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ON A CERTAIN CLASS OF HARMONIC FUNCTIONS ASSOCIATED WITH A CONVOLUTION STRUCTURE

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Abstract. Making use of a convolution structure, we introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. Among the results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for this generalized class of functions.

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1. INTRODUCTION AND PRELIMINARIES

A continuous function $f = u + \mathrm{i}v$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g}, \tag{1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$. Thus, for $f = h+\overline{g} \in \mathcal{H}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m,\ \ g(z) = \sum_{m=1}^{\infty} b_m z^m \qquad (0 \le b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \qquad (0 \le b_1 < 1). \tag{2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

For functions $f \in \mathcal{H}$ given by (2) and $F(z) \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=1}^{\infty} B_m z^m}, \tag{3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=1}^{\infty} b_m B_m z^m} \quad (z \in \mathcal{U}). \tag{4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f*F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (4), we consider the following examples.

(I) If

$$F(z) = z + \sum_{m=2}^{\infty} \sigma_m \, z^m + \sum_{m=1}^{\infty} \sigma_m \, \overline{z}^m \tag{5}$$

and σ_m is defined by

$$\sigma_m = \frac{\Theta \cdot \Gamma(\alpha_1 + A_1(m-1)) \ldots \Gamma(\alpha_p + A_p(m-1))}{(m-1)!\Gamma(\beta_1 + B_1(m-1)) \ldots \Gamma(\beta_q + B_q(m-1))} \tag{6}$$

where Θ is given by

$$\Theta = \left(\prod_{n=1}^{p} \Gamma(\alpha_n) \right)^{-1} \left(\prod_{n=1}^{q} \Gamma(\beta_n) \right), \tag{7}$$

then the convolution (4) gives the Wright's operator for harmonic functions [9]. The Wrights hypergeometric functions [13] ${}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] = {}_p\Psi_q[(\alpha_m, A_m)_{1,p}(\beta_m, B_m)_{1,q}; z]$ is defined by

$$\begin{aligned} &{}_p\Psi_q[(\alpha_m, A_m)_{1,p}(\beta_m, B_m)_{1,q}; z] \\ &= \sum_{m=0}^{\infty} \left\{ \prod_{n=1}^{p} \Gamma(\alpha_n + mA_n) \right\} \left\{ \prod_{n=1}^{q} \Gamma(\beta_n + mB_n) \right\}^{-1} \frac{z^m}{m!} \quad (z \in \mathbb{U}). \end{aligned}$$

(II) If $A_n = 1$ $(n = 1, ..., p)$ and $B_n = 1$ $(n = 1, ..., q)$, then we have the following obvious relationship

$$F(z) = z + \sum_{m=2}^{\infty} \Gamma_m z^m + \sum_{m=1}^{\infty} \Gamma_m \overline{z}^m, \tag{8}$$

where

$$\Gamma_m = \frac{(\alpha_1)_{m-1} \ldots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \ldots (\beta_q)_{m-1}} \, \frac{1}{(m-1)!},$$

then the convolution (4) gives the Dziok–Srivastava operator for harmonic functions [7]: $\Lambda(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z)f(z) \equiv \mathcal{H}_q^p(\alpha_1, \beta_1)f(z)$, where α_1,

$\cdots, \alpha_p;\ \beta_1, \cdots, \beta_q$ are positive real numbers, $p \leq q+1;\ p,\ q \in \mathbb{N} \cup \{0\}$, and $(\alpha)_m$ denotes the familiar Pochhammer symbol (or shifted factorial).

REMARK 1. When $p = 1,\ q = 1;\ \alpha_1 = a,\ \alpha_2 = 1;\ \beta_1 = c$, then (8) corresponds to the operator due to Carlson-Shaffer operator [2], for harmonic functions given by

$$\mathrm{L}(a,c)f(z) := (f * F)(z),$$

where

$$F(z) := z + \sum_{m=2}^{\infty} \frac{(a)_{m-1}}{(c)_{m-1}} z^m + \sum_{m=1}^{\infty} \frac{(a)_{m-1}}{(c)_{m-1}} \overline{z}^m \quad (c \neq 0, -1, -2, \cdots). \tag{9}$$

REMARK 2. When $p = 1,\ q = 0;\ \alpha_1 = m+1,\ \alpha_2 = 1;\ \beta_1 = 1$, then (8) yields the Ruscheweyh derivative operator [10] for harmonic functions given by $\mathrm{D}^k f(z) := (f * F)(z)$ where

$$F(z) = z + \sum_{m=2}^{\infty} \begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix} z^m + \sum_{m=1}^{\infty} \begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix} \overline{z}^m \tag{10}$$

which was initially studied for harmonic functions by Murugusundaramoorthy [8] (see also [5]).

(III) Lastly, the operator $\mathrm{D}^l f(z) = f * F$, where

$$F(z) = z + \sum_{m=2}^{\infty} m^l z^m + (-1)^l \sum_{m=1}^{\infty} m^l \overline{z}^m \quad (\ l \geq 0), \tag{11}$$

was initially studied by Jahangiri et al. [6]

For the purpose of this paper, we introduce here a subclass of $\mathcal{H}$ denoted by $S_H(F;\gamma)$, for $0 \leq \gamma < 1$, which involves the convolution (4) and consist of functions of the form (1) satisfying the inequality:

$$\frac{\partial}{\partial \theta} \left(\arg\left[(f * F)(z)\right]\right) > \gamma \tag{12}$$

$0 \leq \theta < 2\pi$ and $z = r\,\mathrm{e}^{\mathrm{i}\theta}$. Equivalently

$$\mathrm{Re}\ \left\{ \frac{z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}}{h(z) * H(z) + \overline{g(z) * G(z)}} \right\} \geq \gamma, \tag{13}$$

where $z \in \mathcal{U}$.

We also let $\mathcal{V}_{\mathcal{H}}(F;\gamma) = \mathcal{S}_{\mathcal{H}}(F;\gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [4], consisting of functions f of the form (1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\phi \equiv \pi \pmod{2\pi},\ \delta_m + (m+1)\phi \equiv 0 \pmod{2\pi},\ m \geq 2, \tag{14}$$

where $\eta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$.

We deem it proper to mention below some of the function classes which emerge from the function class $\mathcal{S}_{\mathcal{H}}(F;\gamma)$ defined above. Indeed, we observe that if we specialize the function $F(z)$ by means of (5) to (11), and denote the corresponding reducible classes of functions of $\mathcal{S}_{\mathcal{H}}(F;\gamma)$, respectively, by $\mathcal{W}_q^p(\gamma)$, $\mathcal{G}_q^p(\gamma)$, $\mathcal{L}_c^a(\gamma)$, $\mathcal{R}(k,\gamma)$, $\Omega(\gamma)$ and $\mathcal{S}(l,\gamma)$, then we obtain:

(i) If F(z) is given by (5), we have $f*F = \mathrm{W}_q^p[\alpha_1]f(z)$, hence we define a class $\mathcal{W}_q^p(\gamma)$ satisfying the criteria

$$\mathrm{Re}\left\{\frac{z(\mathrm{W}_q^p\, h(z))' - \overline{z(\mathrm{W}_q^p\, g(z))'}}{\mathrm{W}_q^p\, h(z) + \overline{(\mathrm{W}_q^p\, g(z)}}\right\} \geq \gamma,$$

where $\mathrm{W}_q^p([\alpha_1])$ is the Wright's generalized operator on harmonic functions [9].

(ii) If F(z) is given by (8) we have $f*F = \mathrm{H}_q^p[\alpha_1]f(z)$, hence we define a class $\mathcal{G}_q^p(\gamma)$ satisfying the criteria

$$\mathrm{Re}\left\{\frac{z\,\mathrm{H}_q^p[\alpha_1]h(z))' - \overline{z\,\mathrm{H}_q^p[\alpha_1]g(z))'}}{\mathrm{H}_q^p[\alpha_1]h(z) + \overline{\mathrm{H}_q^p[\alpha_1]g(z)}}\right\} \geq \gamma,$$

where $\mathrm{H}_q^p[\alpha_1]$ is the Dziok-Srivastava operator on harmonic functions [7] .

For special choices of $p, q, \alpha_1, \beta_1, \alpha_2, \beta_2$ as stated in Remarks 1 and 2 we state the following subclasses of harmonic functions.

(iii) $H_1^2([a,1;c]) = \mathrm{L}(a,c)f(z)$, hence we define a class $\mathcal{L}_c^a(\gamma)$ satisfying the criteria

$$\mathrm{Re}\left\{\frac{z(\mathrm{L}(a,c)h(z))' - \overline{z(\mathrm{L}(a,c)g(z))'}}{\mathrm{L}(a,c)h(z) + \overline{\mathrm{L}(a,c)g(z)}}\right\} \geq \gamma,$$

where $\mathrm{L}(a,c)$ is the Carlson-Shaffer operator [2].

(iv) $H_1^2([k+1,1;1]) = \mathrm{D}^k f(z)$, hence we define a class $\mathcal{R}(k,\gamma)$ satisfying the criteria

$$\mathrm{Re}\left\{\frac{z(\mathrm{D}^k\, h(z))' - \overline{z(\mathrm{D}^k\, g(z))'}}{\mathrm{D}^k\, h(z) + \overline{(\mathrm{D}^k\, g(z)}}\right\} \geq \gamma,$$

where $\mathrm{D}^k f(z)$ $(k > -1)$ is the Ruscheweyh derivative operator on harmonic functions [5, 8].

(v) $H_1^2([2,1;2-\mu]) = \Omega_z^\mu\, f(z)$ we define another class $\Omega(\gamma)$ satisfying the condition

$$\mathrm{Re}\left\{\frac{z(\Omega_z^\mu\, h(z))' - \overline{z(\Omega_z^\mu\, g(z))'}}{\Omega_z^\mu\, h(z) + \overline{\Omega_z^\mu\, g(z)}}\right\} \geq \gamma,$$

where

$$\Omega_z^\mu\, f(z) = \Gamma(2-\mu)z^\mu\, \mathrm{D}_z^\mu\, f(z) \quad (0 \leq \mu < 1)\ ,$$

and Ω_z^μ is the Srivastava-Owa fractional derivative operator [12].

(vi) If F(z) is given by (11), we have $\mathrm{D}^l f(z) = f * F$, hence we define a class $S(l,\gamma)$ satisfying the criteria

$$\mathrm{Re}\left\{\frac{z(\mathrm{D}^l h(z))' - \overline{z(\mathrm{D}^l g(z))'}}{\mathrm{D}^l h(z) + \overline{\mathrm{D}^l g(z)}}\right\} \geq \gamma,$$

where $\mathrm{D}^l f(z)$ $(l \in N = 0,1,2,3,)$ is the Salagean derivative operator for harmonic functions [6], [11] .

Motivated by the earlier works of [5, 6, 7, 8, 9] on the subject of harmonic functions,in this paper we obtain a sufficient coefficient condition for functions f given by (2) to be in the class $S_{\mathcal{H}}(F,\gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$. Further, distortion results and extreme points for functions in $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ are also obtained.

For the sake of brevity we denote the corresponding coefficient of $F(z)$ as C_m (for $m \geq 2$) throughout our study unless otherwise stated and suppose $C_m \geq 0$ (for $m \geq 2$) and $C_1 = 1$.

2. THE CLASS $S_{\mathcal{H}}(F,\gamma)$

We begin deriving a sufficient coefficient condition for the functions belonging to the class $S_{\mathcal{H}}(F,\gamma)$.

THEOREM 1. *Let $f = h + \overline{g}$ be given by* (2) *and* $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$. *If*

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right)C_m \leq 1 - \frac{1+\gamma}{1-\gamma}b_1, \tag{15}$$

then $f \in S_{\mathcal{H}}(F;\gamma)$.

Proof. We first show that if the inequality (15) holds for the coefficients of $f = h + \overline{g}$, then the required condition (13) is satisfied. Using (6) and (13), we can write

$$\mathrm{Re}\left\{\frac{z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}}{h(z) * H(z) + \overline{g(z) * G(z)}}\right\} = \mathrm{Re}\,\frac{A(z)}{B(z)},$$

where

$$A(z) = z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}$$

and

$$B(z) = h(z) * H(z) + \overline{g(z) * G(z)}.$$

In view of the simple assertion that Re $(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{16}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (16), we get

$$\begin{aligned}
&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\\
\geq\ &(2-\gamma)|z|-\sum_{m=2}^{\infty}(m+1-\gamma)C_m|a_m|\ |z|^m-\sum_{m=1}^{\infty}(m-1+\gamma)C_m|b_m|\ |\ |z|^m\\
&-\gamma|z|-\sum_{m=2}^{\infty}(m-1-\gamma)C_m|a_m|\ |z|^m-\sum_{m=1}^{\infty}(m+1+\gamma)C_m|b_m|\ |z|^m\\
\geq\ &2(1-\gamma)|z|\left\{1-\sum_{m=2}^{\infty}\frac{m-\gamma}{1-\gamma}C_m|a_m||z|^{m-1}-\sum_{m=1}^{\infty}\frac{m+\gamma}{1-\gamma}C_m|b_m||z|^{m-1}\right\}\\
\geq\ &2(1-\gamma)|z|\left\{1-\frac{1+\gamma}{1-\gamma}b_1-\left(\sum_{m=2}^{\infty}\left[\frac{m-\gamma}{1-\gamma}C_m|a_m|+\frac{m+\gamma}{1-\gamma}C_m|b_m|\right]\right)\right\}\\
\geq\ &0
\end{aligned}$$

by virtue of the inequality (15). This implies that $f\in\mathcal{S}_{\mathcal{H}}(F,\gamma)$. $\square$

Now we obtain the necessary and sufficient condition for function $f=h+\overline{g}$ be given with condition (14).

THEOREM 2. *Let* $f=h+\overline{g}$ *be given by* (2) *with restrictions* (14) *and* $0\leq b_1<\frac{1-\gamma}{1+\gamma}$, $0\leq\gamma<1$. *Then* $f\in\mathcal{V}_{\mathcal{H}}(F;\gamma)$ *if and only if*

$$\sum_{m=2}^{\infty}\left\{\frac{m-\gamma}{1-\gamma}|a_m|+\frac{m+\gamma}{1-\gamma}|b_m|\right\}C_m\leq 1-\frac{1+\gamma}{1-\gamma}b_1. \tag{17}$$

Proof. Since $\mathcal{V}_{\mathcal{H}}(F;\gamma)\subset S_{\mathcal{H}}(F,\gamma)$, we only need to prove the necessary part of the theorem. Assume that $f\in\mathcal{V}_{\mathcal{H}}(F;\gamma)$, then by virtue of (13), we obtain

$$\mathrm{Re}\ \left\{\left[\frac{z(h(z)*H(z))'-\overline{z(g(z)*G(z))'}}{h(z)*H(z)+\overline{g(z)*G(z)}}\right]-\gamma\right\}\geq 0.$$

The above inequality is equivalent to

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{z+\left(\sum\limits_{m=2}^{\infty}(m-\gamma)C_m a_m z^m-\sum\limits_{m=1}^{\infty}(m+\gamma)C_m\overline{b_m z^m}\right)}{z+\sum\limits_{m=2}^{\infty}C_m a_m z^m+\sum\limits_{m=1}^{\infty}C_m\overline{b_m z^m}}\right\}\\
&=\mathrm{Re}\left\{\frac{(1-\gamma)+\sum\limits_{m=2}^{\infty}(m-\gamma)C_m a_m z^{m-1}-\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty}(m+\gamma)C_m\overline{b_m z^{m-1}}}{1+\sum\limits_{m=2}^{\infty}C_m a_m z^{m-1}+\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty}C_m\overline{b_m z^{m-1}}}\right\}\geq 0.
\end{aligned}$$

This condition must hold for all values of $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, such that $r < 1$. Upon choosing ϕ according to (14) we must have

$$\frac{(1-\gamma)-(1+\gamma)b_1-\left(\sum\limits_{m=2}^{\infty}(m-\gamma)C_m|a_m|r^{m-1}+(m+\gamma)C_m|b_m|r^{m-1}\right)}{1+b_1-\left(\sum\limits_{m=2}^{\infty}C_m|a_m|-\sum\limits_{m=1}^{\infty}C_m|b_m|\right)r^{m-1}} \geq 0. \tag{18}$$

If (17) does not hold, then the numerator in (18) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0 = r_0$ in (0,1) for which the quotient in (18) is negative. This contradicts our assumption that $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. We thus conclude that it is both necessary and sufficient that the coefficient bound inequality (17) holds true when $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. This completes the proof of Theorem 2. □

If we put $\phi = 2\pi/k$ in (14), then Theorem 2 gives the following corollary.

Corollary 1. *A necessary and sufficient condition for $f = h+\overline{g}$ satisfying* (17) *to be starlike is that*

$$\arg(a_m) = \pi - 2(m-1)\pi/k,$$

and

$$\arg(b_m) = 2\pi - 2(m+1)\pi/k\ ,\ (k = 1,2,3,\dots).$$

3. DISTORTION AND EXTREME POINTS

In this section we obtain distortion bounds for the functions $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ that lead to a covering result for the family $\mathcal{V}_{\mathcal{H}}(F;\gamma)$.

Theorem 3. *If $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ with $C_2 > 0$ and $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$, then*

$$|f(z)| \leq (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}b_1\right)r^2$$

and

$$|f(z)| \geq \max\left\{0,\ (1-b_1)r - \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma} - \frac{1+\gamma}{2-\gamma}b_1\right)r^2\right\},\ \ r = |z|.$$

Proof. We will only prove the right-hand inequality of the above theorem. The arguments for the left-hand inequality are similar and so we omit it. Let $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\leq (1+b_1)r + \sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m \\ &\leq (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m|). \end{aligned}$$

This implies that

$$\begin{aligned}
&|f(z)| \\
\leq\ & (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma}\right)\sum_{m=2}^{\infty}\left[\left(\frac{2-\gamma}{1-\gamma}\right)C_2|a_m| + \left(\frac{2-\gamma}{1-\gamma}\right)C_2|b_m|\right]r^2 \\
\leq\ & (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma}\right)\left[1-\frac{1+\gamma}{1-\gamma}b_1\right]r^2 \\
\leq\ & (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma}-\frac{1+\gamma}{2-\gamma}b_1\right)r^2,
\end{aligned}$$

which establishes the desired inequality. □

As a consequence of the above theorem, we state the following covering corollary.

COROLLARY 2. *Let $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ with $C_2 > 0$ and $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$. If $C_2 < C^*$, then*

$$\left\{w : |w| < \frac{(1-b_1)^2(2-\gamma)C_2}{4[1-\gamma-(1+\gamma)b_1]}\right\} \subset f(\mathcal{U})$$

and if $C_2 \geq C^*$, then

$$\left\{w : |w| < \frac{(1-b_1)(2-\gamma)C_2-(1-\gamma)+(1+\gamma)b_1}{(2-\gamma)C_2}\right\} \subset f(\mathcal{U}),$$

where

$$C^* = 2\,\frac{1-\gamma-(1+\gamma)b_1}{(2-\gamma)(1-b_1)}.$$

Proof. Let denote

$$\varphi(r) = (1-b_1)r - \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma}-\frac{1+\gamma}{2-\gamma}b_1\right)r^2.$$

By simple computation (because $\varphi'(r) = 0$ for $r = r_0$ and $r_0 \geq 1$ for $C_2 \geq C^*$) we deduce that if $0 < C_2 < C^*$, then

$$\varphi(r) \leq \varphi(r_0) = \frac{(1-b_1)^2(2-\gamma)C_2}{4[1-\gamma-(1+\gamma)b_1]}, \ \text{ where } \ r_0 = \frac{(1-b_1)(2-\gamma)C_2}{2\,[1-\gamma-(1+\gamma)b_1]},$$

for all $r \in (0,\,1)$ and if $C_2 \geq C^*$, then

$$\varphi(r) \leq \varphi(1) = \frac{(1-b_1)(2-\gamma)C_2-(1-\gamma)+(1+\gamma)b_1}{(2-\gamma)C_2},$$

for all $r \in (0,\,1)$.

□

Theorem 4. *Suppose $C_m > 0$ (for $m \geq 2$) and set $\lambda_m = \frac{1-\gamma}{(m-\gamma)C_m}$ and $\mu_m = \frac{1-\gamma}{(m+\gamma)C_m}$. Then for b_1 fixed, $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$ the extreme points for $\mathcal{V}_{\mathcal{H}}(F;\gamma)$, $0 \leq \gamma < 1$ are*

$$\left\{z + \lambda_m x z^m + \overline{b_1 z}\right\} \cup \left\{z + \overline{b_1 z + \mu_m x z^m}\right\}, \tag{19}$$

where $m \geq 2$ and $|x| = 1 - (1+\gamma)b_1/(1-\gamma)$.

Proof. Any function f in $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ can be expressed as

$$f(z) = z + \sum_{m=2}^{\infty} |a_m| e^{i\eta_m} z^m + \overline{b_1 z} + \overline{\sum_{m=2}^{\infty} |b_m| e^{i\delta_m} z^m},$$

where the coefficients satisfy the inequality (15). Set

$$h_1(z) = z,\ g_1(z) = b_1 z,\ h_m(z) = z + \lambda_m e^{i\eta_m} z^m,\ g_m(z) = b_1 z + \mu_m e^{i\delta_m} z^m$$

for $m = 2, 3, \cdots$. Writing $X_m = \frac{|a_m|}{\lambda_m}$, $Y_m = \frac{|b_m|}{\mu_m}$, $m = 2, 3, \cdots$ and $X_1 = 1 - \sum\limits_{m=2}^{\infty} X_m$; $Y_1 = 1 - \sum\limits_{m=2}^{\infty} Y_m$, we get

$$f(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m \overline{g_m(z)}).$$

In particular, putting $f_1(z) = z + \overline{b_1 z}$ and $f_m(z) = z + \lambda_m x z^m + \overline{b_1 z} + \overline{\mu_m y z^m}$, $(m \geq 2, |x| + |y| = 1 - |b_1|)$ we see that the extreme points of $\mathcal{V}_{\mathcal{H}}(F;\gamma) \subset \{f_m(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \lambda_2(1 - |b_1|)z^2\} + \frac{1}{2}\{f_1(z) - \lambda_2(1 - |b_1|)z^2\},$$

a convex linear combination of functions in clco $\mathcal{V}_{\mathcal{H}}(F;\gamma)$. To see that f_m is not an extreme point if both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can then also be expressed as a convex linear combinations of functions in clco $\mathcal{V}_{\mathcal{H}}(F;\gamma)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1 + \epsilon$ and $B = 1 - |\frac{\epsilon x}{y}|$. We then see that both

$$t_1(z) = z + \lambda_m A x z^m + \overline{b_1 z + \mu_m y B z^m}$$

and

$$t_2(z) = z + \lambda_m (2 - A) x z^m + \overline{b_1 z + \mu_m y (2 - B) z^m}$$

are in clco $\mathcal{V}_{\mathcal{H}}(F;\gamma)$, and that

$$f_m(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}.$$

The extremal coefficient bounds show that functions of the form (19) are the extreme points for clco $\mathcal{V}_{\mathcal{H}}(F;\gamma)$, and so the proof is complete. $\square$

4. INCLUSION RELATION

Following Avici and Zlotkiewicz [1], we refer to the the δ-neighborhood of the function $f(z)$ defined by (2) to be the set $N_\delta(f)$ which contains the functions φ of the form

$$\varphi(z) = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} \overline{B_m z^m}, \quad z \in \mathcal{U}, \tag{20}$$

and

$$\sum_{m=2}^{\infty} m(|a_m - A_m| + |b_m - B_m|) + |b_1 - B_1| \leq \delta.$$

In our case, let us define the generalized δ-neighborhood of f to be the set $N_\delta(f)$ which contains the functions φ of the form (20) and satisfy the restriction

$$\sum_{m=2}^{\infty} C_m[(m-\gamma)(|a_m - A_m| + (m+\gamma)|b_m - B_m|] + (1-\gamma)|b_1 - B_1| \leq (1-\gamma)\delta.$$

Theorem 5. *Let f be given by* (2). *If f satisfies the conditions*

$$\sum_{m=2}^{\infty} m(m-\gamma)|a_m|C_m + \sum_{m=1}^{\infty} m(m+\gamma)|b_m|C_m \leq (1-\gamma), \tag{21}$$

$0 \leq \gamma < 1$ *and*

$$\delta = \frac{1-\gamma}{2-\gamma}\left(1 - \frac{1+\gamma}{1-\gamma}\, b_1\right), \tag{22}$$

then $N_\delta(f) \subset S_{\mathcal{H}}(F,\gamma)$.

Proof. Let f satisfies (21) and $\varphi(z)$ be given by (20) which belongs to $N(f)$. We obtain

$$\begin{aligned}
& (1+\gamma)|B_1| + \sum_{m=2}^{\infty} ((m-\gamma)|A_m| + (m+\gamma)|B_m|)\, C_m \leq (1+\gamma)|B_1 - b_1| \\
+ \quad & (1+\gamma)|b_1| + \sum_{m=2}^{\infty} C_m\left[(m-\gamma)|A_m - a_m| + (m+\gamma)|B_m - b_m|\right] \\
+ \quad & \sum_{m=2}^{\infty} C_m\left[(m-\gamma)|a_m| + (m+\gamma)|b_m|\right] \\
\leq \quad & (1-\gamma)\delta + (1+\gamma)|b_1| + \frac{1}{2-\gamma}\sum_{m=2}^{\infty} mC_m\left((m-\gamma)|a_m| + (m+\gamma)|b_m|\right) \\
\leq \quad & (1-\gamma)\delta + (1+\gamma)|b_1| + \frac{1}{2-\gamma}\left[(1-\gamma) - (1+\gamma)|b_1|\right] \leq 1-\gamma.
\end{aligned}$$

Hence for $\delta = \frac{1-\gamma}{2-\gamma}\left(1 - \frac{1+\gamma}{1-\gamma}|b_1|\right)$, we infer that $\varphi(z) \in S_{\mathcal{H}}(F,\gamma)$ which concludes the proof of Theorem 5. $\square$

Now, we will examine the closure properties of the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathrm{L}_c(f)$ which is defined by

$$\mathrm{L}_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\mathrm{d}t, \quad c > -1.$$

THEOREM 6. *Let* $f(z) \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. *Then* $\mathrm{L}_c(f(z)) \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$.

Proof. From the representation of $\mathrm{L}_c(f(z))$, it follows that

$$\begin{aligned}
\mathrm{L}_c(f) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]\mathrm{d}t \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t+\sum_{m=2}^{\infty} a_m t^n\right)\mathrm{d}t + \overline{\int_0^z t^{c-1}\left(\sum_{m=1}^{\infty} b_m t^m\right)\mathrm{d}t}\right) \\
&= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^m,
\end{aligned}$$

where

$$A_m = \frac{c+1}{c+m}\, a_m; \;\; B_m = \frac{c+1}{c+m}\, b_m.$$

Therefore,

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left\{\frac{m-\gamma}{1-\gamma}\,\frac{c+1}{c+m}|a_m| + \frac{m+\gamma}{1-\gamma}\,\frac{c+1}{c+m}|b_m|\right\} C_m \\
\le\ & \sum_{m=1}^{\infty}\left\{\frac{m-\gamma}{1-\gamma}|a_m| + \frac{m+\gamma}{1-\gamma}|b_m|\right\} C_m \\
\le\ & 1 - \frac{1+\gamma}{1-\gamma}.
\end{aligned}$$

Since $f(z) \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$, therefore by Theorem 2, $\mathrm{L}_c(f(z)) \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. $\square$

Concluding remarks. The various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes (see [5, 6, 7, 8, 9]). The details involved in the derivations of such specializations of the results presented in this paper are fairly straight-forward.

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