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ON HARMONIC MAPPINGS LIFTING TO MINIMAL SURFACES

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Abstract. The projection on the base plane of a regular minimal surface S in $\mathbb{R}^3$ with isothermal parameters defines a complex-valued univalent harmonic function f. We obtain distortion theorems for the Weierstrass-Enneper parameters and the Gaussian curvature of the minimal surface S, provided that the corresponding univalent harmonic function f belongs to the class $\mathcal{S}_{\mathrm{H}}^{*}$.

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1. INTRODUCTION

Minimal surfaces are most commonly known as which have the minimum area amongst all other surfaces spanning a given closed curve in $\mathbb{R}^3$. Geometrically, the definition of a minimal surface is that the mean curvature H is zero at every point of the surface. If locally one can write the minimal surface in $\mathbb{R}^3$ as $(x, y, \Phi(x, y))$ the minimal surface equation H = 0 is equivalent to

$$(1 + \Phi_y^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (1 + \Phi_x^2)\Phi_{yy} = 0.$$

There exists a choice of isothermal parameters $(u, v) \in \Omega \subset \mathbb{R}^2$ so that the surface $X(u, v) = (x(u, v), y(u, v), \Phi(u, v)) \in \mathbb{R}^3$ satisfying the minimal surface equation is given by

$$E = |X_u|^2 = |X_v|^2 = G > 0, \quad F =< X_u, X_v >= 0, \quad \triangle_{(u,v)}X = 0$$

(where Δ denotes the Laplacian operator). The general solution of such an equation is called the local Weierstrass-Enneper representation [2].

A complex-valued function f which is harmonic in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ has the canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(z_0) = 0$ for some prescribed point $z_0 \in \mathbb{D}$. According to a theorem of H. Lewy [1]; f is locally univalent if and only if its Jacobian $(|f_z|^2 - |f_{\overline{z}}|^2 = |h^{'}(z)|^2 - |g^{'}(z)|^2)$ does not vanish. f is said to be sense-preserving if its Jacobian is positive. In this case $h^{'}(z)$ does not vanish and the analytic function $\omega(z) = \frac{g'(z)}{h'(z)}$, called the second dilatation of f, has the property $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Throughout this paper we will assume that f is locally univalent sense -preserving, and we call f a harmonic mapping.

A harmonic mapping $f = h + \overline{g}$ can be lifted locally to a regular minimal surface given by conformal (or isothermal) parameters if and only if its dilatation is the square of an analytic function $\omega(z) = q^2(z)$ for some analytic function q with $|q(z)| < 1$. Equivalently, the requirement is that any zero of ω be of even order, unless $\omega \equiv 0$ on its domain, so that there is no loss of generality in supposing that z ranges over the unit disc $\mathbb{D}$, because any other isothermal representation can be precomposed with a conformal map from the unit disc $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem. For such a harmonic mapping $f = u + \mathrm{i}v$, the minimal surface has the Weierstrass-Enneper representation with parameters (u, v, t) given by

$$\begin{aligned} u &= \mathrm{Re}\,\{f(z)\} = \mathrm{Re}\left\{\int_0^z \varphi_1(\zeta)\mathrm{d}\zeta\right\}, \\ v &= \mathrm{Im}\,\{f(z)\} = \mathrm{Re}\left\{\int_0^z \varphi_2(\zeta)\mathrm{d}\zeta\right\}, \\ t &= \mathrm{Re}\left\{\int_0^z \varphi_3(\zeta)\mathrm{d}\zeta\right\}, \end{aligned} \tag{1}$$

for $z \in \mathbb{D}$ with

$$\begin{aligned} &\varphi_1 = h' + g' = p(1+q^2) = \frac{\partial u}{\partial z}, \\ &\varphi_2 = -\mathrm{i}(h' - g') = -\mathrm{i}p(1-q^2) = \frac{\partial v}{\partial z}, \\ &\varphi_3 = -2\mathrm{i}pq = \frac{\partial t}{\partial z}, \quad \varphi_3^2 = -4\omega(h')^2 \quad \text{and} \quad h' = p. \end{aligned} \tag{2}$$

See [1] and [4, p. 176].

The metric of the surface has the form $\mathrm{d}s = \lambda|\mathrm{d}z|$, where $\lambda = \lambda(z) > 0$. Here, the function λ takes the form

$$\lambda = |h'| + |g'| = |h'|(1 + |\omega|) = |p|(1 + |q|^2)\,. \tag{3}$$

A general theorem of differential geometry says that if any regular surface is represented by conformal parameters (or isothermal parameters) so that its metric has the form $\mathrm{d}s = \lambda|\mathrm{d}z|$ for some positive function λ, then the Gauss curvature of the surface is $\mathrm{K} = -\lambda^{-2}\Delta(\log\lambda)$. This quantity K is also known as the curvature of the metric. In our special case of a minimal surface associated with a harmonic mapping $f = h + \overline{g}$, the formula for curvature reduces to

$$\mathrm{K} = -\frac{4|q'|^2}{|p|^2(1 + |q|^2)^4}\,. \tag{4}$$

Since the underlying harmonic mapping f has dilatation $\omega = \frac{g'}{h'} = q^2$ and $h' = p$. An equivalent expression is the following

$$\mathrm{K} = -\frac{|\omega'|^2}{|h'g'|(1 + |\omega|)^4}. \tag{5}$$

Now we define the following class of harmonic functions [2], which is used throughout this paper.

Let $h(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ be analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The class of all sense-preserving harmonic functions in $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by $\mathcal{S}_{\mathrm{H}}$. Thus $\mathcal{S}_{\mathrm{H}}$ contains the standard class $\mathcal{S}$ of analytic functions. See [3] and [4].

Let $s(z) = z + c_2 z + c_3 z^2 + \cdots$ be analytic function in the open unit disc $\mathbb{D}$. If $s(z)$ satisfies the condition

$$\mathrm{Re}\left[z\frac{s'(z)}{s(z)}\right] > 0, \quad (z \in \mathbb{D}). \tag{6}$$

then $s(z)$ is called starlike function in $\mathbb{D}$, and the class of starlike functions in $\mathbb{D}$ is denoted by $\mathcal{S}^*$.

Let Ω be the family of functions $\phi(z)$ which are regular and satisfy the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for every $z \in \mathbb{D}$, and let $\Omega(a)$, where $a = |b_1|$, be the class of functions $\omega(z)$ which are analytic in $\mathbb{D}$ and satisfy $\omega(0) = b_1 \neq 0$, $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. We note that $\Omega_{\cup}$ be the union of all classes $\Omega(a)$ whereas a ranges over $(0, 1)$.

We denote by $\mathcal{S}^*_{\mathrm{H}}$ the subclass of $\mathcal{S}_{\mathrm{H}}$ consisting of all univalent harmonic functions whose analytic part is starlike.

2. MAIN RESULTS

Lemma 1. *Let* ω *be an element of* $\Omega_{\cup}$. *Then*

$$\frac{|a-r|}{1-ar} \leq |\omega(z)| \leq \frac{a+r}{1+ar}. \tag{7}$$

Proof. The inequality (7) is clear for $z = 0$, whence $r = |z| = 0$. Now, let $z \in \mathbb{D}\setminus\{0\}$, and define $b_1 = a\mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in \mathbb{R}$. Now we consider the function

$$\phi(z) = \frac{\mathrm{e}^{-\mathrm{i}\theta}\omega(z) - a}{1 - a\mathrm{e}^{-\mathrm{i}\theta}\omega(z)}, \quad z \in \mathbb{D}.$$

This function satisfies the conditions of Schwarz's lemma. The estimation of Schwarz's lemma, $|\phi(z)| \leq |z| = r$, gives

$$|\phi(z)| = \left|\frac{\mathrm{e}^{-\mathrm{i}\theta}\omega(z) - a}{1 - a\mathrm{e}^{-\mathrm{i}\theta}\omega(z)}\right| \leq r \Rightarrow |\mathrm{e}^{-\mathrm{i}\theta}\omega(z) - a| \leq r|1 - a\mathrm{e}^{-\mathrm{i}\theta}\omega(z)|. \tag{8}$$

The inequality (8) is equivalent to

$$\left|\mathrm{e}^{-\mathrm{i}\theta}\omega(z) - \frac{a(1-r^2)}{1-a^2r^2}\right| \leq \frac{r(1-a^2)}{1-a^2r^2}. \tag{9}$$

The equality holds in the inequality (9) only for the function

$$\omega(z) = \mathrm{e}^{\mathrm{i}\theta}.\frac{\mathrm{e}^{\mathrm{i}\varphi}z + a}{1 + a\mathrm{e}^{\mathrm{i}\varphi}z}, \quad z \in \mathbb{D},\ \varphi \in \mathbb{R}.$$

If we use the triangle inequality in the inequality (9), we get

$$\left||\mathrm{e}^{-\mathrm{i}\theta}\omega(z)| - \left|\frac{a(1-r^2)}{1-a^2r^2}\right|\right| \leq \left|\mathrm{e}^{-\mathrm{i}\theta}\omega(z) - \frac{a(1-r^2)}{1-a^2r^2}\right| \leq \frac{r(1-a^2)}{1-a^2r^2}.$$

Therefore, we have

$$-\frac{r(1-a^2)}{1-a^2r^2} \le |\mathrm{e}^{-\mathrm{i}\theta}\omega(z)| - \left|\frac{a(1-r^2)}{1-a^2r^2}\right| \le \frac{r(1-a^2)}{1-a^2r^2},$$

$$-\frac{r(1-a^2)}{1-a^2r^2} + \left|\frac{a(1-r^2)}{1-a^2r^2}\right| \le |\mathrm{e}^{-\mathrm{i}\theta}\omega(z)| \le \frac{r(1-a^2)}{1-a^2r^2} + \left|\frac{a(1-r^2)}{1-a^2r^2}\right|,$$

and this last inequalities are equivalent to

$$\frac{a-r}{1-ar} \le |\omega(z)| = |\mathrm{e}^{-\mathrm{i}\theta}\omega(z)| \le \frac{a+r}{1+ar}\,. \tag{10}$$

Similarly, if we replace a with r in the inequality (8), we finally get

$$\frac{r-a}{1-ar} \le |\omega(z)| \le \frac{a+r}{1+ar}. \tag{11}$$

From the inequalities (10) and (11), we obtain (7). □

Corollary 1. *If $\omega \in \Omega_{\cup}$, then*

$$\frac{(1-a)(1-r)}{1+ar} \le (1-|\omega(z)|) \le \frac{1-ar-|a-r|}{1-ar} \tag{12}$$

and

$$\frac{1-ar+|a-r|}{1-ar} \le 1+|\omega(z)| \le \frac{(1+a)(1+r)}{1+ar} \tag{13}$$

for all $|z| = r < 1$.

Proof. These inequalities are simple consequences of Lemma 1. □

Corollary 2. *Let $f = h + \overline{g}$ be an element of $\mathcal{S}^*_{\mathrm{H}}$. Then*

$$\frac{(1-r)|a-r|}{(1+r)^3(1-ar)} \le |g'(z)| \le \frac{(1+r)(a+r)}{(1-r)^3(1+ar)}. \tag{14}$$

Proof. Recall that if the analytic part h of f is starlike, then we have

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^3}. \tag{15}$$

On the other hand, if we consider Lemma 1 and the definition of the second dilatation of f, then we can write

$$\frac{|a-r|}{1-ar} \le \left|\frac{g'(z)}{h'(z)}\right| \le \frac{a+r}{1+ar}. \tag{16}$$

Considering the inequalities (15) and (16) together, we obtain (14). □

3. APPLICATIONS TO MINIMAL SURFACES

THEOREM 1. *Let the functions* φ_k *for* $k = 1, 2, 3$, *be the Weierstrass-Enneper parameters of a regular minimal surface* S *and* $f = (h + \overline{g}) \in \mathcal{S}_{\mathrm{H}}^*$ *lifts to the minimal surface* S, *then*

$$\frac{(1-a)(1-r)^2}{(1+ar)(1+r)^3} \leq |\varphi_1| \leq \frac{(1+a)(1+r)^2}{(1+ar)(1-r)^3}, \tag{17}$$

$$\frac{(1-a)(1-r)^2}{(1+ar)(1+r)^3} \leq |\varphi_2| \leq \frac{(1+a)(1+r)^2}{(1+ar)(1-r)^3}, \tag{18}$$

and

$$\frac{4(1-r)^2|a-r|}{(1-ar)(1+r)^6} \leq |\varphi_3|^2 \leq \frac{4(a+r)(1+r)^2}{(1+ar)(1-r)^6}. \tag{19}$$

Proof. Using the formulas (2) and the Corollary 1. we obtain (17), (18) and (19). □

THEOREM 2. *Let* K *be the Gaussian curvature of the regular minimal surface* S *and* $f = (h + \overline{g}) \in \mathcal{S}_{\mathrm{H}}^*$ *lifts to the minimal surface* S, *then*

$$|\mathrm{K}| \leq \frac{(1-ar-|a-r|)^2(1+r)^6(1-ar)^3(1+a)^2}{(1-ar+|a-r|)^4(1-r)^4(1+ar)^2|a-r|}. \tag{20}$$

Proof. Using the Corollary 2. and after the simple calculations we get

$$\frac{(1-r)^6(1+ar)}{(1+r)^2(a+r)} \leq \frac{1}{|g'(z)h'(z)|} \leq \frac{(1+r)^6(1-ar)}{(1-r)^2|a-r|}, \tag{21}$$

and

$$|\mathrm{K}| = \frac{|\omega'(z)|^2}{|g'(z)h'(z)|(1+|\omega(z)|)^4} \leq \frac{|\omega'(z)|^2(1+r)^6(1-ar)}{(1+|\omega(z)|)^4(1-r)^2|a-r|}. \tag{22}$$

On the other hand , if we use the Schwarz-Pick's Lemma for the function

$$\phi(z) = \frac{\omega(z) - \omega(0)}{1 - \overline{\omega(0)}\omega(z)},$$

we obtain

$$|\omega'(z)|^2 \leq \frac{(1-|\omega(z)|^2)^2}{(1-r^2)^2} = \frac{(1-|\omega(z)|)^2(1+|\omega(z)|)^2}{(1-r)^2(1+r)^2}. \tag{23}$$

Considering the inequalities (12), (13), (22) and (23), we obtain (20). □

EXAMPLE 1. Consider the function $f(z) = z - \frac{1}{2}\frac{\bar{z}}{2-\bar{z}}$, $z \in \mathbb{D}$. Since $\triangle f = \frac{4\partial^2 f}{\partial z \partial \bar{z}} = 0$, then f is harmonic.

The functions $h(z) = z$ and $g(z) = -\frac{1}{2}.\frac{z}{2-z}$, the analytic and co- analytic parts of f are analytic in $\mathbb{D}$ and they satisfy $h(0) = g(0) = 0$.

$\mathrm{J}_f(z) = |h^{'}(z)|^2 - |g^{'}(z)|^2 = 1 - \frac{1}{|2-z|^2} > 0$ in $\mathbb{D}$, so f is sense-preserving and univalent.

Furthermore, the analytic part $h(z) = z$ of f is starlike, so f belongs to the class $\mathcal{S}^*_{\mathrm{H}}$. The second dilatation of f is $\omega(z) = \frac{g'(z)}{h'(z)} = -(\frac{1}{2-z})^2$. Since $|\omega(0)| = \frac{1}{4} \in (0,1)$ and $|\omega(z)| < 1$, in $\mathbb{D}$, so $\omega \in \Omega_{\cup}$.

On the other hand $\omega(z)$ is the square of the analytic function $q(z) = \frac{\mathrm{i}}{2-z}$ in $\mathbb{D}$. Thus univalent harmonic function f can lift locally to a (regular) minimal surface.

Now, let find the minimal surface, using by formulas (1), we get $p(z) \equiv 1$ and $q(z) = \frac{\mathrm{i}}{2-z}$. We know that these functions p and q are the Weierstrass-Enneper parameters of the minimal surface Catenoid.

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