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# IDEMPOTENT-NILPOTENT UNITS OF COMMUTATIVE GROUP RINGS

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**Abstract.** Suppose that $R$ is a commutative unital ring and $G$ is a multiplicative abelian group. We find a criterion when the decomposition of normalized invertible elements $V(RG) = Id(RG) \times (1 + I(N(R)G; G))$ holds. In particular, when $supp(G) \cap inv(R) \neq \emptyset$, we establish such a necessary and sufficient condition only in terms of $R$ and $G$. This strengthens a result due to Karpilovsky (Arch. Math., 1983) as well as results of the author (Bull. Greek Math. Soc., 2009), (Comm. Algebra, 2010) and (Comment. Math. Univ. Carolin., 2012).

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## 1. INTRODUCTION

Throughout the paper, let it be agreed that $R$ is a commutative ring with identity $1_R$ (called *unital*) and $G$ is an abelian group written multiplicatively as is the custom when discussing group rings. As usual, $RG$ denotes the group ring of $G$ over $R$ with unit group $U(RG)$ and its subgroup of normalized units $V(RG)$; it is easily observed that the equality $U(RG) = V(RG) \times U(R)$ is valid, where $U(R)$ is the multiplicative group of $R$. Standardly, $G_0 = \coprod_p G_p$ will always denote the torsion part of $G$ with $p$-primary component $G_p$.

Imitating [9], we define the sets $supp(G) = \{p | G_p \neq 1\}, inv(R) = \{p | p \cdot 1_R \in U(R)\}, zd(R) = \{p | \exists r \in R \setminus \{0\} : pr = 0\}$ and recollect once again the ideal $N(R) = \{r \in R | \exists n \in \mathbb{N}\colon\ r^n = 0\}$ of $R$ called nil-radical. Likewise, define the set $np(R) = \{p | \exists r \in R \setminus N(R) : pr \in N(R)\}$ (e.g., see [5]).

Moreover, $I(N(R)G; G)$ is the fundamental (augmentation) ideal of the subalgebra $N(R)G \subseteq N(RG)$ of $RG$, $id(R) = \{e \in R | e^2 = e\}$ is the set of all idempotents in $R$, and

$$Id(RG) = \{e_1 g_1 + \cdots + e_s g_s | e_1, ..., e_s \in id(R), e_1 + \cdots + e_s = 1,$$
$$e_i \cdot e_j = 0 (i \neq j); g_1, ..., g_s \in G\}.$$

It is worthwhile noticing that if $id(R) = \{0, 1\}$, i.e., $R$ is indecomposable, then $Id(RG) = G$ and vice versa.

All other notions and notations are in agreement with [8, 9].

A problem of central interest in the commutative group ring theory is to describe $V(RG)$ up to an isomorphism in terms associated only with $R$ and

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$G$. There are too many investigations in this theme (e.g., see [2, 3, 4, 5] as well as the bibliography in [9]). In that aspect, Karpilovsky proved in [6, 7] (see also [8, 9]) that $V(RG)$ can be decomposed like $Id(RG) \times (1 + I(N(R)G; G))$ whenever $G_0 = 1$. In particular, when $G$ is torsion-free and $R$ is both indecomposable and reduced, $V(RG) = G$ holds, which generalizes a classical result due to Higman for trivial units.

The purpose of this paper is to extend the theorem of Karpilovsky by finding a suitable criterion when the above decomposition is true without any additional restriction on $R$ and $G$. We shall do that into two statements, where in the second one we will require that there are no invertible primes whenever there are nonidentity primary components of the group.

## 2. MAIN RESULTS

We foremost begin with a few well-known technicalities from ring theoretical aspect.

It is well known the classical fact of lifting idempotents modulo nil-ideals, namely if $f \in id(R/N(R))$, then there is $e \in id(R)$ such that $f = e + N(R)$. More generally, the following folklore affirmation is fulfilled:

Proposition 1. *Let* $1 = f_1 + \cdots + f_k$ *be a decomposition of* $1$ *as a sum of orthogonal idempotents* $f_1, \ldots, f_k$ *in* $R/I$ *where* $I$ *is a nil-ideal of* $R$. *Then there exist orthogonal idempotents* $e_1, \ldots, e_k \in R$ *with* $1 = e_1 + \cdots + e_k$ *such that* $e_1 + I = f_1, \ldots, e_k + I = f_k$.

Next, by taking $I = N(R)$, we will obtain our pivotal reduction tool.

Lemma 2. *Let* $\phi : R \to R/N(R)$ *be the natural map. Then all five maps presented below, defined by*

$$\Phi\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \phi(\alpha_g) g = \sum_{g\in G} (\alpha_g + N(R))g,$$

*are epimorphisms (i.e., surjective homomorphisms):*

(i)$\Phi : RG \to (R/N(R))G$.
(ii)$\Phi : N(RG) \to N((R/N(R))G)$.
(iii)$\Phi : V(RG) \to V((R/N(R))G)$.
(iv)$\Phi : V_p(RG) \to V_p((R/N(R))G)$.
(v)$\Phi : Id(RG) \to Id((R/N(R))G)$.

*Proof.* (i) This is straightforward since $\phi$ is linearly extended to $\Phi$. Notice that the kernel of this homomorphism is $N(R)G$.

(ii) Given $x \in N((R/N(R))G)$, by (i) there is $y \in (R/N(R))G$ such that $\Phi(y) = x$. But $x^t = 0$ for some $t \in \mathbb{N}$ and hence $(\Phi(y))^t = \Phi(y^t) = 0$. Therefore, $y^t \in \ker \Phi = N(R)G \subseteq N(RG)$. This forces that $y \in N(RG)$, as required.

(iii) Choose $x \in V((R/N(R))G)$. Then there is $x' \in V((R/N(R))G)$ with $xx' = 1'$ and $y, y' \in RG$ with $\Phi(y) = x$ and $\Phi(y') = x'$. Consequently, $\Phi(y)\Phi(y') = \Phi(yy') = 1' = \Phi(1)$ and thus $\Phi(yy'-1) = 0$. So, $yy'-1 \in \ker \Phi = N(R)G$ and $yy' \in 1 + N(R)G \subseteq 1 + N(RG) \subseteq U(RG) = V(RG) \times U(R)$. Therefore, $y \in U(RG)$ and we can write $y = vu$ where $v \in V(RG)$ and $u \in U(R)$. Furthermore, $x = \Phi(v)\Phi(u)$. But $\Phi(V(RG)) \subseteq V((R/N(R))G)$ so that $\Phi(v) \in V((R/N(R))G)$. Observe also that $\phi : U(R) \to U(R/N(R))$ is given by $\phi(r) = r + N(R)$ where $r \in R$ and $\ker \phi_{U(R)} = 1 + N(R)$. Since $V((R/N(R))G) \cap U(R/N(R)) = 1$, one can conclude that $x = \Phi(v)$ as wanted. Thus $\Phi_{V(RG)}$ is a surjection, as claimed. Finally, note that the kernel of this homomorphism is $1 + I(N(R)G; G)$.

(iv) This is a direct consequence of (iii).

(v) Given $x \in Id((R/N(R))G)$, we can write $x = (r_1 + N(R))g_1 + \cdots + (r_s + N(R))g_s$, where $r_i + N(R) \in id(R/N(R))$, $r_1 + \cdots + r_s - 1 \in N(R)$ and $r_ir_j \in N(R)$ whenever $i \neq j$; $1 \leq i, j \leq s$. Invoking Proposition 1, there exist $e_1, \ldots, e_s \in id(R)$ with the properties $e_1 + \cdots + e_s = 1$, $e_ie_j = 0$ provided $i \neq j$ and $e_i + N(R) = r_i + N(R)$. Thus, it is readily seen that the element $e_1g_1 + \cdots + e_sg_s \in Id(RG)$ is the wanted pre-image. □

*Remark* 3. In [2] we have proved the same technical assertion, in a little more general form, but when $char(R) = p$ is a prime integer. Moreover, we have also omitted the above trivial arguments, as these in (iii), about the normalization of the existing element.

As a direct consequence, we yield the following:

Corollary 4. *$N((R/N(R))G) = 0 \Leftrightarrow N(RG) = N(R)G$.*

*Proof.* As noticed above, the kernel in Lemma 2 (i) is $N(R)G$. Hence, it easily follows from Lemma 2 (ii) that $N(RG)/N(R)G \cong N((R/N(R))G)$. The final argument is immediate. □

So, we come to our main reduction statement.

Proposition 5. *We have $V(RG) = Id(RG) \times (1 + I(N(R)G; G)) \Leftrightarrow V((R/N(R))G) = Id((R/N(R))G)$.*

*Proof.* The necessity follows by a direct application of Lemma 2 (iii) and (v) since $\Phi$ maps $1 + I(N(R)G; G)$ to 1.

Conversely, the sufficiency follows like this: given $v \in V(RG)$, there is $z \in V((R/N(R))G$ with the property $\Phi(v) = z$. But $z$ lies in $Id((R/N(R)G)$ and in virtue of Lemma 2 (v) there exists $u \in Id(RG)$ such that $\Phi(u) = z$. Thus $\Phi(v) = \Phi(u)$, i.e., $\Phi(v)\Phi(u)^{-1} = \Phi(v)\Phi(u^{-1}) = \Phi(vu^{-1}) = 1$. Furthermore, $vu^{-1} \in \ker \Phi = 1 + I(N(R)G; G)$ which immediately forces that $v \in Id(RG)(1 + I(N(R)G; G))$, as wanted. □

We next establish some elementary but useful relationships between some ring-theoretic sets.

Proposition 6. $np(R) \subseteq zd(R)$.

*Proof.* Assume $p \in np(R)$, whence $pr \in N(R)$ for some $r \in R \backslash N(R)$. Hence there is $k \in \mathbb{N}$ such that $p \cdot p^{k-1} r^k = 0$. Note that $r^k \neq 0$. Denote $r' = p^{k-1} r^k$. If $r' \neq 0$ we are done. Otherwise, if $r' = 0$ we write $p \cdot p^{k-2} r^k = 0$ and denote $r'' = p^{k-2} r^k$. If $r'' \neq 0$ we are done. In the remaining case $p^{k-2} r^k = 0$ etc. we proceed similarly to obtaining that there is some non-zero $f \in R$ with $pf = 0$. Thus $p \in zd(R)$ and the proof is over. □

As an immediate consequence, we derive:

Corollary 7. *Suppose either $N(R) = 0$ or $char(R) = p$ is prime. Then $np(R) = zd(R) = zd(R/N(R))$.*

*Proof.* If $R$ is reduced, the claim follows at once. If now $R$ has prime characteristic $p$, then it plainly follows that $p \cdot 1 = 0$, and hence $q \cdot r \neq 0$ for each $r \neq 0$ because $(p, q) = 1$. That is why, $zd(R) = \{p\} \subseteq np(R)$. Taking into account Proposition 6, the first equality is obtained.

Finally, note that the equality $zd(R/N(R)) = np(R)$ was proved in ([5, Lemma 3]), and hence the second equality follows as well. □

Proposition 8. $V(RG_0 + N(R)G) = V(RG_0)(1 + I(N(R)G; G))$.

*Proof.* It is apparent that both $V(RG_0)$ and $(1 + I(N(R)G; G) \subseteq V(RG) \cap [N(R)G]$ are contained in $V(RG_0 + N(R)G)$, whence the same is valid for their product.

As for the converse, choosing $x \in V(RG_0 + N(R)G)$, we write $x = r_1 g_1 + \cdots + r_s g_s + f_1 a_1 + \cdots + f_k a_k$ with $r_1 + \cdots + r_s + f_1 + \cdots + f_k = 1$, where $r_1 g_1 + \cdots + r_s g_s \in RG_0$, $f_1 a_1 + \cdots + f_k a_k \in N(R)G$. Note only that since $f_1 + \cdots + f_k \in N(R)$ and the sum of a nilpotent and a unit is again a unit, it follows that $r_1 + \cdots + r_s \in U(R)$. Furthermore, $x = r_1 g_1 + \cdots + r_s g_s + f_1 + \cdots + f_k + f_1(a_1 - 1) + \cdots + f_k(a_k - 1) = y + f_1(a_1 - 1) + \cdots + f_k(a_k - 1)$. Observe that $y = x - f_1(a_1 - 1) - \cdots - f_k(a_k - 1) \in V(RG_0)$ since $y = r_1 g_1 + \cdots + r_s g_s + f_1 + \cdots + f_k \in RG_0$, $aug(y) = 1$, $x \in V(RG)$ is a unit and $f_1(a_1 - 1) - \cdots - f_k(a_k - 1) \in I(N(R)G; G)$ is a nilpotent. Finally, writing $x = y(1 + f_1 y^{-1}(a_1 - 1) + \cdots + f_k y^{-1}(a_k - 1))$, we have that $x \in V(RG_0)(1 + I(N(R)G; G))$, as required. □

W. May has shown in [10] that if $id(R) = \{0, 1\}$ and $supp(G) \cap inv(R) = \emptyset$, then $V(RG) = GV(RG_0 + N(RG))$. Later on, the present author generalizes this decomposition in ([4, Proposition 3]), by dropping off the restriction on $R$ that it is indecomposable. In fact, if $supp(G) \cap inv(K) = \emptyset$ for every indecomposable subring $K$ of $R$, then $V(RG) = Id(RG)V(RG_0 + N(RG))$.

So, we are able to prove the following:

Proposition 9. *If $G$ is a group and $R$ is a ring such that $supp(G) \cap (inv(K) \cup np(R)) = \emptyset$ for each indecomposable subring $K$ of $R$, then*

$$V(RG) = Id(RG)V(RG_0)(1 + I(N(R)G; G)).$$

*Proof.* First of all, since $supp(G) \cap np(R) = supp(G) \cap zd(R/N(R)) = \emptyset$, we employ [10] to infer that $N((R/N(R))G) = 0$. Therefore, Corollary 4 assures that $N(RG) = N(R)G$. Consequently, a simple combination of the previously mentioned result from [2] with Proposition 8 gives the desired equality. $\square$

*Remark* 10. It is worthwhile noticing that when $G_0 = 1$ we will deduce the result due to Karpilovsky [6, 7], which is in the focus of our investigation. In fact, if $G_0 = 1$, then since $supp(G) = \emptyset$, one may derive from Proposition 9 that $V(RG) = Id(RG)(1 + I(N(R)G; G))$, which is precisely the aforementioned result of Karpilovsky, because obviously $Id(RG) \cap (1 + I(N(R)G; G)) = 1$.

Now, for the sake of completeness and and for the readers' convenience we now pause to quote the following result from [2].

Theorem 11. *Let $R$ be a commutative unital ring of prime characteristic and let $G$ be a non-identity abelian group. Then $V(RG) = Id(RG) \times (1 + I(N(R)G; G))$ if and only if at most one of the following conditions holds:*
(a) $G_t = 1$;
(b) $|G| = 2$, $\forall\, r \in R : 2r - 1 \in U(R) \iff r^2 - r \in N(R)$;
(c) $|G| = 3$, $\forall\, r, f \in R : 1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R) \iff r^2 - r \in N(R)$, $f^2 - f \in N(R)$ *and* $rf \in N(R)$.

Next, we proceed by proving the following result which somewhat improves [4, Theorem 5].

Theorem 12. *Suppose $G$ is a group and $R$ is a ring. Then $V(RG) = Id(RG) \times (1 + I(N(R)G; G))$ if and only if we have $V(RG_0) = Id(RG_0) \times (1 + I(N(R)G_0; G_0))$ and precisely one of the following points is valid:*
(1) $G = G_0$;
(2) $G \neq G_0$, $supp(G) \cap (inv(K) \cup np(R)) = \emptyset$ *for all indecomposable subrings $K$ of $R$.*

*Proof.* For the necessity, in view of Proposition 5, we have $V((R/N(R))G) = Id((R/N(R))G)$. Note that for any commutative unital ring $P$ and its subring $L$ (even when it does not contain the same identity element as that of $P$) the equality $V(PG) = Id(PG)$ forces that $V(LG) = Id(LG)$. Furthermore, by [4, Theorem 5] we deduce that $V((R/N(R))G_0) = Id((R/N(R))G_0)$, so that again an appeal to Proposition 5 insures the desired decomposition for $V(RG_0)$. Moreover, the same result applies to infer that either $G$ is torsion, or $G$ is not torsion and $supp(G) \cap (inv(K/N(K)) \cup zd(R/N(R))) = \emptyset$. We next apply [5, Lemmas 2 and 3] (together with Corollary 7) to conclude that $supp(G) \cap (inv(K) \cup np(R)) = \emptyset$ whenever $G$ contains an element of infinite order.

For the sufficiency, if $G = G_0$, then there is nothing to prove. So, let $G$ contain an element of infinite order and the intersection $supp(G) \cap (inv(K) \cup np(R))$ is empty for an arbitrary indecomposable subring $K$ of $R$. We therefore can apply Proposition 9 to deduce that $V(RG) = Id(RG)V(RG_0)(1 +$

$I(N(R)G;G))$. But substituting $V(RG_0)$ in the last formula by $Id(RG_0) \times (1 + I(N(R)G_0;G_0))$, we obtain the desired equality. $\square$

We have now at all disposal all the information needed to prove our chief result, which somewhat enlarges Theorem 11 presented above.

THEOREM 13. *Suppose $R$ is a ring and $G$ is a group such that $supp(G) \cap inv(R) \neq \emptyset$ or $supp(G) = \emptyset$. Then $V(RG) = Id(RG) \times (1 + I(N(R)G;G))$ if and only if at most one of the following is true:*
(a) $G_0 = 1$;
(b) $|G| = 2$, $\forall\, r \in R$: $2r - 1 \in U(R) \iff r \in id(R) + N(R) \iff r^2 - r \in N(R)$;
(c) $|G| = 3$, $\forall r, f \in R$: $1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R) \iff r, f \in id(R) + N(R)$, $rf \in N(R)$.

*Proof.* $\Rightarrow$. Referring to Proposition 5, we may write $V((R/N(R))G) = Id((R/N(R))G)$. We next use [4] to infer that either $G$ is torsion-free or $G$ is finite of order 2 or 3. Observe that $G_0 = 1$ exactly when $supp(G) = \emptyset$.

**Case 1**: $|G| = 2$ and $G = \langle g \rangle = \{1, g\}$ with $g^2 = 1$.

For any $x \in V(RG)$ we have $x = rg + 1 - r$ uniquely when $2r - 1 \in U(R)$, for each $r \in R$ (see, e.g., [4]). Hence we can write $rg + 1 - r = (e_1g_1 + \cdots + e_sg_s)(1 + f + \sum_{g\in G\setminus\{1\}} f_g g)$ where $f, f_g \in N(R)$ and $f + \sum_{g\in G\setminus\{1\}} f_g = 0$.

It is obviously seen that $r = e' + f'$ for some $e' \in id(R)$ and $f' \in N(R)$ because the ring coefficients in the right hand-side are combinations of sums of orthogonal idempotents and nilpotents. Thus $r \in id(R) + N(R)$, as expected.

Conversely, $r \in id(R) + N(R)$ ensures that $r^2 - r \in N(R)$, so that $(2r-1)^2 = 4r^2 - 4r + 1 = 4(r^2 - r) + 1 \in 1 + N(R) \subseteq U(R)$ whence $2r - 1 \in U(R)$ as stated.

**Case 2**: $|G| = 3$ and $G = \langle g \rangle = \{1, g, g^2\}$ with $g^3 = 1$.

For every $y \in V(RG)$ we have $y = 1 - r - f + rg + fg^2$ exactly when $1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R)$ for all $r, f \in R$ (see, for instance, [4]). Consequently, we write $1 - r - f + rg + fg^2 = (e_1g_1 + \cdots + e_sg_s)(1 + d + \sum_{a\in G\setminus\{1\}} d_a a)$ where $d, d_a \in N(R)$ and $d + \sum_{a\in G\setminus\{1\}} d_a = 0$. It is readily seen that $r = e' + d'$ and $f = e'' + d''$ for some $e', e'' \in id(R)$ and $d', d'' \in N(R)$ because the ring coefficients in the right hand-side are combinations of sums of orthogonal idempotents and nilpotents.

Conversely, if $r, f \in id(R) + N(R)$ and $rf = 0$ it is an easy technical exercise to check that $r^2 - r \in N(R)$ and $f^2 - f \in N(R)$, whence $1 + 3r^2 + 3f^2 + 3rf - 3r - 3f = 1 + 3(r^2 - r) + 3(f^2 - f) + 3rf \in 1 + N(R) \subseteq U(R)$, as wanted.

$\Leftarrow$. If $G_0 = 1$, then we apply [6] or [7].

**Case 1**: $|G| = 2$ and $G = \langle g \rangle = \{1, g | g^2 = 1\}$.

Suppose $v \in V(RG)$, whence $v = 1 - r + rg$ for some $r \in R$. Thus $2r - 1 \in U(R)$ and hence by assumption $r = e + f$ where $e \in id(R)$ and $f \in N(R)$. Furthermore, since $1 - e + eg$ is obviously a unit, especially $1 - e + eg \in Id(RG)$, one can write $1 - e - f + (e + f)g = 1 - e + eg - f + fg = (1 - e + eg)(1 + (1 -$

$e+eg)^{-1}(-f+fg) = (1-e+eg)(1+(1-e+eg^{-1}(-f+fg) = (1-e+eg)[1-f+fg+ef-efg-efg^{-1}+ef] = (1-e+eg)[1-f+2ef+(f-2ef)g] \in Id(RG)(1+I(N(R)G;G))$, as expected.

**Case 2**: $|G| = 3$ and $G = \langle g \rangle = \{1, g, g^2 | g^3 = 1\}$.

Letting $u \in V(RG)$, we write $u = 1-r-f+rg+fg^2$ for some $r, f \in R$. Thus $1+3r^2+3f^2+3rf-3r-3f \in U(R)$ follows as in [4], whence $r, f \in id(R)+N(R)$ with $rf \in N(R)$. Furthermore, we write $r = e+a$ and $f = t+b$ where $e, t \in id(R)$ with $et = 0$ and $a, b \in N(R)$. Hence $u = 1-e-t-a-b+eg+ag+tg^2+bg^2$ and because $1-e+eg-t+tg^2 = 1-e-t+eg+tg^2 \in Id(RG) \subseteq V(RG)$ with the inverse $1-e-t+eg^{-1}+tg^{-2}$, one can write $u = (1-e-t+eg+tg^2)[1+(1-e-t+eg^{-1}+tg^{-2})(-a-b+ag+bg^2)] \in Id(RG) \times (1+I(N(R)G;G))$, as required.

Finally, for any $r \in R$, we find the following equivalence: $r^2 - r \in N(R) \iff r \in id(R) + N(R)$. In fact, the sufficiency is self-evident. As for the necessity, observe that $r + N(R) \in id(R/N(R))$. Hence, via our previous comments on lifting idempotents, there is $e \in id(R)$ with $r+N(R) = e+N(R)$. Thus $r \in id(R) + N(R)$, and we are done. $\square$

*Remark* 14. It is worth noting that the preceding theorem can also be obtained directly by Proposition 5 and by the corresponding result from [4]. However, the present proof gives another strategy to attack results of this type.

As a valuable consequence, we yield the listed above Theorem 11 of [2].

Corollary 15. *Suppose $R$ is of prime characteristic $p$ and $G \neq 1$. Then $V(RG) = Id(RG) \times (1 + I(N(R)G;G))$ if and only if exactly one of the following holds:*

(i) $G_0 = 1$*;*

(ii) $|G| = 2$*,* $\forall\, r \in R : 2r-1 \in U(R) \iff r^2 - r \in N(R)$*;*

(iii) $|G| = 3$*,* $\forall\, r, f \in R$*:* $1+3r^2+3f^2+3rf-3r-3f \in U(R) \iff r^2 - r \in N(R)$*,* $f^2 - f \in N(R)$ *and* $rf \in N(R)$*.*

*Proof.* Since $char(R) = p$ is a prime, $inv(R)$ contains of all primes but $p$. If $G_0 \neq G_p$, we derive that $supp(G) \cap inv(R) \neq \emptyset$ and so we can apply Theorem 13 to finish the proof. Otherwise, if $G_0 = G_p$, then one can derive that $V(RG) = Id(RG)V_p(RG)$ (see, for example, [1]). Hereafter, we can proceed as in [2] by considering the case $G = G_0$, which leads to $|G| = p = 2$ and $R = \{0, 1\} + N(R)$ that is contained in point (ii), and the case $G \neq G_0$ which is impossible. $\square$

Before stating and proving our next statement, we need one more technicality (cf. [1]).

Lemma 16. *If $char(R) = p$ is prime, then the following equivalence holds:*

$$V(RG) = Id(RG)V_p(RG) \iff V(R(G/G_p)) = Id(R(G/G_p))V_p(R(G/G_p)).$$

*Proof.* Consider the natural map $\psi : G \to G/G_p$. It can be linearly extended in a usual way to the map $\Psi : RG \to R(G/G_p)$. It is easy to see that $\Psi$ is actually an epimorphism (= a surjective homomorphism) with kernel equals the relative augmentation ideal $I(RG; G_p)$ of $RG$ with respect to $G_p$. Since $I(RG; G_p)$ is obviously a nil-ideal, it is not hard to verify that the restrictions $\Psi : V(RG) \to V(R(G/G_p))$ and $\Psi : V_p(RG) \to V_p(R(G/G_p))$ are also surjections. Moreover, it follows directly also that $\Psi : Id(RG) \to Id(R(G/G_p))$ is a surjection.

And so, concerning the necessity, it follows by what we have shown above under taking in both sides the homomorphism $\Psi$.

Dealing now with the sufficiency, $\Psi(V(RG)) = \Psi(Id(RG))\Psi(V_p(RG)) = \Psi(Id(RG)V_p(RG))$. Since $\ker \Psi_{V(RG)} = 1 + I(RG; G_p)$ is a $p$-group that is $1 + I(RG; G_p) \subseteq V_p(RG)$, it follows at once that $V(RG) = Id(RG)V_p(RG)$, as stated. □

Another interesting consequence is that of [1]:

Corollary 17. *Suppose $char(R) = p$ is a prime and $G \neq 1$. Then $V(RG) = Id(RG)V_p(RG)$ if and only if precisely one of the following clauses is valid:*

(a) $G_0 = G_p$;

(b) $G = G_p \times G_2$, $|G_2| = 2$ *and for all* $r \in R$: $2r - 1 \in U(R) \iff r^2 - r \in N(R)$;

(c) $G = G_p \times G_3$, $|G_3| = 3$ *and for all* $r, f \in R$: $1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R) \iff r^2 - r \in N(R)$, $f^2 - f \in N(R)$ *and* $rf \in N(R)$.

*Proof.* By virtue of Lemma 16, we may assume that $G_p = 1$. Since it is plainly checked that $V_p(RG) = 1 + I(N(R)G; G)$, we can write that $V(RG) = Id(RG) \times (1 + I(N(R)G; G))$. Henceforth, we employ Theorem 11 or Corollary 15. □

*Remark* 18. In [2, Theorem 5 (c) and Corollary 6 (c)], the condition $r^2 = r$ should be written and read as $r^2 - r \in N(R)$. Note also that if $p = 2$, then $2r - 1 = -1 \in U(R)$ is always fulfilled for every $r \in R$ and thus in [2, p. 24, Theorem] the condition (3) implies condition (2), so that (2) being decidable from (3) is unnecessary and is listed only for completeness (see also [2, p. 27, Remark]. Compare also with Theorem 11 stated above. Moreover, in [4, Proposition 3 and Theorem 5], $inv(R)$ should be written and read as $inv(K)$ where $K$ is any indecomposable subring of $R$.

On the other hand, Mollov and Nachev established in [11] the above corollary when $Id(RG) = G$. However, there is no part of novelty in their ideas and they duplicated these from [1] and some other previous author's papers, so that their article is at all redundant.

Finally, we shall demonstrate that some of the conditions in results from [2] can be equivalently stated in other suitable forms. This is substantiated via the following:

Proposition 19. *Let $R$ be a commutative unital ring of prime characteristic $p$.*

(1) *Suppose $p = 2$. Then $R = id(R) + N(R)$ if and only if $\forall\ r \in R$ : $2r - 1 \in U(R) \iff r^2 - r \in N(R)$.*

(2) *Suppose $p \neq 2$ and $id(R) = \{0, 1\}$. Then $U(R) = \pm 1 + N(R)$ if and only if $\forall\ r \in R : 2r - 1 \in U(R) \iff r^2 - r \in N(R)$.*

*Proof.* (1) Since $2r - 1 = -1 \in U(R)$ is ever fulfilled for each $r \in R$, the assertion is equivalent to $R = id(R) + N(R)$ precisely when $r^2 - r \in N(R)$ for any $r \in R$. The necessity is straightforward. As for the sufficiency, $r^2 - r \in N(R)$ forces that $(r^2 - r)^{2^n} = r^{2 \cdot 2^n} - r^{2^n} = 0$, i.e., $(r^{2^n})^2 = r^{2^n}$ for some natural $n$. Thus $r^{2^n} = e$ is an idempotent. But $(r - e)^{2^n} = r^{2^n} - e = 0$ and hence $r - e \in N(R)$ with $r = e + \alpha \in id(R) + N(R)$ for some $\alpha \in N(R)$, as required.

(2) First, let $U(R) = \pm 1 + N(R)$, and choose $2r - 1 \in U(R)$. Hence $2r - 1 = 1 + \alpha_1$ or $2r - 1 = -1 + \alpha_2$ where $\alpha_1, \alpha_2 \in N(R)$. Thus $2(r - 1) = \alpha_1$ or $2r = \alpha_2$. Since $(2, p) = 1$ it follows that $2u + pv = 1$ for some integers $u, v$. Furthermore, $2(r - 1)u + p(r - 1)v = r - 1$ that is $\alpha_1 u = r - 1$, i.e., $r = 1 + \alpha_1 u$, or $2ru + prv = r$ that is $\alpha_2 u = r$. In the first case $r = 1 + \alpha_1 u$ and $r^2 = 1 + 2\alpha_1 u + \alpha_1^2 u^2$, whence $r^2 - r = \alpha_1 u + \alpha_1^2 u^2 \in N(R)$, whereas in the second case $r^2 - r = \alpha_2^2 u^2 - \alpha_2 u \in N(R)$, as asserted.

Second, let the equivalence " $\iff$ " hold. Since $(2, p) = 1$, it follows that $2 \in U(R)$. If $p \geq 5$, we observe that $2.\frac{3}{2} - 1 = 2 \in U(R)$ and hence $(\frac{3}{2})^2 - \frac{3}{2} = \frac{9}{4} - \frac{3}{2} = \frac{3}{4} \in N(R)$. Thus $3 \cdot 2^2 \in N(R)$, i.e., $3 \in N(R)$. Moreover, $(3, p) = 1$ whence $3 \in U(R)$. Finally, $1 \in N(R)$ which leads to $1 = 0$, a contradiction. If now $p = 3$, for any $\alpha \in U(R)$ we have that $2(\alpha - 1) - 1 = 2\alpha \in U(R)$ because $2 \in U(R)$. Therefore, $(\alpha - 1)^2 - (\alpha - 1) \in N(R)$ which means that $(\alpha - 1)^{2 \cdot 3^n} = (\alpha - 1)^{3^n}$ for some natural $n$. This ensures that $(\alpha - 1)^{3^n}$ is an idempotent. Consequently, $(\alpha - 1)^{3^n} = 0$ or $(\alpha - 1)^{3^n} = 1$. The first equality assures that $\alpha - 1 \in N(R)$, i.e., $\alpha \in 1 + N(R)$, while the second equality insures that $(\alpha - 1)^{3^n} - 1 = (\alpha - 1 - 1)^{3^n} = (\alpha - 2)^{3^n} = (\alpha + 1)^{3^n} = 0$, i.e., we finally obtain $\alpha + 1 \in N(R)$ and $\alpha \in -1 + N(R)$, as claimed. $\square$

*Remark* 20. Actually, in (2) we should have $p = 3$.

As an immediate consequence we have the following:

Corollary 21. *Let $R$ be a commutative unital ring of prime characteristic $p$.*

(1) *Suppose $p = 2$. Then $|R| = 2$ if and only if $\forall\ r \in R : 2r - 1 \in U(R) \iff r \in \{0, 1\}$.*

(2) *Suppose $p \neq 2$ and $id(R) = \{0, 1\}$. Then $|U(R)| = 2$ if and only if $\forall\ r \in R : 2r - 1 \in U(R) \iff r \in \{0, 1\}$.*

We close the work with a left-open challenging problem.

PROBLEM 22. Suppose $supp(G) \cap inv(R) = \emptyset$. Find a necessary and sufficient condition whenever the following direct decomposition holds:

$$V(RG) = Id(RG) \times (1 + I(N(R)G; G)).$$

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