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ON BLOCKS AND CLIFFORD EXTENSIONS

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Abstract. We give a short proof of a result of E.C. Dade, as stated in [1, Theorem 9] on Clifford extensions for blocks of group algebras (see also [2, Corollary 12.6], avoiding the machinery developed in [2], but making use of the Brauer homomorphism. Moreover, we do not assume that the ground field k is algebraically closed.

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1. INTRODUCTION AND PRELIMINARIES

The Clifford extension of a block was introduced by E.C. Dade in [1], where he also stated that this extension can be computed from local data. This result was proved in [2, Corollary 12.6]. The paper [2] is quite long and technical, and our aim here is to give a short proof of [1, Theorem 9]. We start by introducing the setting and recalling the definitions. The reader is referred to [7] for block theory, and to [3] for notions and results on group graded algebras.

Let p be a prime number, and let $\mathcal{O}$ be a complete discrete valuation ring with residue field k of characteristic p. Note that the situation $\mathcal{O} = k$ is allowed, and we do not make any assumption on the size of $\mathcal{O}$ and k. Let K be a normal subgroup of the finite group H, and denote $G = H/K$. Consider the group algebra $\mathcal{O}H$. This is a strongly G-graded algebra, where for each $\sigma \in G$, $\mathcal{O}H_\sigma = \mathcal{O}\sigma$.

Let b a block of $\mathcal{O}K$; this primitive central idempotent remains central in the G-graded algebra

$$C_{\mathcal{O}H}(\mathcal{O}K) = (\mathcal{O}H)^K = \bigoplus_{\sigma\in G}(\mathcal{O}H)^K_\sigma,$$

where $(\mathcal{O}H)^K_\sigma = (\mathcal{O}\sigma)^K$ for all $\sigma \in G$. Since K is normal in H, the group H acts by conjugation on $(\mathcal{O}H)^K$, and this action induces an action of G on $(\mathcal{O}H)^K$. Let G_b denote the stabilizer of b in G. Then

$$b\mathcal{O}Hb = \bigoplus_{\sigma\in G_b} b\mathcal{O}\sigma = b\mathcal{O}H_b$$

is a strongly G_b-graded algebra.

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Define

$$G[b] = \{\sigma \in G \mid b(\mathcal{O}H)^K_\sigma \cdot b(\mathcal{O}H)^K_{\sigma^{-1}} = b(\mathcal{O}H)^K_1\}.$$

It is easy to see that $G[b]$ is a normal subgroup of G_b, and that

$$A := \bigoplus_{\sigma\in G[b]} b(\mathcal{O}H)^K_\sigma$$

is a strongly $G[b]$-graded G_b-acted subalgebra of $b\mathcal{O}Hb$.

Because $A_1 = b(\mathcal{O}K)^K = bZ(\mathcal{O}K)$ is a local ring, $\hat{k}_1 := A_1/J(A_1)$ is a finite extension of the field k. Consider the strongly $G[b]$-graded algebra $\bar{A} := A/AJ(A_1)$; for all $\sigma \in G[b]$, we have $\bar{A}_\sigma = A_\sigma/A_\sigma J(A_1)$. By definition, the *Clifford extension* of the block b is the group extension

$$1 \to \hat{k}_1^* \to hU(\bar{A}) \to G[b] \to 1 \tag{1}$$

associated to the crossed product $\bar{A}$ of $\hat{k}_1$ and $G[b]$.

2. THE BRAUER HOMOMORPHISM

In order to define the second extension associated to the block b, we consider the Brauer homomorphism

$$\mathrm{Br}_D : (\mathcal{O}H)^D \to kC_H(D)$$

associated to a p-subgroup D of H, and we denote it in the same way when restricted to $(\mathcal{O}K)^D$. Note that $N_H(D)$ acts by conjugation on both $(\mathcal{O}K)^D$ and $kC_H(D)$, and the Brauer homomorphism Br_D is a homomorphism of $N_H(D)$-algebras.

Now choose a defect pointed group D_γ of b in K, so γ is a local point of $(\mathcal{O}K)^D$ determining the maximal Brauer pair (D, e). The idempotent e is primitive in the center of $kC_K(D)$, and $\mathrm{Br}_D(b)e = e$. Also denote $\bar{b} := \mathrm{Br}_D(b)$ the Brauer correspondent of b which is a primitive idempotent of $kC_K(D)^{N_K(D)}$.

The group algebra $kC_H(D)$ is a $N_H(D)$-algebra by conjugation and it is also $C_H(D)/C_K(D)$-graded. Denote by $C_H(D)_{\bar{b}}/C_K(D)$ the stabilizer of $\bar{b}$.

Let $kC_H(D)^{N_K(D)}$ be the subalgebra of $kC_H(D)$ consisting of elements fixed by the action of $N_K(D)$. Consider the centralizer

$$C_{kC_H(D)^{N_K(D)}}(kC_K(D)^{N_K(D)}) = kC_H(D)^{N_K(D)},$$

where for each $\tau \in C_H(D)/C_K(D)$ the component τ is

$$C_{kC_H(D)^{N_K(D)}}(kC_K(D)^{N_K(D)}) \cap k\tau = (k\tau)^{N_K(D)}.$$

This centralizer is a $C_H(D)/C_K(D)$-graded algebra that contains the idempotent $\bar{b}$ in its center. As above

$$\bar{b}kC_H(D)\bar{b} = \bigoplus_{\tau\in C_H(D)_{\bar{b}}/C_K(D)} \bar{b}k\tau = \bar{b}kC_H(D)_{\bar{b}}$$

is a strongly $C_H(D)_{\bar{b}}/C_K(D)$-graded algebra. Its subalgebra

$$(\bar{b}kC_H(D)_{\bar{b}})^{N_K(D)} = \bigoplus_{\tau\in C_H(D)_{\bar{b}}/C_K(D)} (\bar{b}k\tau)^{N_K(D)}$$

needs to be strongly graded.

So we now introduce the following normal subgroup of $C_H(D)_{\bar{b}}/C_K(D)$:

$$\begin{aligned}&C_H(D)_0/C_K(D)\\&=\{\tau\in C_H(D)/C_K(D)\mid \bar{b}(k\tau)^{N_K(D)}\cdot\bar{b}(k\tau^{-1})^{N_K(D)}=\bar{b}(kC_K(D))^{N_K(D)}\}.\end{aligned}$$

Then

$$B := \bigoplus_{\tau\in C_H(D)_0/C_K(D)} \bar{b}(k\tau)^{N_K(D)} = \bar{b}kC_H(D)_0^{N_K(D)}$$

is a strongly $C_H(D)_0/C_K(D)$-graded algebra, and so is $\overline{B}=B/BJ(B_1)$, where $B_1=\bar{b}kC_K(D)^{N_K(D)}$. Clearly we have $B_\tau=\bar{b}(k\tau)^{N_K(D)}$, $\overline{B_\tau}=B_\tau/B_\tau J(B_1)$ for all $\tau\in C_H(D)_0/C_K(D)$.

Moreover, B_1 is a local ring, hence $\hat{k}_2 := B_1/J(B_1)$ is a finite extension of k. Then $\overline{B}$ is a crossed product of $\hat{k}_2$ and $C_H(D)_0/C_K(D)$, acted upon by $N_H(D)_{\bar{b}}$, and it corresponds to the *Clifford extension* of $\bar{b}$, that is

$$1\to\hat{k}_2^*\to hU(\bar{B})\to C_H(D)_0/C_K(D)\to 1. \qquad (2)$$

Note that $N_K(D)_{\bar{b}} := N_H(D)_{\bar{b}}\cap K$ acts trivially on both $\bar{A}$ and $\bar{B}$. Finally, relying on the inclusions

$$\begin{aligned}&kC_K(D)^{N_K(D)}\subseteq Z(kC_K(D)),\\&C_{kC_H(D)^{N_K(D)}}(kC_K(D)^{N_K(D)})=kC_H(D)^{N_K(D)}\subseteq C_{kC_H(D)}(kC_K(D)),\end{aligned}$$

we can replace $\bar{b}$ with the block e and still have that $e\overline{B}=eB/eBJ(eB_1)$ is a crossed product of $\hat{k}_2$ by $C_H(D)_0/C_K(D)$. Of course this last factor algebra is acted upon by $N_H(D)_e$.

3. ISOMORPHISM OF THE CLIFFORD EXTENSIONS

We are now ready to give an alternative proof of [2, Corollary 12.6].

THEOREM 1. *With the above notations, the following statements hold:*

1) *G_b equals $N_H(D)_eK/K$.*
2) *The group $G[b]$ equals $C_H(D)_0K/K$.*
3) *The extensions* (1) *and* (2) *are isomorphic.*
4) *The isomorphism between the extensions* (1) *and* (2) *is compatible with the natural isomorphism*

$$G[b]\to C_H(D)_0/C_K(D),\qquad \sigma\mapsto\sigma\cap C_H(D)_0,$$

and preserves the conjugation action of the subgroup $N_H(D)_e/N_K(D)_e$ of $G_b\simeq N_H(D)_e/N_K(D)_e$ on the two extensions.

Proof. Since H acts on the inclusion relation between pointed groups, it follows that H_b acts on $D_\gamma \leq K_{\{b\}}$, so it generates the class $\{(D_\gamma)^h \mid h \in H_b\}$ of defect pointed groups of b. Since $K_{\{b\}}$ is projective relative to $(D_\gamma)^h$ for some $h \in H_b$, by [7, Lemma 18.2] it follows that there is $g \in K$ satisfying $(D_\gamma)^g \leq (D_\gamma)^h$. Equivalently, we have $D_\gamma \leq (D_\gamma)^{hg^{-1}}$, and using the maximality of D_γ, the last inclusion is actually an equality, hence $hg^{-1} \in N_{H_b}(D)$. Together with $N_{H_b}(D) = N_H(D)_b$, we obtain $H_b \subseteq N_H(D)_b K$. Since the other inclusion is trivial, we conclude that $H_b = N_H(D)_b K$. Clearly, $N_H(D)_b = N_H(D)_e N_K(D)$, and this implies the equality $G_b = N_H(D)_e K/K$.

For the remaining statements we argue as follows.

By [5, Lemma 3.4], we know that A_1 maps onto $\mathrm{Br}_D(b)kC_K(D)^{N_K(D)}$. Multiplying by e, we obtain an epimorphism of algebras from A_1 onto

$$eB_1 = Z(ekC_K(D)^{N_K(D)}) = ekC_K(D)^{N_K(D)}.$$

Moreover, the composition

$$f : b\mathcal{O}H_b^K \to \bar{b}kC_H(D)_b^{N_K(D)} = \bar{b}kC_H(D)_{\bar{b}}^{N_K(D)} \to ekC_H(D)_{\bar{b}}^{N_K(D)}$$

is an epimorphism of $N_H(D)_e$-algebras. This can easily be shown using the same [5, Lemma 3.4] taking into consideration that $b \in \mathcal{O}K_D^K \subseteq (\mathcal{O}H_b)_D^K$. Also it is quite clear that this morphism carries the surjection componentwise, i.e. if $\sigma \in G_g$ and $\tau = \sigma \cap C_H(D)_{\bar{b}}$ then $b(\mathcal{O}\sigma)^K \to e(k\tau)^{N_K(D)}$. Of course, $A \subseteq b\mathcal{O}H_b^K$ and $eB \subseteq ekC_H(D)_{\bar{b}}^{N_K(D)}$, and choosing $\sigma \in G[b]$ we have

$$eB_1 = f(A_1) = f(A_\sigma \cdot A_{\sigma^{-1}}) = f(A_\sigma) \cdot f(A_{\sigma^{-1}}) = eB_\tau \cdot eB_{\tau^{-1}}.$$

So the restriction of f sends A to eB. Now if $\tau \in C_H(D)_0/C_K(D)$ then using [6, Lemma 1.1] there is an invertible element $\bar{d} \in eB_\tau \cap (eB)^*$ such that $eB_\tau = \bar{d}(eB_1) = (eB_1)\bar{d}$ which lifts to an invertible element $d \in A_\sigma \cap A^*$ where $\sigma \cap C_H(D)_0 = \tau$. Then

$$A_1 = A_\sigma \cdot A_{\sigma^{-1}},$$

hence $\sigma \in G[b]$. This means that the restriction to A of the homomorphism f remains surjective onto eB.

The inclusion $f(J(A_1))) \subseteq J(eB_1)$ allows us to consider the $G[b]$-graded algebra homomorphism

$$\bar{f} : \bar{A} \to \overline{eB}, \quad \bar{a} \mapsto \overline{f(a)}.$$

Also, the same inclusion, together with the fact that A_1 and eB_1 are local rings, applying [4, Proposition 3.23], we obtain

$$A_1/J(A_1) \simeq eB_1/J(eB_1),$$

that is, $\hat{k}_1 \simeq \hat{k}_2$ as extensions of k.

If $\tau \in C_H(D)_0/C_K(D)$ we have $eB_\tau \cdot eB_{\tau^{-1}} = eB_1$, since eB_1 is local. Then

$$f^{-1}(eB_\tau) \cdot f^{-1}(eB_{\tau^{-1}}) \nsubseteq J(A_1),$$

because otherwise

$$eB_\tau \cdot eB_{\tau^{-1}} = f(f^{-1}(eB_\tau) \cdot f^{-1}(eB_{\tau^{-1}})) \subseteq f(J(A_1)) \subseteq J(eB_1),$$

which is false. We have

$$J(A_1) + f^{-1}(eB_\tau) \cdot f^{-1}(eB_{\tau^{-1}}) = A_1,$$

and since

$$f^{-1}(eB_\tau) \cdot f^{-1}(eB_{\tau^{-1}}) = A_{\tau K} \cdot A_{\tau^{-1}K} \subseteq A_1,$$

it follows that for $\sigma = \tau K$ we obtain

$$J(A_1) + A_\sigma \cdot A_{\sigma^{-1}} = A_1.$$

Consequently, $\sigma \in G[b]$, and this proves the inclusion $C_H(D)_0K/K \leq G[b]$.

Conversely, if $\sigma \in G[b]$ then the equality

$$\bar{f}(\bar{A}_\sigma) \cdot \bar{f}(\bar{A}_{\sigma^{-1}}) = \hat{k}_2$$

forces $\bar{f}(\bar{A}_\sigma) \neq 0$. Consequently, $(\mathcal{O}\sigma)^K \nsubseteq (\mathcal{O}\sigma)^K J(bZ(\mathcal{O}K))$, and even more, $\sigma \cap C_H(D)_0 \neq \emptyset$. Hence $\sigma \in C_H(D)_0K/K$.

Finally, the conjugation action of $N_K(D)_e$ on $\hat{k}_1$ on $\hat{k}_2$, as well as on the two crossed product algebras corresponding to the extensions (1) and (2) is trivial, and use also that the Brauer homomorphism is a map of $N_H(D)_e$-algebras. □

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