]_q}\right\|_r\right]\right\}.$$

Since by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, we get $\|P_q(f)\|_r>0$.

Indeed, supposing the contrary it follows that $P_q(f)(z)=0$ for all $z\in\overline{\mathbb{D}}_r=\{z\in\mathbb{C};|z|\le r\}$.

Since simple calculation shows that $P_q(f)(z)=z\cdot\frac{D_q(f)(z)-f'(z)}{q-1}$, $P_q(f)(z)=0$ implies $D_q(f)(z)=f'(z)$, for all $z\in\overline{\mathbb{D}}_r\setminus\{0\}$. Taking into account the representation of f as $f(z)=\sum_{k=0}^{\infty}c_kz^k$, the last equality immediately leads to $c_k=0$, for all $k\ge 2$, which means that f is linear in $\overline{\mathbb{D}}_r$, a contradiction with the hypothesis.

Now, by Theorem 2 we have $[n]_q^2\left\|L_{n,q}(f)-f+\frac{P_q(f)}{[n]_q}\right\|_r\le Q_{r_1,q}(f)$, where $Q_{r_1,q}(f)$ is a positive constant depending only on f, r_1 and q.

Since $\frac{1}{[n]_q}\to 0$ as $n\to\infty$, there exists an index n_0 depending only on f, r, r_1 and q, such that for all $n>n_0$ we have

$$\|P_q(f)\|_r-\frac{1}{[n]_q}\left[[n]_q^2\left\|L_{n,q}(f)-f+\frac{P_q(f)}{[n]_q}\right\|_r\right]\ge\frac{1}{2}\left\|\frac{P_q(f)}{2}\right\|_r,$$

which immediately implies that

$$\|L_{n,q}(f)-f\|_r \geq \frac{1}{[n]_q}\cdot\frac{1}{2}\left\|P_q(f)\right\|_r, \forall n>n_0.$$

For $n\in\{1,...,n_0\}$ we obviously have $\|L_{n,q}(f)-f\|_r \geq \frac{M_{r,r_1,n,q}(f)}{[n]_q}$ with $M_{r,r_1,n,q}(f)=[n]_q\cdot\|L_{n,q}(f)-f\|_r>0$ (if $\|L_{n,q}(f)-f\|_r$ would be equal to 0, this would imply that f is a linear function, a contradiction).

Therefore, finally we get $\|L_{n,q}(f)-f\|_r \geq \frac{C_{r,r_1,q}(f)}{n}$ for all $n\in\mathbb{N}$, where

$$C_{r,r_1,q}(f)=\min\left\{M_{r,r_1,1,q}(f),...,M_{r,r_1,n_0,q}(f),\frac{1}{2}\left\|P_q(f)\right\|_r\right\},$$

which completes the proof. □

Combining now Theorem 3 with Theorem 1 (i), we immediately get the following.

Corollary 4. *Let $R>q^4>1$, $f:\mathbb{D}_R\to\mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{D}_R$, and $1\leq r<\frac{r_1}{q^3}<\frac{R}{q^4}$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then for all $n\in\mathbb{N}$ we have $\|L_{n,q}(f)-f\|_r\sim\frac{1}{[n]_q}$, where the constants in the equivalence depend on f, r, r_1 and q but are independent of n.*

Concerning the simultaneous approximation we present the following.

Theorem 5. *Let $R>q^4>1$, $f:\mathbb{D}_R\to\mathbb{C}$ be analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in\mathbb{D}_R$, and $1\leq r<r^*<\frac{r_1}{q^3}<\frac{R}{q^4}$ be arbitrary fixed. Also, let $p\in\mathbb{N}$. If f is not a polynomial of degree $\leq\max\{1,p-1\}$, then for all $n\in\mathbb{N}$ we have $\|L^{(p)}_{n,q}(f)-f^{(p)}\|_r\sim\frac{1}{[n]_q}$, where the constants in the equivalence depend on f, r, r^*, r_1, p and q but are independent of n.*

Proof. Since by Theorem 1 (ii), we have the upper estimate for $\|L^{(p)}_{n,q}(f)-f^{(p)}\|_r$, it remains to prove the lower estimate for $\|L^{(p)}_{n,q}(f)-f^{(p)}\|_r$.

For this purpose, denoting by Γ the circle of radius r^* and center 0, we have the inequality $|v-z|\geq r^*-r$ valid for all $|z|\leq r$ and $v\in\Gamma$. The Cauchy's formula is expressed by

$$L^{(p)}_{n,q}(f)(z)-f^{(p)}(z)=\frac{p!}{2\pi i}\int_{\Gamma}\frac{L_{n,q}(f)(v)-f(v)}{(v-z)^{p+1}}\mathrm{d}v.$$

Now, as in the proof of Theorem 1 (ii), for all $v\in\Gamma$ and $n\in\mathbb{N}$ we have

$$\begin{aligned}&L_{n,q}(f)(v)-f(v)\\&=\frac{1}{[n]_q}\left\{-P_q(f)(v)+\frac{1}{[n]_q}\left[[n]_q^2\left(L_{n,q}(f)(v)-f(v)+\frac{P_q(f)(v)}{[n]_q}\right)\right]\right\},\end{aligned}$$

which replaced in the above Cauchy's formula implies

$$L_{n,q}^{(p)}(f)(z)-f^{(p)}(z)=\frac{1}{[n]_q}\left\{\frac{p!}{2\pi i}\int_\Gamma -\frac{P_q(f)(v)}{(v-z)^{p+1}}\mathrm{d}v\right.$$

$$\left.+\frac{1}{[n]_q}\cdot\frac{p!}{2\pi i}\int_\Gamma\frac{[n]_q^2\left(L_{n,q}(f)(v)-f(v)+\frac{P_q(f)(v)}{[n]_q}\right)}{(v-z)^{p+1}}\mathrm{d}v\right\}$$

$$=\frac{1}{[n]_q}\left\{[-P_q(f)(z)]^{(p)}+\frac{1}{[n]_q}\cdot\frac{p!}{2\pi i}\int_\Gamma\frac{[n]_q^2\left(L_{n,q}(f)(v)-f(v)+\frac{P_q(f)(v)}{[n]_q}\right)}{(v-z)^{p+1}}\mathrm{d}v\right\}.$$

Passing now to $\|\cdot\|_r$, for all $n\in\mathbb{N}$ it follows

$$\|L_{n,q}^{(p)}(f)-f^{(p)}\|_r\geq\frac{1}{[n]_q}\left\{\left\|[-P_q(f)]^{(p)}\right\|_r\right.$$

$$\left.-\frac{1}{[n]_q}\left\|\frac{p!}{2\pi}\int_\Gamma\frac{[n]_q^2\left(L_{n,q}(f)(v)-f(v)+\frac{P_q(f)(v)}{[n]_q}\right)}{(v-z)^{p+1}}\mathrm{d}v\right\|_r\right\},$$

where by using Theorem 2, for all $n\in\mathbb{N}$ we get

$$\left\|\frac{p!}{2\pi}\int_\Gamma\frac{[n]_q^2\left(L_{n,q}(f)(v)-f(v)+\frac{P_q(f)(v)}{[n]_q}\right)}{(v-z)^{p+1}}\mathrm{d}v\right\|_r$$

$$\leq\frac{p!}{2\pi}\cdot\frac{2\pi r^*[n]_q^2}{(r^*-r)^{p+1}}\left\|L_{n,q}(f)-f+\frac{P_q(f)}{[n]_q}\right\|_{r^*}\leq Q_{r_1,q}(f)\cdot\frac{p!r^*}{(r^*-r)^{p+1}}.$$

But by hypothesis on f, we have $\left\|-[P_q(f)]^{(p)}\right\|_{r^*}>0$. Indeed, supposing the contrary would follow that $[P_q(f)]^{(p)}(z)=0$, for all $|z|\leq r^*$, where by the statement of Theorem 2 we have

$$P_q(f)(z)=\sum_{k=2}^{\infty}c_k([1]_q+[2]_q+...+[k-1]_q)z^k.$$

Firstly, supposing that $p=1$, by $P_q'(f)(z)=\sum_{k=2}^{\infty}c_kk([1]_q+[2]_q+...+[k-1]_q)z^{k-1}=0$, for all $|z|\leq r^*$, would follow that $c_k=0$, for all $k\geq 2$, that is f would be a polynomial of degree $1=\max\{1,p-1\}$, a contradiction with the hypothesis.

Taking $p=2$, we would get $P_q''(z)=\sum_{k=2}^{\infty}c_kk(k-1)([1]_q+[2]_q+...+[k-1]_q)z^{k-2}=0$, for all $|z|\leq r^*$, which immediately would imply that $c_k=0$, for all $k\geq 2$, that is f would be a polynomial of degree $1=\max\{1,p-1\}$, a contradiction with the hypothesis.

Now, taking $p > 2$, for all $|z| \le r^*$ we would get

$$P_q^{(p)}(f)(z) = \sum_{k=p}^{\infty} c_k k(k-1)...(k-p+1)([1]_q + [2]_q + ... + [k-1]_q) z^{k-p} = 0,$$

which would imply $c_k = 0$ for all $k \ge p$, that is f would be a polynomial of degree $p - 1 = \max\{1, p-1\}$, a contradiction with the hypothesis.

In continuation, reasoning exactly as in the proof of Theorem 3, we immediately get the desired conclusion. $\square$

REMARK 6. Taking into account that for $q > 1$ we have the inequalities $(q-1) \cdot \frac{1}{q^n} \le \frac{1}{[n]_q} \le q \cdot \frac{1}{q^n}$, for all $n \in \mathbb{N}$, it follows that the exact order of approximation in Corollary 4 and Theorem 5 is q^{-n}, which is essentially better than the order of approximation $1/n$, obtained in the case $q = 1$, that is for $L_{n,1}(f)(z) := L_n(f)(z)$, in [2].

5. APPROXIMATION BY ITERATES

For f analytic in $\mathbb{D}_R$ that is of the form $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, let us define the iterates of complex Lorentz polynomial $L_{n,q}(f)(z)$, by $L_{n,q}^{(1)}(f)(z) = L_{n,q}(f)(z)$ and $L_{n,q}^{(m)}(f)(z) = L_{n,q}[L_{n,q}^{(m-1)}(f)](z)$, for any $m \in \mathbb{N}$, $m \ge 2$. Since we have $L_{n,q}(f)(z) = \sum_{k=0}^{\infty} c_k L_{n,q}(e_k)(z)$, by recurrence for all $m \ge 1$, we easily get that $L_{n,q}^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k L_{n,q}^{(m)}(e_k)(z)$, where $L_{n,q}^{(m)}(e_k)(z) = 1$ if $k = 0$, $L_{n,q}^{(m)}(e_k)(z) = z$ if $k = 1$, $L_{n,q}^{(m)}(e_k)(z) = 0$ if $k \ge n+1$ and

$$L_{n,q}^{(m)}(e_k)(z) = \left(1 - \frac{[1]_q}{[n]_q}\right)^m \left(1 - \frac{[2]_q}{[n]_q}\right)^m ... \left(1 - \frac{[k-1]_q}{[n]_q}\right)^m z^k,\ 2 \le k \le n.$$

The main result of this section is the following.

THEOREM 7. *Let f be analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, with $R > q > 1$. Let $1 \le r < \frac{r_1}{q} < \frac{R}{q}$. We have*

$$\|L_{n,q}^{(m)}(f) - f\|_r \le \frac{m}{[n]_q} \cdot \frac{q+1}{(q-1)^2} \sum_{k=0}^{\infty} |c_k|(k+1) r_1^k,$$

and therefore if $\lim_{n\to\infty} \frac{m_n}{[n]_q} = 0$, then $\lim_{n\to\infty} \|L_{n,q}^{(m_n)}(f) - f\|_r = 0$.

Proof. For all $|z| \le r$, we easily obtain

$$\begin{aligned} &|f(z) - L_{n,q}^{(m)}(f)(z)| \\ &\le \sum_{k=2}^{n} |c_k| r^k \left[1 - \left(1 - \frac{[1]_q}{[n]_q}\right)^m \left(1 - \frac{[2]}{[n]_q}\right)^m ... \left(1 - \frac{[k-1]_q}{[n]_q}\right)^m\right] \\ &+ \sum_{k=n+1}^{\infty} |c_k| \cdot r^k. \end{aligned}$$

Denoting $A_{k,n}=\left(1-\frac{[1]_q}{[n]_q}\right)\left(1-\frac{[2]}{[n]_q}\right)...\left(1-\frac{[k-1]_q}{[n]_q}\right)$, we get $1-A_{k,n}^m=(1-A_{k,n})(1+A_{k,n}+A_{k,n}^2+...+A_{k,n}^{m-1})\leq m(1-A_{k,n})$ and therefore since $1-A_{k,n}\leq\frac{(k-1)[k-1]_q}{[n]_q}$, for all $|z|\leq r$ we obtain

$$\sum_{k=2}^{n}|c_k|r^k\left[1-\left(1-\frac{[1]_q}{[n]_q}\right)^m\left(1-\frac{[2]_q}{[n]_q}\right)^m...\left(1-\frac{[k-1]_q}{[n]_q}\right)^m\right]$$
$$\leq m\sum_{k=2}^{\infty}|c_k|r^k[1-A_{k,n}]\leq\frac{m}{[n]_q}\sum_{k=2}^{\infty}|c_k|(k-1)[k-1]_qr^k$$
$$\leq\frac{m}{[n]_q}\sum_{k=2}^{\infty}|c_k|\cdot\frac{kq^k}{q-1}\cdot r^k\leq\frac{m}{[n]_q}\cdot\frac{1}{q-1}\sum_{k=2}^{\infty}|c_k|(k+1)(rq)^k$$
$$\leq\frac{m}{[n]_q}\cdot\frac{1}{q-1}\sum_{k=2}^{\infty}|c_k|(k+1)r_1^k.$$

On the other hand, following exactly the reasonings in the proof of Theorem 1, we get the estimate

$$\sum_{k=n+1}^{\infty}|c_k|\cdot r^k\leq\frac{1}{[n]_q}\cdot\frac{2\sum_{k=0}^{\infty}|c_k|(k+1)r_1^k}{(q-1)^2}\leq\frac{m}{[n]_q}\cdot\frac{2\sum_{k=0}^{\infty}|c_k|(k+1)r_1^k}{(q-1)^2}.$$

Collecting now all the estimates and taking into account that $\frac{1}{q-1}+\frac{2}{(q-1)^2}=\frac{q+1}{(q-1)^2}$, we arrive at the desired estimate. □

Remark 8. Taking into account the equivalence $\frac{1}{[n]_q}\sim\frac{1}{q^n}$, from Theorem 7 it follows the conclusion that if $\lim_{n\to\infty}\frac{m_n}{q^n}=0$, then

$$\lim_{n\to\infty}\|L_{n,q}^{(m_n)}(f)-f\|_r=0.$$

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