MATHEMATICA, Tome 54 (77), N° Spécial, 2012, pp. 165–170

SOME PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

SHIGEYOSHI OWA and EMEL YAVUZ DUMAN

Abstract. Applying the subordinations for analytic functions $f(z)$ normalized by $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{U}$, some properties of $f(z)$ are discussed.
MSC 2010. 34C40.
Key words. Analytic, subordination, starlike, convex.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{2}$$

for some real α $(0 \leqq \alpha < 1)$. Further, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{3}$$

for some real α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$.

Mocanu [2] has shown that

THEOREM A. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|f'(z) - 1\right| < \frac{2\sqrt{5}}{5} \qquad (z \in \mathbb{U}), \tag{4}$$

then $f(z) \in \mathcal{S}^*$.

Applying the above theorem, Nunokawa, Owa, Polatoğlu, Çağlar and Yavuz Duman [3] have proved

THEOREM B. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|f''(z)\right| < \frac{\sqrt{5}}{5} = 0.4472\cdots \qquad (z \in \mathbb{U}), \tag{5}$$

then $f(z) \in \mathcal{K}$.

2. SOME PROPERTIES

Let functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0, |w(z)| \leqq |z|$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \quad \prec \quad g(z) \qquad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Now, we derive

Theorem 1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|f'(z) - 1\right| < \alpha \qquad (z \in \mathbb{U}) \tag{6}$$

for some real α $(0 < \alpha \leqq 1)$, then

$$1 - \frac{\alpha}{2}|z| \leqq \mathrm{Re}\left(\frac{f(z)}{z}\right) \leqq 1 + \frac{\alpha}{2}|z| \qquad (z \in \mathbb{U}), \tag{7}$$

$$\left|\frac{f(z)}{z} - 1\right| \leqq \frac{\alpha}{2}|z| \qquad (z \in \mathbb{U}), \tag{8}$$

$$\frac{2(1 - \alpha|z|)}{2 + \alpha|z|} \leqq \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{2(1 + \alpha|z|)}{2 - \alpha|z|} \quad (z \in \mathbb{U}), \tag{9}$$

and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \frac{3\alpha|z|}{2 - \alpha|z|} \quad (z \in \mathbb{U}). \tag{10}$$

Proof. Note that the inequality (6) implies that

$$f'(z) - 1 \ \prec \ \alpha z, \tag{11}$$

which is equivalent to

$$f'(z) - 1 = \alpha w(z), \tag{12}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$. Integrating both side of (12), we have that

$$f(z) = z + \alpha \int_0^z w(t)\mathrm{d}t = z + \alpha \int_0^{|z|} w(\rho \mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\rho.$$

Therefore, we see that

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) = 1 + \alpha \mathrm{Re}\left(\frac{1}{z}\int_0^{|z|} w(\rho \mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\rho\right) \leqq 1 + \frac{\alpha}{|z|}\int_0^{|z|} \left|w(\rho \mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\rho$$

$$\leqq 1 + \frac{\alpha}{|z|}\int_0^{|z|} \rho \mathrm{d}\rho = 1 + \frac{\alpha}{2}|z|$$

and

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) = 1 + \alpha \operatorname{Re}\left(\frac{1}{z}\int_0^{|z|} w(\rho e^{i\theta})e^{i\theta}d\rho\right) \geqq 1 - \frac{\alpha}{|z|}\int_0^{|z|}\left|w(\rho e^{i\theta})\right|d\rho$$
$$\geqq 1 - \frac{\alpha}{|z|}\int_0^{|z|}\rho d\rho = 1 - \frac{\alpha}{2}|z|.$$

Also, we know that

$$\left|\frac{f(z)}{z} - 1\right| = \left|\frac{\alpha}{z}\int_0^z w(t)dt\right| \leqq \frac{\alpha}{|z|}\int_0^{|z|}\left|w(\rho e^{i\theta})\right|d\rho = \frac{\alpha}{2}|z|.$$

Furthermore, we have that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1+\alpha w(z)}{1+\dfrac{\alpha}{z}\displaystyle\int_0^z w(t)dt}\right) \geqq \frac{1-\alpha|w(z)|}{1+\dfrac{\alpha}{|z|}\displaystyle\int_0^{|z|}\left|w(\rho e^{i\theta})\right|d\rho}$$
$$\geqq \frac{2(1-\alpha|z|)}{2+\alpha|z|}$$

and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{1+\alpha|w(z)|}{1-\dfrac{\alpha}{|z|}\displaystyle\int_0^{|z|}\left|w(\rho e^{i\theta})\right|d\rho} \leqq \frac{2(1+\alpha|z|)}{2-\alpha|z|}.$$

Finally, we obtain that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\alpha w(z) - \dfrac{\alpha}{z}\displaystyle\int_0^z w(t)dt}{1+\dfrac{\alpha}{z}\displaystyle\int_0^z w(t)dt}\right| \leqq \frac{\alpha|z| + \dfrac{\alpha}{|z|}\displaystyle\int_0^{|z|}\left|w(\rho e^{i\theta})\right|d\rho}{1-\dfrac{\alpha}{|z|}\displaystyle\int_0^{|z|}\left|w(\rho e^{i\theta})\right|d\rho}$$
$$\leqq \frac{3\alpha|z|}{2-\alpha|z|}.$$

□

REMARK 1. The inequality (10) implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{3\alpha}{2-\alpha} = 1 - \frac{2(1-2\alpha)}{2-\alpha} \quad (z \in \mathbb{U})$$

for $0 < \alpha \leqq \frac{1}{2}$. This means that $f(z) \in \mathcal{S}^*\left(\frac{2(1-2\alpha)}{2-\alpha}\right)$.

Next, we derive

THEOREM 2. *If $f(z) \in \mathcal{A}$ satisfies*

$$|f''(z)| < \alpha \qquad (z \in \mathbb{U}) \tag{13}$$

for some real $\alpha > 0$*, then*

$$1 - \frac{\alpha}{2}|z|^2 \leqq \mathrm{Re} f'(z) \leqq 1 + \frac{\alpha}{2}|z| \qquad (z \in \mathbb{U}), \tag{14}$$

and

$$\left| \frac{zf''(z)}{f'(z)} \right| \leqq \frac{2\alpha|z|^2}{2 - \alpha|z|^2} \quad (z \in \mathbb{U}). \tag{15}$$

Proof. In view of (13), we have that $f''(z) \prec \alpha z$, so that $f''(z) = \alpha w(z)$, where $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$. Therefore, we have that

$$f'(z) - 1 = \alpha \int_0^z w(t)\mathrm{d}t.$$

It follows that

$$\mathrm{Re} f'(z) = \mathrm{Re}\left(1 + \alpha \int_0^z w(t)\mathrm{d}t\right) \leqq 1 + \alpha \int_0^{|z|} \left|w(\rho e^{i\theta})\right| \mathrm{d}\rho \leqq 1 + \frac{\alpha}{2}|z|^2$$

and

$$\mathrm{Re} f'(z) \geqq 1 - \frac{\alpha}{2}|z|^2.$$

Furthermore, we have that

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{\alpha z w(z)}{1 + \alpha \displaystyle\int_0^z w(t)\mathrm{d}t} \right| \leqq \frac{\alpha|z|^2}{1 - \alpha \displaystyle\int_0^{|z|} \left|w(\rho e^{i\theta})\right| \mathrm{d}\rho} \leqq \frac{2\alpha|z|^2}{2 - \alpha|z|^2}.$$

□

Remark 2. If $0 < \alpha \leqq \frac{2}{3}$, the inequality (15) implies that

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \frac{2 - 3\alpha}{2 - \alpha} \qquad (z \in \mathbb{U}).$$

Thus we say that $f(z) \in \mathcal{K}\left(\frac{2 - 3\alpha}{2 - \alpha}\right)$.

3. APPLICATIONS OF SALAGEAN AND ALEXANDER OPERATORS

For $f(z) \in \mathcal{A}$, Salagean [4] has defined the following Salagean differential operator:

$$\mathrm{D}^0 f(z) = f(z), \qquad \mathrm{D}^1 f(z) = zf'(z),$$

and

$$\mathrm{D}^j f(z) = \mathrm{D}^1\left(\mathrm{D}^{j-1} f(z)\right) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 1, 2, 3, \ldots).$$

On the other hand, Alexander [1] has given the following integral operator $\mathrm{D}^{-1} f(z)$ which is called Alexander integral operator.

$$\mathrm{D}^{-1} f(z) = \int_0^z \frac{f(t)}{t}\mathrm{d}t = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n.$$

Furthermore, we introduce

$$\mathrm{D}^{-j} f(z) = \mathrm{D}^{-1}\left(\mathrm{D}^{-j+1} f(z)\right) = z + \sum_{n=2}^{\infty} \frac{1}{n^j} a_n z^n \quad (j = 1, 2, 3, \dots).$$

In view of the definitions for $\mathrm{D}^j f(z)$ and $\mathrm{D}^{-j} f(z)$, we define

$$\mathrm{D}^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n$$

for $f(z) \in \mathcal{A}$, where j is the integer.

Our result for the above operator is contained in

THEOREM 3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{\mathrm{D}^{j+1} f(z)}{z} - 1\right| < \alpha \qquad (z \in \mathbb{U}) \tag{16}$$

for some real $\alpha > 0$, then

$$\left|\frac{\mathrm{D}^{j} f(z)}{z} - 1\right| \leqq \frac{\alpha}{2}|z| \qquad (z \in \mathbb{U}). \tag{17}$$

Proof. By means of the condition (16), we can write

$$\frac{\mathrm{D}^{j+1} f(z)}{z} - 1 = \alpha w(z)$$

with some analytic function $w(z)$ in $\mathbb{U}$ which satisfies $w(0) = 0$ and $|w(z)| \leqq |z|$ $(z \in \mathbb{U})$. Noting that $\mathrm{D}^{j+1} f(z) = z\left(\mathrm{D}^j f(z)\right)'$, we see that

$$\mathrm{D}^j f(z) - z = \alpha \int_0^z w(t)\mathrm{d}t.$$

It follows from the above that

$$\left|\frac{\mathrm{D}^{j} f(z)}{z} - 1\right| \leqq \frac{\alpha}{|z|} \int_0^{|z|} \left|w(\rho \mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\rho \leqq \frac{\alpha}{2}|z|.$$

□

Making $j = 0$ and $j = 1$ in Theorem 3.1, we have

COROLLARY 1. *If $f(z) \in \mathcal{A}$ satisfies $|f'(z) - 1| < \alpha$ $(z \in \mathbb{U})$ for some real $\alpha > 0$, then*

$$\left|\frac{f(z)}{z} - 1\right| \leqq \frac{\alpha}{2}|z| \qquad (z \in \mathbb{U}).$$

Also, if $f(z) \in \mathcal{A}$ *satisfies*

$$\left|zf''(z) + f'(z) - 1\right| < \alpha \qquad (z \in \mathbb{U})$$

for some real $\alpha > 0$*, then*

$$\left|f'(z) - 1\right| \leqq \frac{\alpha}{2}|z| \quad (z \in \mathbb{U}).$$

REFERENCES

[1] Alexander, J.W., *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17** (1915), 12–22.

[2] Mocanu, P.T., *Some starlikeness conditions for analytic functions*, Rev. Roumaine Math. Pures. Appl., **33** (1971), 117–124.

[3] Nunokawa, M., Owa, S., Polatoğlu, Y., Çağlar M. and Yavuz Duman, E., *Some sufficient conditions for starlikeness and convexity*, Turk. J. Math., **34** (2010), 333–337.

[4] Salagean, G.S., *Subclasses of univalent functions,* Complex analysis-fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981); Lecture Notes in Math., **1013**, Springer, Berlin (1983), 362–372.

Kinki University
Department of Mathematics
Higashi-Osaka, Osaka 577-8502, Japan
E-mail: owa@math.kindai.ac.jp

İstanbul Kültür University
Department of Mathematics and Computer Science
34156 İstanbul, Turkey
E-mail: e.yavuz@iku.edu.tr