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$Q_{K,\omega,\log}(p,q)$-TYPE SPACES OF ANALYTIC AND MEROMORPHIC FUNCTIONS

A. EL-SAYED AHMED and ALAA KAMAL

Abstract. In this paper, we define the space $Q_{K,\omega,\log}(p,q)$ of analytic functions on the unit disk. We obtain some characterizations for the space $Q_{K,\omega,\log}(p,q)$ by the help of the nondecreasing function K and the reasonable function ω. Moreover, the meromorphic $Q^{\#}_{K,\omega,\log}(p,q)$ space is also considered and studied.

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1. INTRODUCTION

We start here with some terminology, notation and the definition of various classes of analytic functions defined on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$ with boundary $\partial\mathbb{D}$. $\mathrm{d}A(z)$ be the normalized area measure on $\mathbb{D}$ so that $A(\mathbb{D}) \equiv 1$. Recall that the weighted logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\log}$ (see [15]) is defined as follows:

$$\mathcal{B}^{\alpha}_{\log} = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha} \left(\log \frac{2}{1-|z|^2}\right) |f'(z)| < \infty \right\}.$$

The little weighted logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\log,0}$ (see [15]) is a subspace of $\mathcal{B}^{\alpha}_{\log}$ consisting of all $f \in \mathcal{B}^{\alpha}_{\log}$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2) \left(\log \frac{2}{1-|z|^2}\right) |f'(z)| = 0.$$

Denote by $\mathcal{D} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}A(z) < \infty\}$ the Dirichlet space. Let $0 < q < \infty$. Then the Besov-type spaces

$$\mathbf{B^q} = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^2 \mathrm{d}A(z) < \infty \right\}$$

are introduced and studied intensively by Stroethoff (cf. [11]). Here, $\varphi_a(z)$ stands for the Möbius transformation of $\mathbb{D}$ and it is given by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, where $a \in \mathbb{D}$. In [3] a class of holomorphic functions, the so called $\mathcal{Q}_p$-space is introduced as follows:

$$\mathcal{Q}_p = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a) \mathrm{d}A(z) < \infty \right\},$$

where $0 < p < \infty$ and the weight function $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation φ_a. The weight function $g(z,a)$ is actually Green's function in $\mathbb{D}$ with pole at $a \in \mathbb{D}$.

For a point $a \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(a,r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $D(a,r) = \varphi_a(rD)$. The pseudo-hyperbolic disk $D(a,r)$ is also an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{1-r^2|a|^2}$ and $\frac{(1-|a|^2)r}{1-r^2|a|^2}$, respectively (see [11]). Let A denote the normalized Lebesgue area measure on $\mathbb{D}$, and for a Lebesgue measurable set $K_1 \subset \mathbb{D}$, denote by $|K_1|$ the measure of K_1 with respect to A. It follows immediately that:

$$|D(a,r)| = \frac{(1-|a|^2)^2}{(1-r^2|a|^2)^2} r^2.$$

Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. For $0 < p < \infty$, $-2 < q < \infty$, we say that a function f analytic in $\mathbb{D}$ belongs to the space $Q_K(p,q)$ (cf. [14]), if

$$\|f\|^p_{Q_K(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) < \infty.$$

Using the above mentioned function K, several authors have been studied some classes of holomorphic and meromorphic function spaces (see [1, 2, 5, 6, 8, 9, 13, 14] and others).

Now, given a reasonable function $\omega : (0,1] \to (0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ (see [4]) is defined as the set of all analytic functions f on $\mathbb{D}$ satisfying

$$(1-|z|)|f'(z)| \le C\omega(1-|z|), \quad z \in \mathbb{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$. Here, the word "reasonable" is a non-mathematical term; it was just intended to mean that the "not too bad" and the function satisfy some natural conditions.

We introduce the following definitions:

Definition 1.1. For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$, an analytic function f on $\mathbb{D}$ is said to belong to the weighted logarithmic α-Bloch space $\mathcal{B}^\alpha_{\omega,\log}$ if

$$\|f\|_{\mathcal{B}^\alpha_{\omega,\log}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^\alpha}{\omega(1-|z|)} |f'(z)| \left(\log\frac{2}{1-|z|^2}\right) < \infty.$$

Definition 1.2. For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$, an analytic function f on $\mathbb{D}$ is said to belong to the little weighted logarithmic α-Bloch space $\mathcal{B}^\alpha_{\omega,0}$ if

$$\|f\|_{\mathcal{B}^\alpha_{\omega,\log,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha}{\omega(1-|z|)} |f'(z)| \left(\log\frac{2}{1-|z|^2}\right) = 0.$$

Throughout this paper and for some techniques, we consider the case of $\omega \not\equiv 0$.

The logarithmic order (log-order) of the function $K(r)$ is defined as

$$\rho = \limsup_{r\to\infty} \frac{\ln^+ \ln^+ K(r)}{\ln r},$$

where $\ln^+ x = \max\{\ln x, 0\}$. If $0 < \rho < \infty$, the logarithmic type (log-type) of the function $K(r)$ is defined as

$$\sigma = \limsup_{r\to\infty} \frac{\ln^+ K(r)}{r^\rho} \ .$$

Note that if f is an entire function, then the growth order of f is just the log-order of $M(r)$, the maximum modulus function of f.

Definition 1.3. Let $0 < p < \infty$ and $-2 < q < \infty$. For a nondecreasing function $K : [0,\infty) \to [0,\infty)$ and for a given reasonable function $\omega : (0,1] \to (0,\infty)$, an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega,\log}(p,q)$ if

$$\|f\|^p_{Q_{K,\omega,\log}(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) < \infty.$$

Definition 1.4. Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. For a given reasonable function $\omega : (0,1] \to (0,\infty)$ an analytic function f in $\mathbb{D}$ is said to belong to the spaces $F_{\omega,\log}(p,q,s)$ if

$$\|f\|^p_{F_{\omega,\log}(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) < \infty.$$

Moreover, if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) = 0,$$

then $f \in F_{\omega,\log,0}(p,q,s)$.

We assume throughout the paper that

$$\int_0^1 (1-r^2)^{-2} K\left(\log \frac{1}{r}\right) r\mathrm{d}r < \infty.$$

We can define an auxiliary function as follows:

$$\varphi_K(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)}, \qquad 0 < s < \infty.$$

Remark. It should be remarked that our $Q_{K,\omega,\log}(p,q)$ classes are more general than many classes of analytic functions. If $\omega \equiv 1$, and $\log \frac{2}{1-|z|^2} = 1$, then we obtain $Q_K(p,q)$ type spaces. If $p = 2, q = 0$, $\omega \equiv 1$, and $\log \frac{2}{1-|z|^2} = 1$, we obtain Q_K space. If $p = 2, q = 0$, $\omega \equiv 1$, and $\log \frac{2}{1-|z|^2} = 1$, we obtain

Q_p spaces as studied in [3]. If $\omega \equiv 1$, $\log \frac{2}{1-|z|^2} = 1$ and $K(t) = t^s$, then $Q_{K,\omega,\log} = F(p,q,s)$ classes.

Throughout this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a right continuous and nondecreasing function. Moreover, we suppose that $\omega : (0,1] \to (0,\infty)$ is a nondecreasing function.

2. ANALYTIC CLASSES

We first give some basic properties of analytic $Q_{K,\omega,\log}(p,q)$ spaces.

PROPOSITION 2.1. *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and $\omega : (0,1] \to (0,\infty)$, where $\omega(\lambda t) = \lambda\omega(t)$. For $0 < p < \infty$ and $-2 < q < \infty$, we have that the spaces $Q_{K,\omega,\log}(p,q)$ are subsets of the weighted logarithmic Bloch spaces $\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}$.*

Proof. For a fixed $r \in (0,1)$ and $a \in \mathbb{D}$, let $E(a,r) = \{z \in \mathbb{D},\ |z-a| < r(1-|a|)\}$. Also, suppose that $f \in Q_{K,\omega,\log}(p,q)$. We obtain:

$$\begin{aligned}
\|f\|^p_{Q_{K,\omega,\log}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&\geq \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&\geq \int_{D(a,r)} |f'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&\geq K\left(\log \frac{1}{r}\right) \int_{D(a,r)} |f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&\geq K\left(\log \frac{1}{r}\right) \int_{E(a,r)} |f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z).
\end{aligned}$$

We know that $E(a,r) \subset D(a,r)$ and for any $z \in E(a,r)$, we have

$$(1-r)(1-|a|) \leq 1-|z| \leq (1+r)(1-|a|).$$

Now, since we assume that ω is non-decreasing, we obtain:

$$\begin{aligned}
\|f\|^p_{Q_{K,\omega,\log}(p,q)} &\geq K\left(\log \frac{1}{r}\right) \int_{E(a,r)} |f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&\geq \frac{C(r)(1-|a|)^q \left(\log \frac{2}{(1+r)(1-|a|)}\right)}{\omega^p((1-r)(1-|a|))} \int_{E(a,r)} |f'(z)|^p \mathrm{d}A(z),
\end{aligned}$$

where $C(r)$ is a constant depends on r. Since $|f'(z)|^p$ is a subharmonic function, we have:

$$\int_{E(a,r)} |f'(z)|^p \,\mathrm{d}A(z) \geq |E(a,r)|\ |f'(a)|^p = r^2(1-|a|)^2 |f'(a)|^p.$$

Then, we obtain

$$\|f\|^p_{Q_{K,\omega,\log}(p,q)} \geq \frac{C_1(r)(1-|a|)^{q+2}|f'(a)|^p\big(\log\frac{2}{(1-|a|)}\big)}{\omega^p(1-|a|)},$$

where $C_1(r)$ is a constant depends on r. Then, we deduce that,

$$\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}} \leq \frac{\|f\|^p_{Q_{K,\omega,\log}(p,q)}}{C_1(r)}. \tag{1}$$

Our proposition is therefore established. □

Next we give the following proposition.

Proposition 2.2. *Let* $\omega : (0,1] \to (0,\infty)$ *and* $0 < p < \infty, -2 < q < \infty$. *If the log-order* ρ *and the log-type* σ *of a nondecreasing function* $K(r)$ *satisfy one of the following conditions:*
(i) $\rho > 1$;
(ii) $\rho = 1$ *and* $0 < \sigma < \infty$, *then* $\|f\|^p_{Q_{K,\omega,\log}(p,q)} \subset \|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}}$.

Proof. By Proposition 2.1, it suffices to show that each non-constant weighted logarithmic α-Bloch function f can not belong to the spaces $Q_{K,\omega,\log}(p,q)$.

In fact, if either the log-order ρ of $K(r)$ is greater than 0, or the log-order ρ of $K(r)$ equals 1 and the log-type σ of $K(r)$ is greater than 2, then there exists a sequence $\{r_n\}$ with $r_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\ln^+\ln^+ K(r_n)}{\ln r_n} = \rho > 1 \tag{2}$$

or

$$\sigma = \lim_{n\to\infty} \frac{\ln^+ K(r_n)}{r_n} = \lambda > 0. \tag{3}$$

In the case (2) or (3), we obtain

$$\lim_{n\to\infty} \frac{K(r_n)}{\mathrm{e}^{\lambda r_n}} = \text{const}. \tag{4}$$

Let f be a non-constant weighted logarithmic α-Bloch function. Then

$$\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}} = \sup_{z\in\mathbb{D}} \left\{ \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} \left(\log\frac{2}{(1-|z|)}\right) |f'(z)|^p : z\in\mathbb{D} \right\} \neq 0.$$

However, by (1) and (4) we have

$$\begin{aligned}\|f\|^p_{Q_{K,\omega,\log}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log\frac{2}{(1-|z|)}\right) \mathrm{d}A(z)\\ &\geq \pi\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}} (1-t_n)^p K\left(\log\frac{1}{t_n}\right) \not\to \infty.\end{aligned}$$

Hence $f \in Q_{K,\omega,\log}(p,q)$. □

Theorem 2.3. *Let* $K:[0,\infty)\to[0,\infty)$ *be a nondecreasing function and* $\omega:(0,1]\to(0,\infty)$, *satisfying both of the following:*

(A) *There exists a constant* $p>1$ *such that* $\lim_{n\to\infty}\frac{K(r)}{r^p}=c\neq 0$;
(B) *The log-order* ρ *and the log-type* σ *satisfy one of the following cases:*
 (i) $0\le\rho<1$;
 (ii) $\rho=1$ *and* $0<\sigma<\infty$.

Then $Q_{K,\omega,\log}(p,q)=\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}$.

Proof. Let $\lim_{r\to\infty}\frac{K(r)}{r^p}=C\neq 0$, for some $p\in(1,\infty)$. Then there exists a fixed $r_1\in(0,1)$ such that

$$\frac{c}{2}\le\frac{K(r)}{r^p}\le c+1,\quad 0<r<r_1. \tag{5}$$

We may choose $r_0\in(0,1)$ such that

$$z\in\mathbb{D}\backslash D(a,r_0)\Rightarrow g(z,a)=\log\frac{1}{|\varphi_a(z)|}<r_1. \tag{6}$$

Now we first suppose that $f\in Q_{K,\omega,\log}(p,q)$ with

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)=C,$$

and write

$$\begin{aligned}&\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q\frac{(g(z,a))^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\&=\int_{D(a,r_0)}|f'(z)|^p(1-|z|^2)^q\frac{(g(z,a))^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\&+\int_{\mathbb{D}\backslash D(a,r_0)}|f'(z)|^p(1-|z|^2)^q\frac{(g(z,a))^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\&=I_1+I_2.\end{aligned} \tag{7}$$

Since $Q_{K,\omega,\log}(p,q)\subset\mathcal{B}^{\frac{p+2}{q}}_{\omega,\log}$ from Proposition 2.1, we have

$$\begin{aligned}I_1&=\int_{D(a,r_0)}|f'(z)|^p(1-|z|^2)^q\frac{(g(z,a))^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\&\le\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}}\int_{D(a,r_0)}(1-|z|^2)^{-2}\left(\log\frac{1}{\varphi_a(z)}\right)^p\mathrm{d}A(z)\\&=2\pi\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}}\int_0^{r_0}r(1-r^2)^{-2}\left(\log\frac{1}{r}\right)^p\mathrm{d}r\\&=2\pi\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}}I(r_0,p),\end{aligned} \tag{8}$$

where the integral

$$I(r_0,p)=\int_0^{r_0} r(1-r^2)^{-2}\left(\log\frac{1}{r}\right)^p \mathrm{d}r<\infty$$

for $0<r_0<1$ and $1<p<\infty$. On the other hand, by (5) and (6), we get with $\mathbb{D}_0=\mathbb{D}\backslash D(a,r_0)$:

$$\begin{aligned}
I_2&=\int_{\mathbb{D}_0}|f'(z)|^p(1-|z|^2)^q\frac{(g(z,a))^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z).\\
&\le\frac{2}{c}\int_{\mathbb{D}_0}|f'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)<\infty.
\end{aligned}\tag{9}$$

Consequently, by (7), (8) and (9), we obtain that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q\frac{(g(z,a))^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\le\sup\{I_1+I_2\}<\infty.$$

Thus $f\in Q_{K,\omega,\log}(p,q)$. Hence $Q_{K,\omega,\log}(p,q)\subset\mathcal{B}^{\frac{p+2}{q}}_{\omega,\log}$. Since $f\in Q_{K,\omega,\log}(p,q)$, f must be a weighted logarithmic Bloch function in $\mathbb{D}$, and

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q\frac{(1-|\varphi_a(z)|^2)^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)=C<\infty.$$

With $\mathbb{D}_0=\mathbb{D}\backslash D(a,r_0)$, it follows from (5) and (6) that

$$\begin{aligned}
J_2&=\int_{\mathbb{D}_0}|f'(z)|^p(1-|z|^2)^q\frac{K\left(\log\frac{1}{\varphi_a(z)}\right)}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\
&\le(c+1)\int_{\mathbb{D}_0}|f'(z)|^p(1-|z|^2)^q\frac{\left(\log\frac{1}{\varphi_a(z)}\right)^p}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\
&=(c+1)C.
\end{aligned}\tag{10}$$

Since $f\in Q_{K,\omega,\log}(p,q)$, f must be a weighted logarithmic Bloch function in $\mathbb{D}$. Similar to (8), we have

$$\begin{aligned}
J_1&=\int_{D(a,r_0)}|f'(z)|^p(1-|z|^2)^q\frac{K\left(\log\frac{1}{\varphi_a(z)}\right)}{\omega^p(1-|z|)}\left(\log\frac{2}{(1-|z|)}\right)\mathrm{d}A(z)\\
&\le 2\pi\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}}\int_0^{r_0}(1-r^2)^{-2}K\left(\log\frac{1}{r}\right)r\mathrm{d}r\\
&\le\frac{2\pi}{(1-r_0^2)^2}\|f\|^p_{\mathcal{B}^{\frac{q+2}{p}}_{\omega,\log}}\int_0^{r_0}K\left(\log\frac{1}{r}\right)r\mathrm{d}r.
\end{aligned}\tag{11}$$

Now we show that the integral $\int_0^{r_0} K(\log\frac{1}{r})r\mathrm{d}r$ in (11) is convergent. Setting $t=\log\frac{1}{r}$, we have

$$J(K)=\int_0^{r_0} K\left(\log\frac{1}{r}\right)r\mathrm{d}r=\int_{t_0}^{+\infty}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t.$$

If $K(t)$ satisfies condition (i), then for given $\epsilon>0$ with $\rho+\epsilon<1$, there exists $t_1>t_0$ such that $K(t)<\mathrm{e}^{t\rho+\epsilon}<\mathrm{e}^t$, $t\geq t_1$. Therefore,

$$J(K)=\int_{t_0}^{t_1}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t+\int_{t_1}^{+\infty}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t\leq\int_{t_0}^{t_1}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t+\int_{t_1}^{+\infty}\frac{1}{\mathrm{e}^{2t}}\mathrm{d}t<\infty. \tag{12}$$

If $K(t)$ satisfies condition (ii), then for given $\epsilon>0$ with $0<\sigma+2\varepsilon<2$, there exists $t_2>t_0$ such that $K(t)<\mathrm{e}^{(\sigma+\epsilon)t}<\mathrm{e}^{(2-\varepsilon)t}$, $t\geq t_2$. Thus

$$\begin{aligned}J(K)&=\int_{t_0}^{t_2}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t+\int_{t_2}^{+\infty}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t\leq\int_{t_0}^{t_2}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t+\int_{t_2}^{+\infty}\frac{\mathrm{e}^{(2-\varepsilon)t}}{\mathrm{e}^{2t}}\mathrm{d}t\\&=\int_{t_0}^{t_2}\frac{K(t)}{\mathrm{e}^{2t}}\mathrm{d}t+\int_{t_2}^{+\infty}\mathrm{e}^{-\varepsilon t}\mathrm{d}t<\infty.\end{aligned}$$

Therefore, by (10) and (11), we get

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}\left(\log\frac{2}{1-|z|}\right)\mathrm{d}A(z)=\sup_{a\in\mathbb{D}}\{J_1+J_2\}<\infty.$$

This implies that $f\in Q_{K,\omega,\log}(p,q)$. The proof is therefore completed. □

3. MEROMORPHIC CLASSES

A natural analogue of $|f'(z)|$ is the spherical derivative

$$f^{\#}(z)=\frac{|f'(z)|}{(1+|f(z)|^2)}.$$

We define the classes $F^{\#}_{\omega,\log}(p,q,s)$ and $F^{\#}_{\omega,\log,0}(p,q,s)$ as follows.

DEFINITION 3.1. Let $0<p<\infty$, $-2<q<\infty$ and $0\leq s<\infty$. A function f meromorphic in $\mathbb{D}$ is said to belong to the class $F^{\#}_{\omega,\log}(p,q,s)$ if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(f^{\#}(z)\right)^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega^p(1-|z|)}\left(\log\frac{2}{1-|z|^2}\right)\mathrm{d}A(z)<\infty,$$

that is, $\|f\|^p_{F^{\#}_{\omega,\log}(p,q,s)}<\infty$. Moreover, if

$$\lim_{|a|\to1^-}\int_{\mathbb{D}}\left(f^{\#}(z)\right)^p(1-|z|^2)^q\frac{g^s(z,a)}{\omega^p(1-|z|)}\left(\log\frac{2}{1-|z|^2}\right)\mathrm{d}A(z)=0,$$

that is, $\|f\|^p_{F^{\#}_{\omega,\log,0}(p,q,s)}=0$, then $f\in F^{\#}_{\omega,\log}(p,q,s)$.

Therefore we define the classes $M^{\#}_{\omega,\log}(p,q,s)$ and $M^{\#}_{\omega,\log,0}(p,q,s)$ as follows.

Definition 3.2. Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leq s < \infty$. A function f meromorphic in $\mathbb{D}$ is said to belong to the class $M^{\#}_{\omega,\log}(p,q,s)$ if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) < \infty,$$

that is, $\|f\|^p_{M^{\#}_{\omega,\log}(p,q,s)} < \infty$. Moreover, if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} (f^{\#}(z))^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) = 0,$$

that is, $\|f\|^p_{M^{\#}_{\omega,\log,0}(p,q,s)} = 0$, then $f \in M^{\#}_{\omega,\log,0}(p,q,s)$.

Let $\mathcal{N}^{\alpha}_{\omega,\log}$ be the class of all normal functions in $\mathbb{D}$. A function f meromorphic in $\mathbb{D}$ is said to be logarithmic normal if and only if

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|)} f^{\#}(z) \left(\log \frac{2}{1-|z|^2}\right) < \infty.$$

Now we give the following theorem.

Theorem 3.3. *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leq s < 1$. Then a function f meromorphic in $\mathbb{D}$ is logarithmic normal if and only if*

$$\|f\|^p_{F^{\#}_{\omega,\log}(p,q,s)} < \infty.$$

Proof. The proof of this theorem is very similar to the corresponding result in [12], so it will be omitted. □

In the corresponding way to the analytic case, we define the meromorphic classes $Q^{\#}_{K,\omega,\log}(p,q)$ as follows.

Definition 3.4. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, a function f meromorphic in $\mathbb{D}$ is said to belong to the classes $Q^{\#}_{K,\omega,\log}(p,q)$ if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^p (1-|z^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) < \infty. \tag{13}$$

Remark. Similar to the analytic case, if we take $\omega \equiv 1$ and $K(t) = t^s$ for $0 \leq s < \infty$ and $\log \frac{2}{1-|z|^2} = 1$, then $Q^{\#}_{K,\omega,\log}(p,q) = F^{\#}(p,q,s)$ (see [10]), the corresponding meromorphic of $F(p,q,s)$ spaces. If we take $K(t) = t^p$, $q = 0$ and $\omega \equiv 1$ and $\log \frac{2}{1-|z|^2} = 1$, then $Q^{\#}_{K,\omega,\log}(p,q) = Q^{\#}_p$ (see [3]).

Definition 3.5. [12] A function f meromorphic in $\mathbb{D}$ is said to be a spherical Bloch function, denoted by $f \in \mathcal{B}^{\#}$, if there exists an r, $0 < r < 1$, such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^2 \, \mathrm{d}A(z) < \infty.$$

It is easy to see that a normal function is a spherical Bloch function, that is, $\mathcal{N} \subset \mathcal{B}^{\#}$, but the converse is not true. A counterexample can be found in [7].

PROPOSITION 3.6. *Let $K : [0,\infty) \to [0,\infty)$ be a right continuous and nondecreasing function and suppose that $\omega : (0,1] \to (0,\infty)$ is a nondecreasing function. Then, the classes $Q^{\#}_{K,\omega,\log}(p,q)$ are subsets of the spherical Bloch classes $\mathcal{B}^{\# \frac{q+2}{p}}_{\omega,\log}$, where $0 < p < \infty$ and $-2 < q < \infty$.*

Proof. We can prove the proposition by making the obvious modifications to the proof of Proposition 2.1. □

THEOREM 3.7. *Let $K : [0,\infty) \to [0,\infty)$ be a right continuous, bounded and nondecreasing function and suppose that $\omega : (0,1] \to (0,\infty)$ is a nondecreasing function. Moreover, suppose that f is a logarithmic normal function. Let $0 < p < \infty$ and $-2 < q < \infty$. If $\lim_{r\to 0} \frac{K(r)}{r^s} = c < \infty$ holds for some $0 < s < \infty$, then $f \in Q^{\#}_{K,\omega,\log}(p,q)$.*

Proof. Suppose that $\lim_{r\to 0} \frac{K(r)}{r^s} = c < \infty$ holds for some $0 < s < \infty$. Then there exists a fixed $r_1 \in (0,1)$ such that $\frac{K(r)}{r^s} \le c+1$ for $0 < r < r_0$, we may take $r_0 \in (0,1)$ such that both

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|} < r_1, \quad \log \frac{1}{|\varphi_a(z)|} \le c_1(1 - |\varphi_a(z)|^2)$$

hold for constant $c_1 > 0$ whenever $z \in \mathbb{D}\backslash D(a,r_0)$. Now we have

$$\begin{aligned}
\|f\|^p_{Q^{\#}_{K,\omega,\log}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f_0{}^{\#}(z))^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&= \sup_{D(a,r_0)} \int_{D(a,r_0)} (f_0^{\#}(z))^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&+ \sup_{\mathbb{D}\backslash D(a,r_0)} \int_{\mathbb{D}} (f_0^{\#}(z))^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z).
\end{aligned}$$

For $a \in \mathbb{D}$ and r_0 as above, and $\mathbb{D}_0 = \mathbb{D}\backslash D(a,r_0)$, using Theorem 3.3 we have that

$$\begin{aligned}
L_2 &= \int_{\mathbb{D}_0} (f_0^{\#}(z))^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) \\
&\le (c+1)(c_1)^s \int_{\mathbb{D}_0} (f_0^{\#}(z))^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z)
\end{aligned}$$

Then $L_2 \le (c+1)(c_1)^s \|f\|^p_{F^{\#}_{\omega,\log}(p,q,s)} < \infty$.

On the other hand, since K is bounded, there exists a constant C_2 such that $K(r) \leq C_2$ for all $r, , 0 < r < \infty$. Thus

$$L_1 = \int_{D(a,r_0)} \left(f_0^{\#}(z)\right)^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z)$$
$$\leq \frac{C_2}{(1-r_0^2)^s} \int_{D(a,r_0)} \left(f_0^{\#}(z)\right)^p (1-|z|^2)^q \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z)$$

Then $L_1 \leq \frac{C_2}{(1-r_0^2)^s} \|f\|^p_{F^{\#}_{\omega,\log}(p,q,s)}$.

Therefore, we have

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(f_0^{\#}(z)\right)^p (1-|z|^2)^q \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z)$$
$$= \sup_{a\in\mathbb{D}} \{L_1 + L_2\} < \infty.$$

Thus $f_0 \in \|f\|^p_{Q^{\#}{}_{K,\omega,\log}(p,q)}$, and the proof of our theorem is completed. □

Finally, we consider the harmonic counterpart of $Q_{K,\omega,\log}(p,q)$ as follows.

Definition 3.8. Let $0 < p < \infty$ and $-2 < q < \infty$ and let $K : [0,\infty) \to [0,\infty)$ be a right continuous, bounded and nondecreasing function and suppose that $\omega : (0,1] \to (0,\infty)$ is a nondecreasing function. A real-valued harmonic function u in $\mathbb{D}$ is said to belong to the space $Q_{Kh,\omega,\log}(p,q)$ if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) \mathrm{d}A(z) < \infty,$$

where $\nabla u(z) = (u_x, u_y)$ is the gradient of u and $|\nabla u(z)| = \sqrt{u_x^2 + u_y^2}$.

The harmonic logarithmic weighted α-Bloch space $\mathcal{B}^{\alpha}_{h,\omega,\log}$, is defined by the set

$$\left\{ u : u \text{ harmonic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha} |\nabla u(z)|}{\omega(1-|z|)} \left(\log \frac{2}{1-|z|^2}\right) < \infty \right\}.$$

Remark. It is easy to see that some corresponding results to Propositions 2.1, 2.2, and Theorem 2.3 are also true for $Q_{Kh,\omega,\log}(p,q)$ and the proofs are similar to those of them.

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Sohag University
Faculty of Science
Mathematics Department
82524 Sohag, Egypt

Taif University
Faculty of Science
Mathematics Department
El-Taif 5700, El-Hawiyah, Saudi Arabia
E-mail: ahsayed80@hotmail.com

Majmaah University
Faculty of Science
Department of Mathematics
Saudi Arabia
E-mail: alaa_mohamed1@yahoo.com