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ISSUES OF BASIS PROPERTY RELATED TO THE CHOICE OF ASSOCIATED FUNCTIONS OF DIFFERENTIAL OPERATORS

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Abstract. This publication presents the research conducted on the problem of choosing associated functions of ordinary differential operators of higher orders. Necessary conditions for the unconditional basis of the given transformed system were defined. For systems consisting of root functions of differential operator it is shown that the anti-a priori estimates without the positive power in the right-hand side of inequality are sufficient conditions for unconditional basis property in the space of square summable functions for any choice of associated functions. A new definition of associated functions of linear operators in Hilbert space was proposed.

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1. INTRODUCTION

In the publication [2] Bari N.K. has shown that the complete and minimal in $L_2(G)$ system

$$\psi_{2n-1}(x) = \varphi_{2n-1}(x),$$

$$\psi_{2n}(x) = \frac{1}{\sqrt{1+q_n^2}}[q_n\varphi_{2n-1}(x) + \varphi_{2n}(x)], \quad n = 1, 2, \ldots,$$

where $\{\varphi_n(x)\}$ is a complete orthonormal system, forms a Riesz basis for $|q_n| < M$, $n = 1, 2, \ldots$.

The issue of the basis property of systems of this type also arises in the theory of non-self-adjoint spectral problems with an infinite number of associated functions. The following example of a non-self-adjoint spectral problem [3, 4]:

$$u''(x) + \lambda u(x) = 0, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(0) = u'(1),$$

shows that the complete and minimal system of eigenfunctions and associated functions of this problem $\{x, \sin 2k\pi x, (4k\pi)^{-1} x \cos 2k\pi x + C \sin 2k\pi x\}$ is an unconditional basis in $L_2(0,1)$ only for $C = 0$, and is not a basis for any $C \neq 0$. In other words, for one choice of associated functions ($\{(4k\pi)^{-1} x \cos 2k\pi\}$) the system of eigenfunctions and associated functions $\{x, \sin 2k\pi x, (4k\pi)^{-1} x \cos 2k\pi x\}$ forms a basis, and with a different choice of associated functions ($\{(4k\pi)^{-1} x \cos 2k\pi x + C \sin 2k\pi x\}$) does not. It is clear that associated functions can be determined with accuracy up to terms containing the corresponding eigenfunction.

The eigenvalues of a spectral problem in this example are the numbers $\lambda_k = (2k\pi)^2$, $k = 0, 1, 2, \dots$. For $k \geq 1$ all eigenvalues are double so each eigenvalue corresponds to a two-dimensional root subspace H_k, $k = 1, 2, 3, \dots$. Each such root subspace consists of all possible linear combinations of functions $\sin 2k\pi x$ and $(4k\pi)^{-1} x \cos 2k\pi x$. Usually, in each root subspace a basis is chosen, which corresponds to the Jordan chains:

$$Lg_0 = \lambda g_0, \quad Lg_i = \lambda g_i + g_{i-1}, \quad i = \overline{1, n},$$

and the basis property of a union of all the chains is examined. The above example illustrates the problem of choosing associated functions.

Studying the identified problem, Ilyin V.A. [3] constructed a theory of so-called reduced system of eigenfunctions and associated functions of ordinary differential operators. This reduced system always forms a basis, if there is a basis for at least one choice of associated functions. However, the construction of such a reduced system is complex.

The problem of choosing associated functions naturally leads to the question: how relationships must meet their own and associated functions, in order to avoid this effect?

In the case of a non-self-adjoint Schrödinger operator

$$\mathrm{L}\, u = -u''(x) + q(x)u(x),$$

defined on a finite interval G of the real axis with complex-valued potential $q(x) \in L_1(G)$ the problem has been solved in publications [7, 8].

The current paper presents the solution of the investigated issue in the case of ordinary differential operators of the higher order.

2. A NECESSARY CONDITION FOR THE BASIS IN THE SPACE L_2

For completeness, we give a well-known fact from the theory of bases in Banach spaces.

THEOREM 1. *Let the system $\{u_n(x)\}$ be the basis for the space L_2 and $\{v_n(x)\}$ – biorthogonal conjugate system. Then*

$$(*) \qquad \|u_n\|_{L_2} \, \|v_n\|_{L_2} \leq const.$$

for all n numbers.

Proof. Indeed, if $f(x) \in L_2$ and $f(x) = \sum c_n u_n(x)$; $c_n = (f, v_n)$, then the convergence of the series $\sum c_n u_n(x)$ in L_2 leads to $||c_n u_n|| \to 0$ for $n \to \infty$, i.e. $(f, \|u_n\| \, v_n) \to 0$. If $(f, \|u_n\| \, v_n) \to 0$ for any $f \in L_2$, then by resonant type theorems [1], the norms in L_2 of $\|u_n\| \, v_n$ functions are limited in the aggregate. The theorem is proved. $\square$

3. MAIN RESULTS

Let us consider an arbitrary system $\{u_{k0}(x), u_{k1}(x)\}$ from $L_2(G)$ class.

THEOREM 2. *Let the system* $\{\ u_{k0}(x),\ \ u_{k1}(x)\}$ *form an unconditional basis of* $L_2(G)$ *space. If the system* $\{\ u_{k0}(x),\ \ u_{k1}(x)+Cu_{k0}(x)\}$ *is an unconditional basis in* $L_2(G)$*, then the following uniform estimate will be made*

$$\|\, u_{k0}\|_{L_2(G)} \le C_1 \|\, u_{k1}\|_{L_2(G)} \tag{1}$$

for all k *numbers.*

Proof. Let the condition of the theorem be met. It is easy to show that the biorthogonally adjoint for the system $\{u_{k0}(x), u_{k1}(x)+Cu_{k0}(x)\}$ is the system $\left\{v_{k0}(x)-\bar{C}v_{k1}, v_{k1}(x)\right\}$.

Let the system $\{u_{k0}(x), u_{k1}(x)+Cu_{k0}(x)\}$ form an unconditional basis in $L_2(G)$. Let us prove that inequality (1) holds. But first we note that for the basis property of the system $\{u_{k0}(x), u_{k1}(x)+Cu_{k0}(x)\}$ as a necessary condition for the basis property of (*) implies that the following two conditions should be met:

$$\|\, u_{k0}\|_{L_2(G)} \cdot \left\|\, v_{k0}-\bar{T}v_{k1}\right\|_{L_2(G)} \le C_2, \tag{2}$$

$$\|\, u_{k1}+Cu_{k0}\|_{L_2(G)} \cdot \|\, v_{k1}\|_{L_2(G)} \le C_3. \tag{3}$$

In addition, we also have the following estimates

$$\|\, u_{k0}\|_{L_2(G)} \cdot \|\, v_{k0}\|_{L_2(G)} \le C_4, \tag{4}$$

$$\|\, u_{k1}\|_{L_2(G)} \cdot \|\, v_{k1}\|_{L_2(G)} \le C_5, \tag{5}$$

which derive from the unconditional basis property of the system $\{u_{k0}(x), u_{k1}(x)\}$ as a necessary condition for the basis property of (*). With the help of these four estimates (2)-(5) we get the estimate of $\|\, u_{k0}\|_{L_2(G)} \cdot \|\, v_{k1}\|_{L_2(G)}$ value. This estimate will be derived from the following chain of inequalities (all norms are taken in $L_2(G)$)

$$|T| \cdot \|\, u_{k0}\| \cdot \|v_{k1}\| = \|u_{k0}\| \cdot \left\|\, \bar{C}v_{k1} - v_{k0} + v_{k0}\right\| \le \|u_{k0}\| \cdot \left\{\left\|v_{k0} - \bar{C}v_{k1}\right\| + \|v_{k0}\|\right\} = \|u_{k0}\| \cdot \left\|v_{k0} - \bar{C}v_{k1}\right\| + \|u_{k0}\| \cdot \|v_{k0}\| \le C_2 + C_4,$$

where C_2, C_4 are constants from (6) and (4) respectively. From here we get another one

$$\|u_{k0}\| \cdot \|v_{k1}\| \le C_6, \tag{6}$$

where $C_6 = (T_2 + T_4)\, |T|^{-1}$. Due to the biorthonormality of the systems $\{u_{k0}(x), u_{k1}(x)\}$, $\{v_{k0}(x), v_{k1}(x)\}$, the estimate (5) can be written in the form $1 = |(u_{k1}(x), v_{k1}(x))| \le \|u_{k1}\| \cdot \|\, v_{k1}\| \le C_5$, which implies $\frac{1}{\|u_{k1}\|} \le \|v_{k1}\| \le$ $\le \frac{C_5}{\|u_{k1}\|}$, (as before, all norms are taken in $L_2(G)$). If we use this relation in (6), then we get $\|\, u_{k0}\|\, (\|\, u_{k1}\|)^{-1} \le C_6$ or $\|\, u_{k0}\| \le C_6 \|\, u_{k1}\|$. Theorem is proved. □

An ordinary differential operation L of general type is generated by a differential expression of the following form

$$\mathrm{L}\,u = u^{(n)} + p_1(x)u^{(n-1)} + p_2(x)u^{(n-2)} + \cdots + p_n(x)u, \quad x \in G = (a,b) \tag{7}$$

and the following boundary conditions

$$U_\nu(u) = \alpha_{\nu 1}u(a) + \alpha_{\nu 2}u'(a) + \cdots + \alpha_{\nu\, n-1}\, u^{\,(n-1)}(a) +$$

$$+\beta_{\nu 1}u(b) + \beta_{\nu 2}u'(b) + \cdots + \beta_{\nu n-1}u^{(n-1)}(b) = 0, \quad \nu = 1, 2, \ldots, n.$$

But the boundary conditions may be non-local in nature, depend on the spectral parameter, which contain integral terms, etc. Therefore, it is convenient to understand the differential operator in a different, generalized, meaning.

A differential operator (in the generalized sense) will be considered as the operator generated by the differential expression (7) and some boundary conditions, the specific form of which is not important for us. Then the eigenfunctions and associated functions (EAF) of the operator can be defined as a regular solution of differential equations with spectral parameter.

A regular on G solution of the following equation

$$\mathrm{L}\,u = \lambda u + f, \tag{8}$$

where $\lambda \in T$ and $f \in L_1(G)$, will be considered as an arbitrary function $u(x)$, which is absolutely continuous together with its derivatives up to n-1 order inclusive on any compact set in G, and almost everywhere on G satisfies the equation (8).

An eigenfunction of L, generated by the differential expression (7) and some boundary conditions,will be considered as any non-trivial regular solution $u_0(x)$ of the following equation $\mathrm{L}\,u_0(x) = \lambda u_0(x)$, belonging to the $L_2(G)$ space. In this case, the complex number λ will be called the *eigenvalue* of the operator L.

Associated functions are defined by induction. *The associated function of the first order,* corresponding to the eigenfunction $u_0(x)$ and eigenvalue λ will be considered as any non-trivial regular solution $u_1(x)$ of the following equation $\mathrm{L}\,u_1(x) = \lambda u_1(x) + u_0(x)$, belonging to the $L_2(G)$ space.

If the associated function $u_{k-1}(x)$ of $k-1 \geq 0$ order is defined, then an *associated function of k order* is understood as any non-trivial solution $u_k(x)$ of the following equation $\mathrm{L}\,u_k(x) = \lambda u_k(x) + u_{k-1}(x)$, belonging to the $L_2(G)$ space.

Note that the consideration of root functions in this sense began with the famous works of Ilyin V.A. (see, for example, [3]) and is widely used in the work of his students and followers.

Along with the eigenvalue λ_k we will use the spectral parameter μ_k, which is defined as follows

$$\mu_k = \begin{cases} \left[(-1)^{\frac{n}{2}}(-\lambda_k)\right]^{\frac{1}{n}}, & \text{if } n - \text{even;} \\ (-\mathrm{i}\lambda_k)^{\frac{1}{n}}, & \text{if } n - \text{uneven and } \operatorname{Im}\lambda_k \geq 0; \\ (\mathrm{i}\lambda_k)^{\frac{1}{n}}, & \text{if } n - \text{uneven and } \operatorname{Im}\lambda_k < 0, \end{cases}$$

where $[\rho\exp(\mathrm{i}\varphi)]^{\frac{1}{n}} = \rho^{\frac{1}{n}}\exp\left(\frac{\mathrm{i}\varphi}{n}\right)$, $-\frac{\pi}{2} < \varphi \leq \frac{3\pi}{2}$.

Further, we will need the following theorem [5] (see also [6]), which we will present in a convenient for us form.

THEOREM 3. *(Kurbanov V.M.). Let* $\{u_k(x)\}$, $\{v_k(x)\}$ *be a pair of biorthogonally adjoint systems of root functions of the operators* L *and* L* *respectively, the following conditions are met:*
- $p_i(x) \in W_1^{n-i}(G)$, $i = \overline{1,n}$*;*
- for any eigenvalueλ_k *there is the following inequality*

$$|\operatorname{Im}\mu_k| \leq const.; \tag{9}$$

- in every chain of eigenfunctions and associated functions of the operators L *and* L* *the following estimates of anti-a priori type are made*

$$\|u_k(x)\|_{L_2(G)} \leq const.(1+|\mu_k|)^{n-1}\|u_{k+1}(x)\|_{L_2(G)},$$

$$\|v_{k+1}(x)\|_{L_2(G)} \leq const.(1+|\mu_k|)^{n-1}\|v_k(x)\|_{L_2(G)}. \tag{10}$$

Then for the unconditional basis property in $L_2(G)$ for system $\{u_k(x)\}$ it is necessary and sufficient to meet the following three conditions

$$\sum_{t\leq|\mu_k|\leq t+1} 1 \leq const., \quad \forall\, t \geq 0, \tag{11}$$

$$\sum_{|\mu_k|\leq t} |u_k(x)|^2\|u_k(x)\|_{L_2(G)}^{-2} \leq const.(1+t), \quad \forall\, t \geq 0, \tag{12}$$

$$\sum_{|\mu_k|\leq t} |v_k(x)|^2\|v_k(x)\|_{L_2(G)}^{-2} \leq const.(1+t); \quad \forall\, t \geq 0, \tag{13}$$

and the following inequality

$$\|u_k(x)\|_{L_2(G)} \cdot \|v_k(x)\|_{L_2(G)} \leq const. \tag{14}$$

Note that Ilyin V.A. was the first to prove sufficiency of the type (14) condition for an unconditional basis property of the root vectors of a second order differential operator.

The estimates of type (1) or (10) were called anti-a priori estimates by Ilyin V.A.

It should be specially emphasized that the condition (14) of this theorem can take place for one choice of associated functions and can be not satisfied with a different choice of associated functions. This means a failure to meet the condition (14) for a system of eigenfunctions and associated functions does not mean that this problem has no system of eigenfunctions and associated functions with a basis property. Basis can be a different system of eigenfunctions and associated functions, as in the case of the example given in the introduction.

Let us further suppose that each eigenfunction corresponds to an associated function, and consider a complete and minimal in $L_2(G)$ system $\{u_{k0}(x), u_{k1}(x)\}$ consisting of eigenfunctions and associated functions of the operator understood in the above-mentioned generalized sense. Due to the minimality and completeness of this system there is a unique system $\{v_{k0}(x), v_{k1}(x)\}$ called biorthogonally adjoint to the previous system, which satisfies the following conditions in the sense of the scalar product in $L_2(G)$

$$(u_{k0}(x), v_{k0}(x)) = 1, \quad (u_{k1}(x), v_{k1}(x)) = 1,$$
$$(u_{k0}(x), v_{k1}(x)) = 0, \quad (u_{k1}(x),\ v_{k0}(x)) = 0.$$

We assume that the system $\{v_{k0}(x), v_{k1}(x)\}$ consists of eigenfunctions and associated functions (understood in the above-mentioned sense) of a formally adjoint operator

$$\mathrm{L}^*(z) = (-1)^n (z)^{(n)} + (-1)^{(n-1)} (\overline{p_1} z)^{(n-1)} + (-1)^{(n-2)} (\overline{p_2} z)^{(n-2)} + \cdots + \overline{p_n} z.$$

i.e. eigenfunctions $v_{k1}(x)$ satisfy the equations $\mathrm{L}^*\, v_{k1} = \bar{\lambda}_k v_{k1}(x)$, and associated functions $v_{k0}(x)$ satisfy the equations $\mathrm{L}^*\, v_{k0} = \bar{\lambda}_k v_{k0}(x) + v_{k1}(x)$.

In biorthogonally adjoint systems, together with the eigenfunction of the given operator there is an associated function of the corresponding chain of an adjoint operator, and together with the associated function $u_{k1}(x)$ of the operator L there is an eigenfunction of the operator L^*. We consider the case with two functions in each chain: one eigenfunction and one associated function.

Under these assumptions, the following theorem holds.

THEOREM 4. *Let the system $\{u_{k0}(x), u_{k1}(x)\}$ consisting of eigenfunctions and associated functions of the operator* L *form an unconditional basis of $L_2(G)$ space, and let the following conditions be met:*

- $p_i(x) \in W_1^{n-i}(G)$, $i = \overline{1, n}$;

- for any eigenvalue λ_k the following inequality holds

$$|\mathrm{Im}\, \mu_k| \le const.; \tag{9}$$

- for any knumbers

$$\|\, u_{k0}\|_{L_2(G)} \le C_1 \|\, u_{k1}\|_{L_2(G)}. \tag{15}$$

In this case, the system $\{u_{k0}(x), u_{k1}(x) + C u_{k0}(x)\}$ is an unconditional basis of $L_2(G)$ space.

Proof. Let us show that when the conditions of the theorem are met we have the relations (10)-(14). The first estimate (10) is directly derived from the relation (15). The second estimate is derived from the following two inequalities $1 \le \| u_{k0}\| \cdot \| v_{k0}\| \le C_1\|u_{k1}\| \|v_{k0}\|, \quad 1 \le \| u_{k1}\| \cdot \| v_{k1}\| \le C_3$.

Due to the unconditional basis property of the system $\{u_{k0}(x), u_{k1}(x)\}$ and the estimate (15), according to the Kurbanov V.M. theorem, the conditions (11)-(13) will be met for the system $\{u_{k0}(x), u_{k1}(x) + Cu_{k0}(x)\}$ as well. The unconditional basis property of the system $\{u_{k0}(x), u_{k1}(x)\}$ also gives us the following $\| u_{k0}\| \cdot \| v_{k0}\| \le C_2, \quad \| u_{k1}\| \cdot \| v_{k1}\| \le C_3$.

Therefore

$$\| u_{k0}\| \cdot \left\| v_{k0} - \bar{C}v_{k1}\right\| \le \| u_{k0}\| \cdot \| v_{k0}\| + |C_1| \cdot \| u_{k0}\| \cdot \| v_{k1}\| \le$$
$$\le \| u_{k0}\| \cdot \| v_{k0}\| + |C|C_1\| u_{k1}\| \cdot \| v_{k1}\| \le C_4 + |C| \cdot C_1 \cdot C_5.$$

Similarly, we prove the inequality

$$\| u_{k1} + Cu_{k0}\|_{L_2(G)} \cdot \| v_{k1}\|_{L_2(G)} \le C_4.$$

Thus we have established the validity of inequality (14) for the system $\{u_{k0}(x), u_{k1}(x) + Cu_{k0}(x)\}$ in the above-mentioned theorem of Kurbanov V.M.

Therefore, it forms an unconditional basis of the $L_2(G)$ space. The theorem is proved. □

Theorems 2 and 4 are valid in the case when the number of associated functions in each chain may be more than one, but their number is uniformly bounded in all chains. Such a system can be characterized as follows.

Let us consider the system $\{u_{k0}(x), u_{k1}(x), \dots, u_{k\,p}(x)\}$, $p \le p_0$ of eigenfunctions and associated functions of a differential operator L. We will denote a biorthogonally adjoint system as $\{v_{k0}(x), v_{k1}(x), \dots, v_{k,p-1}(x),\ v_{k\,p}\}$, which consists of eigenfunctions and associated function of a formally adjoint operator L*. $u_{k0}(x)$- eigenfunction, $u_{k1}(x), u_{k2}(x), \dots, u_{k\,p}(x)$ – associated functions of the operator L belonging to one chain, and $v_{k\,p}(x)$ - eigenfunction, $v_{k,\,p-1}(x)$,..., $v_{k1}(x)$, $v_{k0}(x)$ – associated functions of the operator L* from one of the chains. Since the associated functions can be defined with an accuracy up to the term that contains an eigenfunction, the system

$$\{u_{k0}(x), u_{k1}(x) + Cu_{k0}(x), \dots, u_{k\,p}(x) + Cu_{k0}\}$$

is also a system of eigenfunctions and associated functions of the operator L. It is easy to verify that its biorthogonally adjoint system is as follows

$$\left\{v_{k0}(x) - \bar{C}v_{k1}(x) - \bar{C}v_{k2}(x) - \dots - \bar{C}v_{kp}(x), v_{k1}(x), \dots, v_{k\,p}(x)\right\}.$$

The elements of this system are the root functions of the operator L*. In this system $v_{k\,p}(x)$ are the eigenfunctions, and $v_{k,\,p-1}(x), \dots, v_{k0}(x)$ are the associated functions of the operator L*.

The estimates of type (15) (when there is no spectral parameter with a positive degree in the right hand side) can be not met in case of differential operators, if there is an infinite of associated functions and they are defined by the formula $\mathrm{L}\, u_{kj} = \lambda_k u_{kj}(x) + u_{k,j-1}(x)$, $j = 1, 2, \ldots, p$. It may be that there are estimates of the type (10), in the right side of which there is a spectral parameter in a positive degree. It is the lack of estimates of the type (15) and the existence of relations of type (10) (when there is a spectral parameter with a positive degree in the right hand side of the inequality) in each chain of associated functions that raises the issue of choosing associated functions. Therefore, researchers are forced to seek new formulas to determine the associated functions. In this respect, we note the works of Ionkin N.I. [4], Lomov I.S. [6], which show different ways of constructing associated functions of an ordinary differential operator of order.

Based on the conducted researches the authors of this paper find it natural for each root subspace of any linear operator L in Hilbert space to choose a basis of elements that form the analogue Jordan chains of the following form

$$\mathrm{L}\, u_{k0} = \lambda_k u_{k0}(x), \quad \mathrm{L}\, u_{kj} = \lambda_k (u_{kj}(x) + u_{k,j-1}(x)), \quad j = 1,\ 2, \ldots, p.$$

As shown in [7, 8], such a definition of associated functions does not rise the problem of choosing associated functions at least for a second order differential operator.

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