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STABILITY OF AQ-FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN $\mathcal{L}$-FUZZY NORMED SPACES

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Abstract. In this paper we prove the generalized Hyers-Ulam stability of the mixed type additive and quadratic functional equation

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+2f(3x)-2f(x)$$

in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

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Key words. Stability, quadratic functional equation, non-Archimedean $\mathcal{L}$-fuzzy normed space.

1. INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [53] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [37]. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Th. M. Rassias [50] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias has provided a lot of influence in the development of what we now call a generalized Hyers-Ulam stability of functional equations. We refer the interested readers for more information on such problems to the papers [4, 36, 39, 49].

In 1991, Z. Gajda [23] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [19, 24, 38]). On the other hand, J.M. Rassias [43]–[48] considered the Cauchy difference controlled by a product of different powers of norm. This stability phenomenon is called the Ulam-Găvruta-Rassias stability (see also [23]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is related to symmetric bi-additive function (see [1, 40]). Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1) between Banach spaces was proved by Skof (see [3, 4, 34, 51]).

A triangular norm (shortly, t-norm) is a binary operation $T : [0,1]\times[0,1] \to [0,1]$ which is commutative, associative, monotone and has 1 as the unit element. A t-norm T can be extended (by associativity) in a unique way to an

n-ary operation taking, for all $(x_1, ..., x_n) \in [0,1]^n$, the value $T(x_1, ..., x_n)$ defined by $T_{i=1}^0 x_i = 1$, $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, ..., x_n)$. A t-norm T can also be extended to a countable operation taking, for any sequence $\{x_n\}_{n\in N}$ in $[0,1]$, the value $T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^n x_i$. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a nonempty set called the universe. An $\mathcal{L}$-fuzzy set in U is defined as a mapping $A : U \to L$. For each u in U, $A(u)$ represents the degree (in L) to which u is an element of U.

Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : \ (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1, x_2 \geq y_2,$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice (see [5]).

A triangular norm (t-norm) on L is a mapping $T : L^2 \to L$ satisfying the following conditions:

(1) $T(x, 1_L) = x$, for all $x \in L$; (boundary condition).

(2) $T(x, y) = T(y, x)$, for all $(x, y) \in L^2$; (commutativity).

(3) $T(x, T(y, z)) = T(T(x, y), z)$, for all $(x, y, z) \in L^3$; (associativity).

(4) $x \leq_L x', y \leq_L y' \Longrightarrow T(x, y) \leq_L T(x', y')$, for all $(x, x', y, y') \in L^4$; (monotonicity).

A t-norm T on $\mathcal{L}$ is said to be continuous if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$, $\{y_n\}$ which converge to x and y, respectively, $\lim_{n\to\infty} T(x_n, y_n) = T(x, y)$. A t-norm T can also be defined recursively as an $(n+1)$-ary operation ($n \in N$) by $T^1 = T$ and $T^n(x_1, ..., x_{n+1}) = T(T^{n-1}(x_1, ..., x_n), x_{n+1})$, for all $n \geq 2$ and $x_i \in L$.

(1) A negator on $\mathcal{L}$ is any decreasing mapping $N : L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$.

(2) If $N(N(x)) = x$, for all $x \in L$, then N is called an involutive negator.

(3) The negator N_s on $([0,1], \leq)$ defined as $N_s(x) = 1 - x$, for all $x \in [0,1]$, is called the standard negator on $([0,1], \leq)$.

Definition 1. The triple (X, M, T) is said to be an $\mathcal{L}$-fuzzy metric space if X is an arbitrary (non-empty) set, T is a continuous t-norm on L and M is an $\mathcal{L}$-fuzzy set on $X^2 \times]0, +\infty[$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s \in]0, +\infty[$,

(1) $M(x, y, t) >_L 0_L$;

(2) $M(x, y, t) = 1_L$, for all $t > 0$ if and only if $x = y$;

(3) $M(x, y, t) = M(y, x, t)$;

(4) $T(M(x, y, t), M(y, z, s)) \leq_L M(x, z, t + s)$;

(5) $M(x, y, .) :]0, +\infty[\to L$ is continuous.

In this case, M is called an $\mathcal{L}$-fuzzy metric.

Definition 2. The triple (V, P, T) is said to be an $\mathcal{L}$-fuzzy normed space if V is a vector space, T is a continuous t-norm on L and P is an $\mathcal{L}$-fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,

(1) $P(x,t) >_L 0_L$;
(2) $P(x,t) = 1_L$ if and only if $x = 0$;
(3) $P(\alpha x, t) = P(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(4) $T(P(x,t), P(y,s)) \leq_L P(x+y, t+s)$;
(5) $P(x,.) :]0, +\infty[\to L$ is continuous.
(6) $\lim_{t\to 0} P(x,t) = 0_L$ and $\lim_{t\to\infty} P(x,t) = 1_L$.

In this case, P is called an $\mathcal{L}$-fuzzy norm.

A sequence $\{x_n\}_{n\in N}$ in an $\mathcal{L}$-fuzzy normed space (V, P, T) is called a Cauchy sequence if, for each $\epsilon \in L\backslash\{0_L\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $P(x_n - x_m, t) >_L N(\epsilon)$, where N is a negator on $\mathcal{L}$.

A sequence $\{x_n\}_{n\in N}$ is said to be convergent to $x \in V$ in the $\mathcal{L}$-fuzzy normed space (V, P, T), which is denoted by $x_n \to x$ if $P(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$, for all $t > 0$.

An $\mathcal{L}$- fuzzy normed space (V, P, T) is said to be complete if and only if every Cauchy sequence in V is convergent.

Note that, if P is an $\mathcal{L}$-fuzzy norm on V, then the following are satisfied:

(1) $P(x,t)$ is nondecreasing with respect to t, for all $x \in V$.

(2) $P(x-y, t) = P(y-x, t)$, for all $x, y \in V$ and $t \in]0, +\infty[$.

Let (V, P, T) be an $\mathcal{L}$-fuzzy normed space. If we define $M(x,y,t) = P(x-y, t)$, for all $x, y \in V$ and $t \in]0, +\infty[$, then M is an $\mathcal{L}$-fuzzy metric on V, which is called the $\mathcal{L}$-fuzzy metric induced by the $\mathcal{L}$-fuzzy norm P.

In 1897, Hensel [35] introduced a field with a valuation in which does not have the Archimedean property. Let K be a field. A non-Archimedean absolute value on K is a function $|.| : K \to [0, +\infty[$ such that, for any $a, b \in K$,

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(2) $|ab| = |a||b|$,
(3) $|a + b| \leq \max\{|a|, |b|\}$ (the strict triangle inequality).

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|.|$ is non-trivial, i.e., there exists an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

Definition 3. A non-Archimedean $\mathcal{L}$-fuzzy normed space is a triple (V, P, T), where V is a vector space, T is a continuous t-norm on L and P is an $\mathcal{L}$-fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,

(1) $0_L <_L P(x,t)$;
(2) $P(x,t) = 1_L$ if and only if $x = 0$;
(3) $P(\alpha x, t) = P(x, \frac{t}{|\alpha|})$, for all $\alpha \neq 0$;
(4) $T(P(x,t), P(y,s)) \leq_L P(x+y, \max\{t,s\})$;
(5) $P(x,.) :]0, \infty[\to L$ is continuous;
(6) $\lim_{t\to 0} P(x,t) = 0_L$ and $\lim_{t\to\infty} P(x,t) = 1_L$.

Recently, S. Shakeri, R. Saadati and C. Park in [52], proved the generalized Hyers-Ulam stability of functional equation (1) in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

In this paper we deal with the following mixed type additive-quadratic functional equation (briefly AQ-functional equation):

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+2f(3x)-2f(x)$$

and prove the generalized Hyers-Ulam stability in non-Archimedean $\mathcal{L}$-fuzzy normed spaces. The stability problems of several mixed type functional equations have been extensively investigated by a number of authors and there are many interesting results concerning them (see [6]–[20], [26]–[33], [41, 42]).

2. GENERALIZED $\mathcal{L}$-FUZZY HYERS-ULAM STABILITY

Throughout this paper, assume that Ψ is an $\mathcal{L}$-fuzzy set on $X\times X\times[0,\infty)$ such that $\Psi(x,y,.)$ is nondecreasing,

$$\Psi(cx,cx,t)\geq_L \Psi\left(x,x,\frac{t}{|c|}\right),\quad \forall x\in X,\ c\neq 0$$

and

$$\lim_{t\to\infty}\Psi(x,y,t)=1_{\mathcal{L}},\quad \forall x,y\in X,\ t>0.$$

THEOREM 4. *Let K be a non-Archimedean field, X a vector space over K and (Y,P,T) a non-Archimedean $\mathcal{L}$-fuzzy Banach space over K. Suppose that $f:X\to Y$ is an odd mapping satisfying*

$$P(f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x),t)\geq_L \Psi(x,y,t), \tag{2}$$

for all $x,y\in X$ and $t>0$. If there exist an $\alpha\in\mathbb{R}$ and an integer k, $k\geq 2$ with $|2^k|<\alpha$ and $|2|\neq 0$ such that

$$\Psi(2^{-k}x,2^{-k}y,t)\geq_L \Psi(x,y,\alpha t),\quad \forall x\in X,\ t>0, \tag{3}$$

$$\lim_{n\to\infty}T_{j=n}^{\infty}M\left(x,\frac{\alpha^j t}{|2|^{kj}}\right)=1_{\mathcal{L}},\quad \forall x\in X,\ t>0,$$

then there exists a unique additive mapping $A:X\to Y$ such that

$$P(f(x)-A(x),t)\geq T_{i=0}^{\infty}M\left(x,\frac{\alpha^{i+1}t}{|2|^{ki}}\right),\quad \forall x\in X,\ t>0, \tag{4}$$

where

$$M(x,t):=T\Big(T\Big(T\Big(\Psi\Big(\frac{x}{4},\frac{x}{4},t\Big),\Psi\Big(\frac{x}{4},\frac{3x}{4},t\Big)\Big),T\Big(\Psi\Big(\frac{x}{4},\frac{x}{4},t\Big),\Psi\Big(\frac{x}{4},\frac{5x}{4},t\Big)\Big)\Big),$$
$$T\Big(T\Big(\Psi\Big(\frac{2x}{4},\frac{2x}{4},t\Big),\Psi\Big(\frac{2x}{4},\frac{3.2x}{4},t\Big)\Big),T\Big(\Psi\Big(\frac{2x}{4},\frac{2x}{4},t\Big),\Psi\Big(\frac{2x}{4},\frac{5.2x}{4},t\Big)\Big)\Big),...,$$
$$T\Big(T\Big(\Psi\Big(\frac{2^{j-1}x}{4},\frac{2^{j-1}x}{4},t\Big),\Psi\Big(\frac{2^{j-1}x}{4},\frac{3.2^{j-1}x}{4},t\Big)\Big),T\Big(\Psi\Big(\frac{2^{j-1}x}{4},\frac{2^{j-1}x}{4},t\Big),$$
$$\Psi\Big(\frac{2^{j-1}x}{4},\frac{5.2^{j-1}x}{4},t\Big)\Big)\Big)\Big)\Big),$$

for all $x\in X$, $t>0$.

Proof. We show by induction on j that, for all $x \in X$, $t > 0$, $j \geq 1$, we have

$$\begin{aligned}(5)\quad & P(f(2^j x) - 2^j f(x), t) \geq_L M_j(x,t) \\ & := T\Big(T\Big(T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{3x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{5x}{4}, t\Big)\Big)\Big), \\ & \ldots, T\Big(T\Big(\Psi\Big(\frac{2^{j-1}x}{4}, \frac{2^{j-1}x}{4}, t\Big), \Psi\Big(\frac{2^{j-1}x}{4}, \frac{3.2^{j-1}x}{4}, t\Big)\Big), \\ & T\Big(\Psi\Big(\frac{2^{j-1}x}{4}, \frac{2^{j-1}x}{4}, t\Big), \Psi\Big(\frac{2^{j-1}x}{4}, \frac{5.2^{j-1}x}{4}, t\Big)\Big)\Big)\Big).\end{aligned}$$

Putting $y = x$ in (2), we obtain

$$(6)\qquad P(f(4x) - 2f(3x) + 2f(x), t) \geq_L \Psi(x, x, t),$$

for all $x \in X$ and $t > 0$. If we let $y = 3x$ in (2), we get by the oddness of f,

$$(7)\qquad P(f(6x) - 2f(3x) - f(4x) + 2f(x) + f(2x), t) \geq_L \Psi(x, 3x, t),$$

for all $x \in X$ and $t > 0$. It follows from (6) and (7) that

$$\begin{aligned}(8)\quad & P(f(6x) - 2f(4x) + f(2x), t) \\ & \geq_L T(P(f(4x) - 2f(3x) + 2f(x), t), P(f(6x) - 2f(3x) - f(4x) + 2f(x) \\ & + f(2x), t)) \geq_L T(\Psi(x, x, t), \Psi(x, 3x, t)),\end{aligned}$$

for all $x \in X$ and $t > 0$. Once again, by letting $y = 5x$ in (2), we get by the oddness of f,

$$(9)\qquad P(f(8x) - f(2x) - f(6x) + f(4x) - 2f(3x) + 2f(x), t) \geq_L \Psi(x, 5x, t),$$

for all $x \in X$ and $t > 0$. By (6) and (9), we get

$$(10)\qquad P(f(8x) - f(6x) - f(2x), t) \geq_L T(\Psi(x, x, t), \Psi(x, 5x, t)),$$

for all $x \in X$ and $t > 0$. By (8) and (10), we obtain

$$P(f(8x) - 2f(4x), t) \geq_L T(T(\Psi(x, x, t), \Psi(x, 3x, t)), T(\Psi(x, x, t), \Psi(x, 5x, t))),$$

for all $x \in X$ and $t > 0$. If we replace x by $\frac{x}{4}$, we get

$$\begin{aligned}(11)\quad & P(f(2x) - 2f(x), t) \\ & \geq_L T\Big(T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{3x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{5x}{4}, t\Big)\Big)\Big),\end{aligned}$$

for all $x \in X$ and $t > 0$. This proves (5) for $j = 1$.

Let (5) hold for some $j > 1$. Replacing x by $2^j x$ in (2.11), we obtain

$$\begin{aligned}P(f(2^{j+1}x) - 2f(2^j x), t) \geq_L & T\Big(T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{3.2^j x}{4}, t\Big)\Big), \\ & T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{5.2^j x}{4}, t\Big)\Big)\Big),\end{aligned}$$

for all $x \in X$ and $t > 0$. Since $|2| < 1$, it follows that

$$\begin{aligned}
&P(f(2^{j+1}x) - 2^{j+1}f(x), t) \\
&\geq_L T(P(f(2^{j+1}x) - 2f(2^j x), t), P(2f(2^j x) - 2^{j+1}f(x), t)) \\
&= T\Big(P(f(2^{j+1}x) - 2f(2^j x), t), P\Big(f(2^j x) - 2^j f(x), \frac{t}{|2|}\Big)\Big) \\
&\geq_L T(P(f(2^{j+1}x) - 2f(2^j x), t), P(f(2^j x) - 2^j f(x), t)) \\
&\geq_L T\Big(T\Big(T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{3.2^j x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big), \\
&\Psi\Big(\frac{2^j x}{4}, \frac{5.2^j x}{4}, t\Big)\Big)\Big), M_j(x, t)) = M_{j+1}(x, t),
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus (5) holds for all $j \geq 1$. In particular, we have

$$P(f(2^k x) - 2^k f(x), t) \geq_L M(x, t), \tag{12}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^{-(kn+k)}x$ in (12) and using the inequality (3), we obtain

$$P\Big(f\Big(\frac{x}{2^{kn}}\Big) - 2^k f\Big(\frac{x}{2^{kn+k}}\Big), t\Big) \geq_L M\Big(\frac{x}{2^{kn+k}}, t\Big) \geq_L M(x, \alpha^{n+1}t),$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Thus we have

$$P\Big((2^k)^n f\Big(\frac{x}{(2^k)^n}\Big) - (2^k)^{n+1} f\Big(\frac{x}{(2^k)^{n+1}}\Big), t\Big) \geq_L M\Big(x, \frac{\alpha^{n+1}}{|(2^k)^n|}t\Big),$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Hence it follows that

$$\begin{aligned}
&P\Big((2^k)^n f\Big(\frac{x}{(2^k)^n}\Big) - (2^k)^{n+p} f\Big(\frac{x}{(2^k)^{n+p}}\Big), t\Big) \\
&\geq_L T_{j=n}^{n+p-1} P\Big((2^k)^j f\Big(\frac{x}{(2^k)^j}\Big) - (2^k)^{j+1} f\Big(\frac{x}{(2^k)^{j+1}}\Big), t\Big) \\
&\geq_L T_{j=n}^{n+p-1} M\Big(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\Big),
\end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Since $\lim_{n\to\infty} T_{j=n}^{\infty} M\Big(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\Big) = 1_{\mathcal{L}}$, for all $x \in X$ and $t > 0$, $\Big\{(2^k)^n f\Big(\frac{x}{(2^k)^n}\Big)\Big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space (Y, P, T). Hence we can define a mapping $A : X \to Y$ such that

$$\lim_{n\to\infty} P\Big((2^k)^n f\Big(\frac{x}{(2^k)^n}\Big) - A(x), t\Big) = 1_{\mathcal{L}}, \tag{13}$$

for all $x \in X$ and $t > 0$. Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned} &P\Big(f(x) - (2^k)^n f\Big(\frac{x}{(2^k)^n}\Big), t\Big) \\ &= P\Big(\sum_{i=0}^{n-1} \Big[(2^k)^i f\Big(\frac{x}{(2^k)^i}\Big) - (2^k)^{i+1} f\Big(\frac{x}{(2^k)^{i+1}}\Big)\Big], t\Big) \\ &\geq_L T_{i=0}^{n-1}\Big(P\Big((2^k)^i f\Big(\frac{x}{(2^k)^i}\Big) - (2^k)^{i+1} f\Big(\frac{x}{(2^k)^{i+1}}\Big), t\Big)\Big) \\ &\geq_L T_{i=0}^{n-1} M\Big(x, \frac{\alpha^{i+1}}{|2^k|^i} t\Big) \end{aligned}$$

and so

$$\begin{aligned} &P(f(x) - A(x), t) \\ &\geq_L T(P(f(x) - (2^k)^n f(\frac{x}{(2^k)^n}), t), P((2^k)^n f(\frac{x}{(2^k)^n}) - A(x), t)) \\ &\geq_L T(T_{i=0}^{n-1} M(x, \frac{\alpha^{i+1}}{|2^k|^i} t), P((2^k)^n f(\frac{x}{(2^k)^n}) - A(x), t)). \end{aligned} \tag{14}$$

Taking the limit as $n \to \infty$ in (14), $P(f(x) - A(x), t) \geq_L T_{i=0}^{\infty} M\Big(x, \frac{\alpha^{i+1} t}{|2^k|^i}\Big)$, which proves (4). Replacing x, y by $2^{-kn}x, 2^{-kn}y$ in (2) and (3), we get

$$\begin{aligned} &P\Big(2^{kn} f\Big(\frac{3x+y}{2^{kn}}\Big) + 2^{kn} f\Big(\frac{3x-y}{2^{kn}}\Big) - 2^{kn} f\Big(\frac{x+y}{2^{kn}}\Big) \\ &- 2^{kn} f\Big(\frac{x-y}{2^{kn}}\Big) - 2.2^{kn} f\Big(\frac{3x}{2^{kn}}\Big) + 2.2^{kn} f\Big(\frac{x}{2^{kn}}), t\Big) \\ &\geq_L \Psi\Big(2^{-kn}x, 2^{-kn}y, \frac{t}{|2^{kn}|}\Big) \geq_L \Psi\Big(x, y, \frac{\alpha^n t}{|2^{kn}|}\Big), \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n \to \infty} \Psi(x, y, \frac{\alpha^n t}{|2^k|^n}) = 1_{\mathcal{L}}$, we infer that A is an additive mapping. For the uniqueness of A, let $A' : X \to Y$ be another additive mapping such that $P(A'(x) - f(x), t) \geq_L T_{i=0}^{\infty} M\Big(x, \frac{\alpha^{i+1} t}{|2^k|^i}\Big)$, for all $x \in X$ and $t > 0$. Then we have, for all $x \in X$ and $t > 0$,

$$\begin{aligned} &P(A(x) - A'(x), t) \\ &\geq_L T\Big(P\Big(A(x) - (2^k)^n f\Big(\frac{x}{(2^k)^n}\Big), t\Big), P\Big((2^k)^n f\Big(\frac{x}{(2^k)^n}\Big) - A'(x), t\Big)\Big). \end{aligned}$$

Therefore, from (14), we conclude that $A = A'$. This completes the proof. □

THEOREM 5. *Let K be a non-Archimedean field, X a vector space over K and (Y, P, T) a non-Archimedean $\mathcal{L}$-fuzzy Banach space over K. Suppose that $f : X \to Y$ is an even mapping satisfying*

$$\begin{aligned} &P(f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 2f(3x) + 2f(x), t) \\ &\geq_L \Psi(x, y, t), \end{aligned} \tag{15}$$

for all $x, y \in X$ *and* $t > 0$. *If there exist an* $\alpha \in \mathbb{R}$ *and an integer* k, $k \geq 2$ *with* $|2^k| < \alpha$ *and* $|2| \neq 0$ *such that*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x \in X,\ t > 0, \tag{16}$$

$$\lim_{n\to\infty} T_{j=n}^{\infty} N\Big(x, \frac{\alpha^j t}{|2|^{kj}}\Big) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

then there exists a unique quadratic mapping $Q : X \to Y$ *such that*

$$P(f(x) - Q(x), t) \geq T_{i=0}^{\infty} N\Big(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\Big), \quad \forall x \in X,\ t > 0, \tag{17}$$

where

$$N(x,t) := T\Big(T\Big(T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{x}{4}, \frac{5x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{-3x}{4}, t\Big)\Big)\Big),$$
$$T\Big(T\Big(\Psi\Big(\frac{2x}{4}, \frac{2x}{4}, t\Big), \Psi\Big(\frac{2x}{4}, \frac{2x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{2x}{4}, \frac{5.2x}{4}, t\Big), \Psi\Big(\frac{2x}{4}, \frac{-3.2x}{4}, t\Big)\Big)\Big), ...,$$
$$T\Big(T\Big(\Psi\Big(\frac{2^{j-1}x}{4}, \frac{2^{j-1}x}{4}, t\Big), \Psi\Big(\frac{2^{j-1}x}{4}, \frac{2^{j-1}x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{2^{j-1}x}{4}, \frac{5.2^{j-1}x}{4}, t\Big),$$
$$\Psi\Big(\frac{2^{j-1}x}{4}, \frac{-3.2^{j-1}x}{4}, t\Big)\Big)\Big)\Big),$$

for all $x \in X$, $t > 0$.

Proof. We show by induction on j that, for all $x \in X$, $t > 0$, $j \geq 1$, we have

$$\begin{aligned} &P(f(2^j x) - 2^{2j} f(x), t) \\ &\geq_L N_j(x,t) := T\Big(T\Big(T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{x}{4}, \frac{5x}{4}, t\Big), \\ &\Psi\Big(\frac{x}{4}, \frac{-3x}{4}, t\Big)\Big)\Big), \ldots, T\Big(T\Big(\Psi\Big(\frac{2^{j-1}x}{4}, \frac{2^{j-1}x}{4}, t\Big), \Psi\Big(\frac{2^{j-1}x}{4}, \frac{2^{j-1}x}{4}, t\Big)\Big), \\ &T\Big(\Psi\Big(\frac{2^{j-1}x}{4}, \frac{5.2^{j-1}x}{4}, t\Big), \Psi\Big(\frac{2^{j-1}x}{4}, \frac{-3.2^{j-1}x}{4}, t\Big)\Big)\Big)\Big). \end{aligned} \tag{18}$$

Replacing y by $x + y$ in (15) we get

$$\begin{aligned} &P(f(4x+y) + f(2x-y) - f(2x+y) - f(y) - 2f(3x) + 2f(x), t) \\ &\geq_L \Psi(x, x+y, t), \end{aligned} \tag{19}$$

for all $x, y \in X$ and $t > 0$. If we replace y by $-y$ in (19), we obtain

$$\begin{aligned} &P(f(4x-y) + f(2x+y) - f(2x-y) - f(y) - 2f(3x) + 2f(x), t) \\ &\geq_L \Psi(x, x-y, t), \end{aligned} \tag{20}$$

for all $x, y \in X$ and $t > 0$. By (19) and (20), we get

$$\begin{aligned} &P(f(4x+y) + f(4x-y) - 2f(y) - 4f(3x) + 4f(x), t) \\ &\geq_L T(\Psi(x, x+y, t), \Psi(x, x-y, t)). \end{aligned} \tag{21}$$

Letting $y = 0$ in (21), we get the inequality

$$P(2f(4x) - 4f(3x) + 4f(x), t) \geq_L T(\Psi(x, x, t), \Psi(x, x, t)), \tag{22}$$

for all $x \in X$ and $t > 0$. Once again, by letting $y = 4x$ in (21), we get

$$P(f(8x) - 2f(4x) - 4f(3x) + 4f(x), t) \geq_L T(\Psi(x, 5x, t), \Psi(x, -3x, t)), \tag{23}$$

for all $x \in X$ and $t > 0$. By (22) and (23), we get

$$\begin{aligned} &P(f(8x) - 4f(4x), t) \\ &\geq_L T(T(\Psi(x, x, t), \Psi(x, x, t)), T(\Psi(x, 5x, t), \Psi(x, -3x, t))), \end{aligned} \tag{24}$$

for all $x \in X$ and $t > 0$. If we replace x in (24) by $\frac{x}{4}$, we get

$$\begin{aligned} &P(f(2x) - 4f(x), t) \\ &\geq_L T\Big(T\Big(\Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{x}{4}, t\Big)\Big), T\Big(\Psi\Big(\frac{x}{4}, \frac{5x}{4}, t\Big), \Psi\Big(\frac{x}{4}, \frac{-3x}{4}, t\Big)\Big)\Big), \end{aligned} \tag{25}$$

for all $x \in X$ and $t > 0$. This proves (18) for $j = 1$.

Let (18) hold for some $j > 1$. Replacing x by $2^j x$ in (25), we obtain

$$\begin{aligned} &P(f(2^{j+1}x) - 4f(2^j x), t) \geq_L T\Big(T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big)\Big), \\ &T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{5.2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{-3.2^j x}{4}, t\Big)\Big)\Big), \end{aligned}$$

for all $x \in X$ and $t > 0$. Since $|2| < 1$, it follows that

$$\begin{aligned} &P(f(2^{j+1}x) - 2^{2(j+1)}f(x), t) \\ &\geq_L T(P(f(2^{j+1}x) - 4f(2^j x), t), P(4f(2^j x) - 2^{2(j+1)}f(x), t)) \\ &= T\Big(P(f(2^{j+1}x) - 4f(2^j x), t), P\Big(f(2^j x) - 2^{2j}f(x), \frac{t}{|4|}\Big)\Big) \\ &\geq_L T(P(f(2^{j+1}x) - 4f(2^j x), t), P(f(2^j x) - 2^{2j}f(x), t)) \\ &\geq_L T\Big(T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{2^j x}{4}, t\Big)\Big), \\ &T\Big(\Psi\Big(\frac{2^j x}{4}, \frac{5.2^j x}{4}, t\Big), \Psi\Big(\frac{2^j x}{4}, \frac{-3.2^j x}{4}, t\Big)\Big), N_j(x, t)) = N_{j+1}(x, t), \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus (18) holds for all $j \geq 1$. In particular, we have

$$P(f(2^k x) - 2^{2k}f(x), t) \geq_L N(x, t), \tag{26}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^{-(kn+k)}x$ in (26) and using the inequality (3), we obtain

$$P\Big(f\Big(\frac{x}{2^{kn}}\Big) - 2^{2k}f\Big(\frac{x}{2^{kn+k}}\Big), t\Big) \geq_L N\Big(\frac{x}{2^{kn+k}}, t\Big) \geq_L N(x, \alpha^{n+1}t),$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Thus we have

$$P\Big((2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big) - (2^{2k})^{n+1} f\Big(\frac{x}{(2^k)^{n+1}}\Big), t\Big) \geq_L N\Big(x, \frac{\alpha^{n+1}}{|(2^{2k})^n|}t\Big)$$
$$\geq_L N\Big(x, \frac{\alpha^{n+1}}{|(2^k)^n|}t\Big),$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Hence it follows that

$$\begin{aligned}
&P\Big((2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big) - (2^{2k})^{n+p} f\Big(\frac{x}{(2^k)^{n+p}}\Big), t\Big)\\
&\geq_L T_{j=n}^{n+p-1} P\Big((2^{2k})^j f\Big(\frac{x}{(2^k)^j}\Big) - (2^{2k})^{j+1} f\Big(\frac{x}{(2^k)^{j+1}}\Big), t\Big)\\
&\geq_L T_{j=n}^{n+p-1} N\Big(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\Big),
\end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Since $\lim_{n\to\infty} T_{j=n}^{\infty} N\Big(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\Big) = 1_{\mathcal{L}}$, for all $x \in X$ and $t > 0$, $\Big\{(2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big)\Big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space (Y, P, T). Hence we can define a mapping $Q : X \to Y$ such that

$$\lim_{n\to\infty} P\Big((2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big) - Q(x), t\Big) = 1_{\mathcal{L}}, \tag{27}$$

for all $x \in X$ and $t > 0$. Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned}
&P\Big(f(x) - (2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big), t\Big)\\
&= P\Big(\sum_{i=0}^{n-1} \Big[(2^{2k})^i f\Big(\frac{x}{(2^k)^i}\Big) - (2^{2k})^{i+1} f\Big(\frac{x}{(2^k)^{i+1}}\Big)\Big], t\Big)\\
&\geq_L T_{i=0}^{n-1}(P\Big((2^{2k})^i f\Big(\frac{x}{(2^k)^i}\Big) - (2^{2k})^{i+1} f\Big(\frac{x}{(2^k)^{i+1}}\Big), t\Big)\Big)\\
&\geq_L T_{i=0}^{n-1} N\Big(x, \frac{\alpha^{i+1}}{|2^k|^i}t\Big)
\end{aligned}$$

and so

$$\begin{aligned}
&P(f(x) - Q(x), t)\\
&\geq_L T\Big(P\Big(f(x) - (2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big), t\Big), P\Big((2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big) - Q(x), t\Big)\Big)\\
&\geq_L T\Big(T_{i=0}^{n-1} N\Big(x, \frac{\alpha^{i+1}}{|2^k|^i}t\Big), P\Big((2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big) - Q(x), t\Big)\Big).
\end{aligned} \tag{28}$$

Taking the limit as $n \to \infty$ in (28), $P(f(x) - Q(x), t) \geq_L T_{i=0}^{\infty} N\Big(x, \frac{\alpha^{i+1} t}{|2^k|^i}\Big)$, which proves (17). Replacing x, y by $2^{-kn}x, 2^{-kn}y$ in (15) and (16), we get

$$\begin{aligned}
&P\Big(2^{2kn} f\Big(\frac{3x+y}{2^{kn}}\Big) + 2^{2kn} f\Big(\frac{3x-y}{2^{kn}}\Big) - 2^{2kn} f\Big(\frac{x+y}{2^k n}\Big) \\
&- 2^{2kn} f\Big(\frac{x-y}{2^{kn}}\Big) - 2.2^{2kn} f\Big(\frac{3x}{2^{kn}}\Big) + 2.2^{2kn} f\Big(\frac{x}{2^{kn}}\Big), t\Big) \\
&\geq_L \Psi\Big(2^{-kn}x, 2^{-kn}y, \frac{t}{|2^{kn}|}\Big) \geq_L \Psi\Big(x, y, \frac{\alpha^n t}{|2^{kn}|}\Big),
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \Psi\Big(x, y, \frac{\alpha^n t}{|2^k|^n}\Big) = 1_{\mathcal{L}}$, we infer that Q is a quadratic mapping. For the uniqueness of Q, let $Q^{'} : X \to Y$ be another quadratic mapping such that $P(Q^{'}(x) - f(x), t) \geq_L T_{i=0}^{\infty} N\Big(x, \frac{\alpha^{i+1} t}{|2^k|^i}\Big)$, for all $x \in X$ and $t > 0$. Then we have, for all $x \in X$ and $t > 0$,

$$\begin{aligned}
&P(Q(x) - Q^{'}(x), t) \\
&\geq_L T\Big(P\Big(Q(x) - (2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big), t\Big), P\Big((2^{2k})^n f\Big(\frac{x}{(2^k)^n}\Big) - Q^{'}(x), t\Big)\Big).
\end{aligned}$$

Therefore, from (27), we conclude that $Q = Q^{'}$. This completes the proof. □

THEOREM 6. *Let K be a non-Archimedean field, X a vector space over K and (Y, P, T) a non-Archimedean $\mathcal{L}$-fuzzy Banach space over K. Suppose that $f : X \to Y$ is a mapping satisfying*

$$\begin{aligned}
&P(f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 2f(3x) + 2f(x), t) \\
&\geq_L \Psi(x, y, t),
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. If there exist an $\alpha \in \mathbb{R}$ and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x \in X,\ t > 0,$$

$$\lim_{n\to\infty} T_{j=n}^{\infty}\Big(T\Big(M\Big(x, \frac{2\alpha^j t}{|2|^{kj}}\Big), M\Big(-x, \frac{2\alpha^j t}{|2|^{kj}}\Big)\Big)\Big) = 1_{\mathcal{L}},$$

$$\lim_{n\to\infty} T_{j=n}^{\infty}\Big(T\Big(N\Big(x, \frac{2\alpha^j t}{|2|^{kj}}\Big), N\Big(-x, \frac{2\alpha^j t}{|2|^{kj}}\Big)\Big)\Big) = 1_{\mathcal{L}},$$

then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
&P(f(x) - C(x) - Q(x), t) \geq_L T\Big(T_{i=0}^{\infty}\Big(T\Big(M\Big(x, \frac{2\alpha^{i+1} t}{|2^k|^i}\Big), \\
&M\Big(-x, \frac{2\alpha^{i+1} t}{|2^k|^i}\Big)\Big)\Big), T_{i=0}^{\infty}\Big(T\Big(N\Big(x, \frac{2\alpha^{i+1} t}{|2^k|^i}\Big), N\Big(-x, \frac{2\alpha^{i+1} t}{|2^k|^i}\Big)\Big)\Big)\Big),
\end{aligned} \tag{29}$$

where

$$M(x,t) := T\Big(T\Big(T\Big(\Psi\Big(\frac{x}{4},\frac{x}{4},t\Big),\Psi\Big(\frac{x}{4},\frac{3x}{4},t\Big)\Big),T\Big(\Psi\Big(\frac{x}{4},\frac{x}{4},t\Big),$$
$$\Psi\Big(\frac{x}{4},\frac{5x}{4},t\Big)\Big)\Big),T\Big(T\Big(\Psi\Big(\frac{2x}{4},\frac{2x}{4},t\Big),\Psi\Big(\frac{2x}{4},\frac{3.2x}{4},t\Big)\Big),$$
$$T\Big(\Psi\Big(\frac{2x}{4},\frac{2x}{4},t\Big),\Psi\Big(\frac{2x}{4},\frac{5.2x}{4},t\Big)\Big)\Big),\ldots,$$
$$T\Big(T\Big(\Psi\Big(\frac{2^{j-1}x}{4},\frac{2^{j-1}x}{4},t\Big),\Psi\Big(\frac{2^{j-1}x}{4},\frac{3.2^{j-1}x}{4},t\Big)\Big),$$
$$T\Big(\Psi\Big(\frac{2^{j-1}x}{4},\frac{2^{j-1}x}{4},t\Big),\Psi\Big(\frac{2^{j-1}x}{4},\frac{5.2^{j-1}x}{4},t\Big)\Big)\Big)\Big),$$

and

$$N(x,t) := T\Big(T\Big(T\Big(\Psi\Big(\frac{x}{4},\frac{x}{4},t\Big),\Psi\Big(\frac{x}{4},\frac{x}{4},t\Big)\Big),T\Big(\Psi\Big(\frac{x}{4},\frac{5x}{4},t\Big),$$
$$\Psi\Big(\frac{x}{4},\frac{-3x}{4},t\Big)\Big)\Big),T\Big(T\Big(\Psi\Big(\frac{2x}{4},\frac{2x}{4},t\Big),\Psi\Big(\frac{2x}{4},\frac{2x}{4},t\Big)\Big),$$
$$T\Big(\Psi\Big(\frac{2x}{4},\frac{5.2x}{4},t\Big),\Psi\Big(\frac{2x}{4},\frac{-3.2x}{4},t\Big)\Big)\Big),\ldots,$$
$$T\Big(T\Big(\Psi\Big(\frac{2^{j-1}x}{4},\frac{2^{j-1}x}{4},t\Big),\Psi\Big(\frac{2^{j-1}x}{4},\frac{2^{j-1}x}{4},t\Big)\Big),$$
$$T\Big(\Psi\Big(\frac{2^{j-1}x}{4},\frac{5.2^{j-1}x}{4},t\Big),\Psi\Big(\frac{2^{j-1}x}{4},\frac{-3.2^{j-1}x}{4},t\Big)\Big)\Big)\Big),$$

for all $x \in X,\ \ t > 0$.

Proof. Let $f_0(x) = \frac{1}{2}[f(x)-f(-x)]$, for all $x \in X$. Then $f_0(0) = 0$, $f_0(-x) = -f_0(x)$, and

$$P(f_0(3x+y) + f_0(3x-y) - f_0(x+y) - f_0(x-y) - 2f_0(3x) + 2f_0(x), t)$$
$$\geq_L T\Big(P\Big(\frac{1}{2}\Big[f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 2f(3x)$$
$$+ 2f(x)\Big],t\Big),P\Big(\frac{-1}{2}\Big[f(-3x-y) + f(-3x+y) - f(-x-y) - f(-x+y)$$
$$- 2f(-3x) + 2f(-x)\Big],t\Big)\Big) \geq_L T(\Psi(x,y,2t),\Psi(-x,-y,2t)),$$

for all $x, y \in X$ and $t > 0$. By Theorem 4, it follows that there exists a unique additive function $A : X \to Y$ satisfying

$$P(f_0(x) - A(x),t) \geq_L T_{i=0}^{\infty}\Big(T\Big(M\Big(x,\frac{2\alpha^{i+1}t}{|2^k|^i}),M\Big(-x,\frac{2\alpha^{i+1}t}{|2^k|^i}\Big)\Big)\Big), \tag{30}$$

for all $x, y \in X$ and $t > 0$.

Let $f_e(x) = \frac{1}{2}[f(x)+f(-x)]$, for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\begin{aligned}
&P(f_e(3x+y) + f_e(3x-y) - f_e(x+y) - f_e(x-y) - 2f_e(3x) + 2f_e(x), t)\\
&\geq_L T\Big(P\Big(\frac{1}{2}\Big[f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 2f(3x)\\
&+ 2f(x)\Big], t\Big), P\Big(\frac{1}{2}[f(-3x-y) + f(-3x+y) - f(-x-y) - f(-x+y)\\
&- 2f(-3x) + 2f(-x)\Big], t\Big)\Big) \geq_L T(\Psi(x, y, 2t), \Psi(-x, -y, 2t)),
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. By Theorem 5, it follows that there exists a unique quadratic function $Q : X \to Y$ satisfying

$$P(f_e(x) - Q(x), t) \geq_L T_{i=0}^{\infty}\Big(T\Big(N\Big(x, \frac{2\alpha^{i+1}t}{|2^k|^i}\Big), N\Big(-x, \frac{2\alpha^{i+1}t}{|2^k|^i}\Big)\Big)\Big), \tag{31}$$

for all $x, y \in X$ and $t > 0$. Hence (29) follows from (30) and (31). □

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