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CONVOLUTION TYPE OPERATORS WITH OSCILLATING SYMBOLS ON WEIGHTED LEBESGUE SPACES ON A UNION OF INTERVALS

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Abstract. We establish Fredholm criteria for convolution type operators W with oscillating symbols, continuous on $\mathbb{R}$ and admitting mixed (slowly oscillating and semi-almost periodic) discontinuities at $\pm\infty$, on weighted Lebesgue spaces on a union of intervals with weights in a subclass of Muckenhoupt weights.

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1. INTRODUCTION

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on a Banach space X, and $\mathcal{K}(X)$ the closed two-sided ideal of all compact operators in $\mathcal{B}(X)$. An operator $\mathrm{A} \in \mathcal{B}(X)$ is called *Fredholm* if $\mathrm{Im}\,\mathrm{A}$ is closed in X and the numbers $n(\mathrm{A}) := \dim \mathrm{Ker}\,\mathrm{A}$ and $d(\mathrm{A}) := \dim(X/\mathrm{Im}\,\mathrm{A})$ are finite (see, e.g., [7]). In that case

$$\mathrm{Ind}\,\mathrm{A} := n(\mathrm{A}) - d(\mathrm{A}).$$

Given $1 \le p \le \infty$, let $L^p(\mathbb{R})$ be the usual Lebesgue space with norm denoted by $\|\cdot\|_p$. A measurable function $w : \mathbb{R} \to [0,\infty]$ is called a *weight* if $w^{-1}(\{0,\infty\})$ has Lebesgue measure zero. For $1 \le p < \infty$ and a weight w, we denote by $L^p(\mathbb{R}, w)$ the weighted Lebesgue space with the norm

$$\|f\|_{p,w} := \left(\int_{\mathbb{R}} |f(x)|^p w^p(x) \mathrm{d}x \right)^{1/p}.$$

Let $L^p_N(\mathbb{R}, w)$ be the Banach space of vector functions $f = (f_k)_{k=1}^N$ with entries $f_k \in L^p(\mathbb{R}, w)$ and the norm $\|f\|_{L^p_N(\mathbb{R},w)} = \big(\sum_{k=1}^N \|f_k\|^p_{p,w}\big)^{1/p}$, where $N \in \mathbb{N}$. If $\mathcal{A}$ is a subalgebra of $L^\infty(\mathbb{R})$, then $\mathcal{A}_{N\times N}$ or $[\mathcal{A}]_{N\times N}$ denote the matrix functions $a : \mathbb{R} \to \mathbb{C}^{N\times N}$ whose entries belong to $\mathcal{A}$.

In what follows we assume that $1 < p < \infty$ and w is a *Muckenhoupt weight* (that is, $w \in \mathrm{A}_p(\mathbb{R})$), which means (see [11] and also [9], [5]) that

$$\sup_I \left(\frac{1}{|I|} \int_I w^p(x)\, \mathrm{d}x \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{-q}(x)\, \mathrm{d}x \right)^{1/q} < \infty,$$

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where $1/p+1/q=1$, I ranges over all bounded intervals $I\subset\mathbb{R}$, and $|I|$ is the length of I.

Let $\mathcal{F}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ denote the *Fourier transform*,

$$(\mathcal{F}f)(x):=\int_{\mathbb{R}} f(t)\mathrm{e}^{\mathrm{i}tx}\mathrm{d}t,\quad x\in\mathbb{R}.$$

A function $a\in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $L^p(\mathbb{R},w)$ if the convolution operator $\mathrm{W}^0(a):=\mathcal{F}^{-1}a\mathcal{F}$ maps $L^2(\mathbb{R})\cap L^p(\mathbb{R},w)$ into itself and extends to a bounded linear operator on $L^p(\mathbb{R},w)$ (notice that $L^2(\mathbb{R})\cap L^p(\mathbb{R},w)$ is dense in $L^p(\mathbb{R},w)$ if $w\in \mathrm{A}_p(\mathbb{R})$). Let $[M_{p,w}]_{N\times N}$ stand for the Banach algebra of all Fourier multipliers a on $L^p_N(\mathbb{R},w)$ equipped with the norm

$$\|a\|_{[M_{p,w}]_{N\times N}}:=\left\|\mathrm{W}^0(a)\right\|_{\mathcal{B}(L^p_N(\mathbb{R},w))}$$

Let χ_+ be the characteristic function of $\mathbb{R}_+=[0,\infty)$. By $L^p(\mathbb{R}_+,w)$ we understand the space $L^p(\mathbb{R}_+,w|\mathbb{R}_+)$. For $a\in M_{p,w}$, the Wiener-Hopf operator $\mathrm{W}(a)$ is defined on the space $L^p(\mathbb{R}_+,w)$ by

$$\mathrm{W}(a)f=\chi_+\mathrm{W}^0(a)\chi_+f,\quad\text{for}\quad f\in L^p(\mathbb{R}_+,w).$$

Let $\dot{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$, $\overline{\mathbb{R}}=[-\infty,+\infty]$, and let PC be the C^*-algebra of all functions on $\mathbb{R}$ having finite one-sided limits at every point $t\in\dot{\mathbb{R}}$. By Stechkin's inequality (see, e.g., [6, Theorem 17.1]), every function $a\in PC$ of finite total variation belongs to $M_{p,w}$. We denote by $C_{p,w}(\dot{\mathbb{R}})$ (resp. $C_{p,w}(\overline{\mathbb{R}})$) the closure in $M_{p,w}$ of the set of all functions $a\in C(\dot{\mathbb{R}})$ (resp. $a\in C(\overline{\mathbb{R}})$) with finite total variation. Obviously, $C_{p,w}(\dot{\mathbb{R}})\subset C(\dot{\mathbb{R}})$, $C_{p,w}(\overline{\mathbb{R}})\subset C(\overline{\mathbb{R}})$.

To study Wiener-Hopf operators with semi-almost periodic (SAP) symbols, we need to consider the set $\mathrm{A}^0_p(\mathbb{R})$ consisting of all weights $w\in\mathrm{A}_p(\mathbb{R})$ for which the functions $e_\lambda:x\mapsto \mathrm{e}^{\mathrm{i}\lambda x}$ belong to $M_{p,w}$ for all $\lambda\in\mathbb{R}$. Let $w\in\mathrm{A}^0_p(\mathbb{R})$. Then the set AP^0 of all almost periodic polynomials $\sum_{\lambda\in\Lambda_0}c_\lambda e_\lambda$, where $c_\lambda\in\mathbb{C}$ and Λ_0 is a finite subset of $\mathbb{R}$, is contained in $M_{p,w}$. We define $AP_{p,w}$ as the closure of AP^0 in $M_{p,w}$. Clearly, $AP_{p,w}$ is a Banach subalgebra of $M_{p,w}$. Let $SAP_{p,w}$ denote the smallest closed subalgebra of $M_{p,w}$ that contains $C_{p,w}(\overline{\mathbb{R}})$ and $AP_{p,w}$. It is clear that

$$AP_{p,w}\subset AP:=AP_{2,1}\subset L^\infty(\mathbb{R}),\quad SAP_{p,w}\subset SAP:=SAP_{2,1}\subset L^\infty(\mathbb{R}).$$

Let $C_b(\mathbb{R})$ be the C^*-algebra of all bounded continuous functions $a:\mathbb{R}\to\mathbb{C}$. Following [18] we denote by SO the C^*-algebra of *slowly oscillating at* ∞ functions,

$$SO:=\Big\{f\in C_b(\mathbb{R}):\lim_{x\to+\infty}\sup_{t,s\in[-2x,-x]\cup[x,2x]}|f(t)-f(s)|=0\Big\}. \tag{1}$$

Consider the commutative Banach algebra

$$SO^3:=\Big\{a\in SO\cap C^3(\mathbb{R}):\lim_{|x|\to\infty}(D^\gamma a)(x)=0,\ \gamma=1,2,3\Big\}$$

equipped with the norm $\|a\|_{SO^3} := \max_{\gamma=0,1,2,3} \|D^\gamma a\|_{L^\infty(\mathbb{R})}$ where $(Da)(x) = xa'(x)$ for $x \in \mathbb{R}$. By [13, Corollary 2.10], $SO^3 \subset M_{p,w}$. For $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, let $SO_{p,w}$ denote the closure of SO^3 in $M_{p,w}$. Clearly, $SO_{p,w}$ is a commutative Banach subalgebra of $M_{p,w}$. Since $M_{p,w} \subset M_2 = L^\infty(\mathbb{R})$, we conclude that $SO_{p,w} \subset SO$.

Let $[\mathcal{A}, \mathcal{B}]$ denote the smallest Banach algebra that contains Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Then $[SO_{p,w},\, SAP_{p,w}]$ is the Banach subalgebra of $M_{p,w}$ generated by all functions in $SO_{p,w}$ and $SAP_{p,w}$. We will omit index w, if $w = 1$.

The Fredholmness in Banach algebras generated by all operators $a\mathrm{W}^0(b)$ with $a \in [SO,\, PC]_{N\times N}$ and $b \in [SO_p,\, PC_p]_{N\times N}$ on unweighted Lebesgue spaces $L^p_N(\mathbb{R})$ was studied in [1], [2]. Wiener-Hopf operators with slowly oscillating matrix symbols on weighted Lebesgue spaces were investigated in [13].

Wiener-Hopf operators with semi-almost periodic symbols on the spaces $L^p(\mathbb{R}_+)$ $(1 < p < \infty)$ were studied by R.V. Duduchava and A.I. Saginashvili [8] (for preceding results on integro-difference operators see [10]). The Fredholm theory for Wiener-Hopf operators with semi-almost periodic matrix symbols on the spaces $L^p_N(\mathbb{R}_+)$ $(1 < p < \infty,\ N > 1)$ based on the concept of almost periodic (AP) factorization was constructed by I.M. Spitkovsky and the first author (see [6], [15] and the references therein). Wiener-Hopf operators with semi-almost periodic matrix symbols on weighted Lebesgue spaces were investigated in [12].

A Fredholm theory for Toeplitz operators with oscillating matrix symbols $a \in [SO,\, SAP]_{N\times N}$ on Hardy spaces H^p_N was constructed in [4]. Fredholm criteria for Wiener-Hopf operators $\mathrm{W}(a)$ with oscillating symbols $a \in [SO_{p,w},\, SAP_{p,w}]_{N\times N}$ on weighted Lebesgue spaces $L^p_N(\mathbb{R}_+, w)$ with Muckenhoupt weights $w \in A^0_p(\mathbb{R})$ were obtained in [14].

Let $J = \bigcup_{m=1}^n J_m$ where $J_m = [a_{m-1}, a_m]$ are intervals of $\mathbb{R}$ admitting only endpoints in common, and $0 = a_0 < a_1 < a_2 < \ldots < a_n < \infty$. In the present paper we establish Fredholm criteria for the convolution type operator

$$\mathrm{W} := \chi_+ \sum\nolimits_{m=1}^n \mathcal{F}^{-1} K_m \mathcal{F} \chi_{J_m} I :\ L^p(J, w) \to L^p(J, w), \tag{2}$$

where $K_m \in [SO_{p,w}, SAP_{p,w}]$, χ_{J_m} are the characteristic functions of J_m, and $f \in L^p(J, w)$ is extended by zero to $\mathbb{R} \setminus J$.

The paper is organized as follows. In Section 2 we collect results on algebras of slowly oscillating and semi-almost periodic functions, their maximal ideal spaces, and present invertibility and Fredholm criteria for Wiener-Hopf operators with almost periodic and semi-almost periodic matrix symbols, respectively.

In Section 3, applying the Allan-Douglas local principle (see, e.g., [7, Section 1.7]) we obtain an intermediate Fredholm criterion for Wiener-Hopf operators $\mathrm{W}(a)$ with symbols $a \in [SO_{p,w},\, SAP_{p,w}]_{N\times N}$ on the space $L^p_N(\mathbb{R}_+, w)$, and also give necessary Fredholm conditions for $\mathrm{W}(a)$ in terms of invertibility of simpler Wiener-Hopf operators with symbols in the algebra $[AP_{p,w}]_{N\times N}$.

In Section 4 we consider an equivalent reduction of the convolution type operator W defined by (2) to the Wiener-Hopf operator $\mathrm{W}(G)$ with some matrix symbol $G \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$ where $N = n+1$, and give its applications to the invertibility and Fredholmness of W.

In Section 5, making use of the results of Sections 3–4, we study the Fredholmness of the convolution type operator (2) on the space $L^p(J,w)$.

In Section 6, applying the concept of canonical generalized AP factorization, we establish a Fredholm criterion for the operator (2) on the space $L^2(J)$.

2. AUXILIARY RESULTS

2.1. Slowly oscillating Fourier multipliers on $L^p(\mathbb{R},w)$. Consider the commutative C^*-algebra SO of slowly oscillating functions defined by (1). Clearly, SO is a subalgebra of $L^\infty(\mathbb{R})$ which contains all functions in $C(\dot{\mathbb{R}})$. Identifying the points $t \in \dot{\mathbb{R}}$ with the evaluation functionals δ_t on $\dot{\mathbb{R}}$, $\delta_t(f) = f(t)$, we see that the maximal ideal space $\mathcal{M}(SO)$ of SO is of the form

$$\mathcal{M}(SO) = \mathbb{R} \cup \mathcal{M}_\infty(SO), \quad \text{where } \mathcal{M}_\infty(SO) := \Big\{\xi \in \mathcal{M}(SO) : \xi|_{C(\dot{\mathbb{R}})} = \delta_\infty\Big\}$$

is the fiber of $\mathcal{M}(SO)$ over ∞. By [4, Proposition 5],

$$\mathcal{M}_\infty(SO) = (\operatorname{clos}_{SO^*}\mathbb{R}) \setminus \mathbb{R},$$

where $\operatorname{clos}_{SO^*}\mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*, the dual space of SO.

Lemma 1 ([13]). *If $1 < p < \infty$ and $w \in \mathrm{A}_p(\mathbb{R})$, then the maximal ideal spaces of $SO_{p,w}$ and SO coincide as sets, that is, $\mathcal{M}(SO_{p,w}) = \mathcal{M}(SO)$.*

Lemma 1 and the Gelfand theory immediately give the following assertion.

Corollary 1. *If $1 < p < \infty$ and $w \in \mathrm{A}_p(\mathbb{R})$, then the Banach algebra $SO_{p,w}$ is inverse closed in the C^*-algebras SO and $L^\infty(\mathbb{R})$, that is, if $a \in SO_{p,w}$ is invertible, then $a^{-1} \in SO_{p,w}$ too.*

2.2. The fiber $\mathcal{M}_\infty([SO,\, SAP])$. By [19], any function $a \in SAP$ can be uniquely represented in the form

$$a = a_+u_+ + a_-u_- + a_0 \tag{3}$$

where $a_\pm \in AP$, $a_0 \in C(\dot{\mathbb{R}})$, $a_0(\infty) = 0$, $u_\pm(x) = (1 \pm \tanh x)/2$, and the mappings $\nu_\pm : a \mapsto a_\pm$ are C^*-algebra homomorphisms of SAP onto AP.

According to [18, Section 3], the C^*-algebras SO and SAP are asymptotically independent, which means the following.

Proposition 1. *The fiber $\mathcal{M}_\infty([SO,\, SAP])$ is naturally homeomorphic to the set $\mathcal{M}_\infty(SO) \times \mathcal{M}_\infty(SAP)$, that is, for every $\mu \in \mathcal{M}_\infty([SO,\, SAP])$ there are characters $\xi \in \mathcal{M}_\infty(SO)$ and $\nu \in \mathcal{M}_\infty(SAP)$ such that $\mu|_{SO} = \xi$ and $\mu|_{SAP} = \nu$.*

Identifying $\mu \in \mathcal{M}_\infty([SO,\, SAP])$ with pairs $(\xi,\nu) \in \mathcal{M}_\infty(SO)\times\mathcal{M}_\infty(SAP)$ due to Proposition 1, for every $\xi \in \mathcal{M}_\infty(SO)$ we obtain a homomorphism

$$\beta_\xi : [SO,\, SAP] \to SAP|_{\mathcal{M}_\infty(SAP)}, \quad (\beta_\xi\, \varphi)(\nu) = (\xi,\nu)\varphi \quad \text{for } \nu \in \mathcal{M}_\infty(SAP).$$

Hence, for every $\varphi \in [SO,\, SAP]$ there exists a non-unique function $\varphi_\xi \in SAP$ with uniquely determined almost periodic representatives $\varphi_{\xi,\pm}$ at $\pm\infty$ such that $\beta_\xi\, \varphi = \varphi_\xi\big|_{\mathcal{M}_\infty(SAP)}$. Since the fiber $\mathcal{M}_\infty(AP)$ is homeomorphic to $\mathcal{M}(AP)$, identifying $\mathcal{M}_\infty(SAP)$ and $\mathcal{M}_\infty(AP)\times\mathcal{M}_\infty(AP)$, we conclude that the maps

$$\gamma_\pm : \ \varphi_\xi\, |_{\mathcal{M}_\infty(SAP)} \mapsto \varphi_{\xi,\pm}|_{\mathcal{M}_\infty(AP)} \ \mapsto \varphi_{\xi,\pm}$$

are Banach algebra homomorphisms of $SAP|_{\mathcal{M}_\infty(SAP)}$ onto AP. Thus the maps

$$\nu_{\xi,\pm} = \gamma_\pm \circ \beta_\xi : \ [SO,\, SAP] \to AP, \quad \nu_{\xi,\pm}\, \varphi = \varphi_{\xi,\pm} \tag{4}$$

are well-defined Banach algebra homomorphisms for every $\xi \in \mathcal{M}_\infty(SO)$.

2.3. Wiener-Hopf operators with almost periodic matrix symbols. Let APW be the Banach algebra of all functions in AP of the form $a = \sum_\lambda a_\lambda e_\lambda$ with $a_\lambda \in \mathbb{C}$, $\lambda \in \mathbb{R}$, and the norm $\|a\|_{\mathrm{W}} := \sum_\lambda |a_\lambda| < \infty$. Let $APW^\pm$ be the closure in APW of the set AP^0 of all almost periodic polynomials $\sum_\lambda a_\lambda e_\lambda$ with $\pm\lambda \ge 0$. Thus, $APW^\pm$ are Banach subalgebras of APW.

For every function $a \in AP$, there exist the quantities

$$M(a) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T a(t)\mathrm{d}t, \quad \Omega(a) := \big\{\lambda \in \mathbb{R} : M(ae_{-\lambda}) \ne 0\big\},$$

$$\kappa(a) := \lim_{x\to\infty} \big(x^{-1}\arg a(x)\big) \ \text{ if } \ a^{\pm 1} \in AP, \quad \mathbf{d}(a) := \mathrm{e}^{M(\ln a)} \ \text{ if } \ \ln a \in AP,$$

which are called, respectively, the *Bohr mean value*, the *Bohr-Fourier spectrum*, the *mean motion*, and the *geometric mean* of a (see [6]).

Given $1 < p < \infty$ and $w \in \mathrm{A}_p^0(\mathbb{R})$, let $APW_{p,w}$ be the Banach subalgebra of $M_{p,w}$ composed by the series $a = \sum_\lambda a_\lambda e_\lambda$ with coefficients $a_\lambda \in \mathbb{C}$ and the norm

$$\|a\|_{\mathrm{W}} := \sum\nolimits_\lambda |a_\lambda|\, \|e_\lambda\|_{M_{p,w}},$$

where $\|e_\lambda\|_{M_{p,w}} = \|v_\lambda\|_{L^\infty(\mathbb{R})}$ for $\lambda \in \mathbb{R}$ and the functions $v_\lambda(x) = \frac{w(x+\lambda)}{w(x)}$ are in $L^\infty(\mathbb{R})$ for weights $w \in \mathrm{A}_p^0(\mathbb{R})$ (see [12, Proposition 2.3]).

Let $AP^\pm_{p,w}$ be the $M_{p,w}$ closure of the set of all almost periodic polynomials $\sum_\lambda a_\lambda e_\lambda$ with $\pm\lambda \ge 0$. Along with the Banach subalgebras $AP^\pm_{p,w}$ of $M_{p,w}$ we consider the Banach subalgebras $APW^\pm_{p,w} := APW_{p,w} \cap AP^\pm_{p,w}$ of $APW_{p,w}$. Clearly,

$$APW_{p,w} \subset AP_{p,w} \subset AP, \quad APW^\pm_{p,w} \subset AP^\pm_{p,w} \subset AP^\pm.$$

Given $p \in (1,\infty)$, consider the weights $w \in \mathrm{A}_p^0(\mathbb{R})$ satisfying the condition

$$\lim_{|t|\to\infty} \operatorname*{ess\,sup}_{x,y\in[t,\ t+1]} \big|\ln w(x) - \ln w(y)\big| = 0. \tag{5}$$

According to [12, Example 2.4], if parameters $\delta, \nu, \eta \in \mathbb{R}$ satisfy the relations

$$-1/p < \delta - |\nu|\sqrt{\eta^2+1} \le \delta + |\nu|\sqrt{\eta^2+1} < 1/q,$$

then the weight

$$w(x) = \begin{cases} \mathrm{e}^{(\delta+\nu\sin(\eta\log(\log|x|)))\log|x|} & \text{if } \ |x| \ge e, \\ \mathrm{e}^{\delta} & \text{if } \ |x| < e, \end{cases}$$

with different indices of powerlikeness at ∞ (see [5, Section 3.6]), gives an example of weights in $\mathrm{A}_p^0(\mathbb{R})$ possessing the property (5). According to [17], a weight $w \in \mathrm{A}_p^0(\mathbb{R})$ is equivalent to the continuous weight $\omega \in C(\mathbb{R})$ given by

$$\omega(x) = \exp\left(\int_{-1/2}^{1/2} \ln w(x+t)\,\mathrm{d}t\right), \tag{6}$$

where the equivalence means that $w/\omega,\ \omega/w \in L^\infty(\mathbb{R})$. Furthermore, by (6),

$$\big|\ln\omega(x) - \ln\omega(y)\big| \le \int_{-1/2}^{1/2} \big|\ln(w(x+t)) - \ln(w(y+t))\big|\mathrm{d}t,$$

whence ω satisfies (5) too. Hence we may without loss of generality assume that $w \in C(\mathbb{R}) \cap \mathrm{A}_p^0(\mathbb{R})$. Then, for every $\lambda \in \mathbb{R}$, we infer from (5) that

$$v_\lambda(x) = \frac{w(x+\lambda)}{w(x)} \in C(\dot{\mathbb{R}}) \quad \text{and} \quad \lim_{|x|\to\infty} v_\lambda(x) = 1. \tag{7}$$

Let $G\mathcal{A}$ denote the group of all invertible elements of a unital algebra $\mathcal{A}$. Since $\|v_\lambda\|_\infty \ge 1$ for all $\lambda \in \mathbb{R}$ due to (7), we conclude that

$$G[APW_{p,w}]_{N\times N} \subset GAPW_{N\times N}, \quad G[APW_{p,w}^{\pm}]_{N\times N} \subset GAPW_{N\times N}^{\pm}$$

for all $N \in \mathbb{N}$ in view of the relation

$$\sum\nolimits_\lambda \|a_\lambda\|_{\mathbb{C}^{N\times N}} \le \sum\nolimits_\lambda \|a_\lambda\|_{\mathbb{C}^{N\times N}}\ \|v_\lambda\|_\infty.$$

Consider now the invertibility of Wiener-Hopf operators $\mathrm{W}(a)$ with matrix symbols $a \in [APW_{p,w}]_{N\times N}$ on weighted Lebesgue spaces $L_N^p(\mathbb{R}_+, w)$ where $1 < p < \infty$, $N \in \mathbb{N}$, $w \in \mathrm{A}_p^0(\mathbb{R})$ and (5) holds. By [12, Section 6.1], in that case the operator $\mathrm{W}(a)$ is invertible on the spaces $L_N^p(\mathbb{R}_+, w)$ and $L_N^p(\mathbb{R}_+)$ only simultaneously. Hence from [6, Corollary 19.11] we obtain the following ([12, Theorem 6.1]):

Theorem 1. *Let $1 < p < \infty$, $N \in \mathbb{N}$, $w \in \mathrm{A}_p^0(\mathbb{R})$, and let condition* (5) *hold. If $a \in [APW_{p,w}]_{N\times N}$, then the Wiener-Hopf operator* $\mathrm{W}(a)$ *is invertible on the space $L_N^p(\mathbb{R}_+, w)$ if and only if a admits a canonical right APW factorization, that is, $a = a^-a^+$ where $a^\pm \in GAPW_{N\times N}^{\pm}$.*

If $a \in APW_{N\times N}$ admits a canonical right APW factorization $a = a^-a^+$, then the matrix $\mathbf{d}(a) = M(a^-)M(a^+) \in \mathbb{C}^{N\times N}$, where the Bohr mean values $M(a^\pm)$ are defined entry-wise, is called the *geometric mean* of a. By [6, Proposition 8.4], $\mathbf{d}(a)$ is uniquely defined by a. Obviously, $\det\mathbf{d}(a) \ne 0$.

2.4. Wiener-Hopf operators: semi-almost periodic matrix symbols. Consider now the Fredholmness of Wiener-Hopf operators $\mathrm{W}(a)$ with matrix symbols $a \in [SAP_{p,w}]_{N\times N}$ on weighted Lebesgue spaces $L^p_N(\mathbb{R}_+, w)$ where $1 < p < \infty$, $N \in \mathbb{N}$, $w \in \mathrm{A}^0_p(\mathbb{R})$ and (5) holds. In what follows, according to (3), we denote by $a_r := a_+$ and $a_l := a_-$ the almost periodic representatives of a at $+\infty$ and $-\infty$, respectively. We also assume that $a_l,\ a_r \in [APW_{p,w}]_{N\times N}$.

According to [6, Definition 3.13], the *Cauchy index* of any function $a \in GSAP$ with $\kappa(a_l) = \kappa(a_r) = 0$ is defined by the formula

$$\operatorname{ind} a := \frac{1}{2\pi} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \big((\arg a)(x) - (\arg a)(-x)\big)\mathrm{d}x$$

where the limit exists, is finite, independent of the particular choice of continuous branch of $\arg a$, and possesses the logarithmic property: for every $f_1, f_2 \in GSAP$ with almost periodic representatives at $\pm\infty$ having zero mean motions,

$$\operatorname{ind}(f_1 f_2) = \operatorname{ind} f_1 + \operatorname{ind} f_2.$$

Theorem 2. [12, Theorem 6.8] *Let $1 < p < \infty$, $N \in \mathbb{N}$, $w \in \mathrm{A}^0_p(\mathbb{R})$, let condition (5) hold. If $a \in [SAP_{p,w}]_{N\times N}$ and $a_l, a_r \in [APW_{p,w}]_{N\times N}$, then the Wiener-Hopf operator $\mathrm{W}(a)$ is Fredholm on the space $L^p_N(\mathbb{R}_+, w)$ if and only if the following three conditions are satisfied:*

(i) *$a \in GSAP_{N\times N}$,*
(ii) *a_l and a_r admit canonical right APW factorizations,*
(iii) *$\dfrac{1}{p} + \dfrac{1}{2\pi}\arg \eta_j \notin \mathbb{Z}$ for all eigenvalues η_j of the matrix $\mathbf{d}^{-1}(a_r)\mathbf{d}(a_l)$.*

If $\mathrm{W}(a)$ is Fredholm, then its index is calculated by the formula

$$\operatorname{Ind} \mathrm{W}(a) = -\operatorname{ind}(\det a) + \frac{N}{p} - \sum_{j=1}^{N} \left\{ \frac{1}{p} + \frac{1}{2\pi} \arg \eta_j \right\}, \tag{8}$$

where $\{x\}$ denotes the fractional part of a number $x \in \mathbb{R}$.

3. AN APPLICATION OF THE ALLAN-DOUGLAS LOCAL PRINCIPLE

Given $p \in (1, \infty)$, $w \in \mathrm{A}^0_p(\mathbb{R})$ and $N \in \mathbb{N}$, we consider the Banach subalgebra $\mathcal{Z}$ of $\mathcal{B}(L^p_N(\mathbb{R}_+, w))$ generated by all Wiener-Hopf operators $\mathrm{W}(c)$ with symbols cI_N where $c \in SO_{p,w}$ and I_N is the $N \times N$ identity matrix.

By [13, Lemma 5.3], the commutators of the multiplication operators aI ($a \in PC$) and the convolution operators $\mathrm{W}^0(b)$ ($b \in SO_{p,w}$) are compact on the space $L^p(\mathbb{R}, w)$. Hence

$$\mathrm{W}(a)\mathrm{W}(b) \simeq \mathrm{W}(ab) \simeq \mathrm{W}(b)\mathrm{W}(a) \quad \text{for all } a \in M_{p,w} \text{ and all } b \in SO_{p,w}, \tag{9}$$

where $\mathrm{A} \simeq \mathrm{B}$ means that the operator $\mathrm{A}-\mathrm{B}$ is compact on the space $L^p(\mathbb{R}_+, w)$.

Let $\Lambda := \Lambda(\mathcal{Z})$ denote the Banach subalgebra of $\mathcal{B} := \mathcal{B}(L_N^p(\mathbb{R}_+, w))$ that consists of all operators of local type (with respect to $\mathcal{Z}$), that is,

$$\Lambda := \Big\{\mathrm{A} \in \mathcal{B} : \ \mathrm{W}(c)A - \mathrm{W}(c) \in \mathcal{K} \ \text{ for all } \ c \in SO_{p,w}\Big\},$$

where $\mathcal{K} := \mathcal{K}(L_N^p(\mathbb{R}_+, w))$ is the ideal of all compact operators in $\mathcal{B}$. The quotient Banach algebra $\Lambda^\pi = \Lambda/\mathcal{K}$ is inverse closed in the Calkin algebra $\mathcal{B}^\pi = \mathcal{B}/\mathcal{K}$, and $\mathcal{Z}^\pi = (\mathcal{Z} + \mathcal{K})/\mathcal{K}$ is a central subalgebra of Λ^π. For $\mathrm{A} \in \mathcal{B}$, let $\mathrm{A}^\pi := \mathrm{A} + \mathcal{K}$. By (9), the Wiener-Hopf operators $\mathrm{W}(a)$ with symbols $a \in [SO_{p,w},\, SAP_{p,w}]_{N\times N}$ belong to the Banach algebra Λ.

For every $\xi \in \mathcal{M}(SO) = \mathcal{M}(SO_{p,w})$, let J_ξ^π denote the closed two-sided ideal of Λ^π generated by the maximal ideal

$$I_\xi^\pi := \Big\{\mathrm{W}^\pi(bI_N) : \ b \in SO_{p,w}, \ \xi(b) = 0\Big\}$$

of the commutative algebra $\mathcal{Z}^\pi$, and let $\Lambda_\xi^\pi := \Lambda^\pi / J_\xi^\pi$ be the corresponding quotient Banach algebra. Consider the cosets

$$\mathrm{W}_\xi^\pi(a) := \mathrm{W}^\pi(a) + J_\xi^\pi \in \Lambda_\xi^\pi.$$

To study the Fredholmness of Wiener-Hopf operators $\mathrm{W}(a)$ with oscillating matrix symbols a on the space $L_N^p(\mathbb{R}_+, w)$, we need to apply the Allan-Douglas local principle (see, e.g., [7, Section 1.7]), which gives the following.

THEOREM 3. *The operator* $\mathrm{W}(a)$ *with a symbol* $a \in [SO_{p,w},\, SAP_{p,w}]_{N\times N}$ *is Fredholm on the space* $L_N^p(\mathbb{R}_+, w)$ *if and only if for every* $\xi \in \mathcal{M}(SO)$ *the coset* $\mathrm{W}_\xi^\pi(a) = \mathrm{W}^\pi(a) + J_\xi^\pi$ *is invertible in the quotient algebra* Λ_ξ^π.

The mappings $\nu_{\xi,\pm}$ defined by (4) allows us to obtain a necessary condition for the Fredholmness of the Wiener-Hopf operators $\mathrm{W}(a)$ with symbols $a \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$ on weighted Lebesgue spaces $L_N^p(\mathbb{R}_+, w)$. We obtain the following corollary ([14, Corollary 4.5]):

COROLLARY 2. *If* $p \in (1, \infty)$, $w \in \mathrm{A}_p^0(\mathbb{R})$, $N \in \mathbb{N}$, *and the Wiener-Hopf operator* $\mathrm{W}(a)$ *with a symbol* $a \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$ *is Fredholm on the space* $L_N^p(\mathbb{R}_+, w)$, *then for every* $\xi \in \mathcal{M}_\infty(SO)$ *the operators* $\mathrm{W}(a_{\xi,\pm})$ *with symbols* $a_{\xi,\pm} = \nu_{\xi,\pm} a \in [AP_{p,w}]_{N\times N}$ *are invertible on the space* $L_N^p(\mathbb{R}_+, w)$ *and the norms of their inverses are uniformly bounded.*

4. CONVOLUTION TYPE OPERATORS

Let $p \in (1, \infty)$, $w \in \mathrm{A}_p^0(\mathbb{R})$, let χ_δ stand for the operator of multiplication by the characteristic function of a set $\delta \subset \mathbb{R}$, and let $J = \bigcup_{m=1}^n J_m$ where J_m are intervals of $\mathbb{R}$ admitting only endpoints in common. Consider the convolution type operator

$$\text{(10)} \qquad \mathrm{W} : L^p(J, w) \to L^p(J, w), \quad f \mapsto \chi_J\Big(\sum\nolimits_{m=1}^n k_m * (\chi_{J_m} f)\Big),$$

where k_m are tempered distributions such that $K_m = \mathcal{F}k_m \in [SO_{p,w}, SAP_{p,w}]$, and $f \in L^p(J,w)$ is extended by zero to $\mathbb{R} \setminus J$. Assume that

(11) $J_m = [a_{m-1}, a_m]$ $(m = 1,2,\dots,n)$, $0 = a_0 < a_1 < a_2 < \ldots < a_n < \infty$.

We say that two bounded linear operators A and B are equivalent if either both operators are not normally solvable or both A and B are normally solvable and

$$\dim \operatorname{Ker} \mathrm{A} = \dim \operatorname{Ker} B, \quad \dim \operatorname{Coker} \mathrm{A} = \dim \operatorname{Coker} B.$$

By analogy with [3, Section 2] (cf. also [20], [16]), we obtain the following result for weighted Lebesgue spaces.

Lemma 2. *The convolution type operator* $\mathrm{W} : L^p(J,w) \to L^p(J,w)$ *given by* (10) *is equivalent to the Wiener-Hopf operator*

$$\mathrm{W}(G) := \chi_+ \mathcal{F}^{-1} G \mathcal{F} : \, L^p_N(\mathbb{R}_+, w) \to L^p_N(\mathbb{R}_+, w) \tag{12}$$

where $N = n+1$,

$$G = \begin{bmatrix} e_{-\gamma_1} & 0 & \ldots & 0 & 0 \\ 0 & e_{-\gamma_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & e_{-\gamma_n} & 0 \\ K_1 & K_2 e_{\varepsilon_1} & \ldots & K_n e_{\varepsilon_{n-1}} & e_{\varepsilon_n} \end{bmatrix} \in [SO_{p,w}, SAP_{p,w}]_{N\times N}, \tag{13}$$

$$\gamma_m = a_m - a_{m-1}, \quad \varepsilon_m = \gamma_1 + \cdots + \gamma_m \quad (m = 1,2,\dots,n), \tag{14}$$

and a_m *are given by* (11).

Indeed, the operator $\mathrm{W} : L^p(J,w) \to L^p(J,w)$ is equivalent to the operator

$$\widetilde{\mathrm{W}} := \chi_+ \sum_{m=1}^{n} \mathcal{F}^{-1} K_m \mathcal{F} \chi_{J_m} I + (1-\chi_J) I : \, L^p(\mathbb{R}_+, w) \to L^p(\mathbb{R}_+, w), \tag{15}$$

where $\chi_+ = \chi_{\mathbb{R}_+}$. Setting for $m = 1,2,\dots,n$, $\chi_m = \chi_{J_m} I$,

$$V_m^{\pm 1} := \chi_+ \mathcal{F}^{-1} e_{\pm\gamma_m} \mathcal{F} : \, L^p(\mathbb{R}_+, w) \to L^p(\mathbb{R}_+, w),$$
$$\mathrm{W}_m := \chi_+ \mathcal{F}^{-1} K_m \mathcal{F} : \, L^p(\mathbb{R}_+, w) \to L^p(\mathbb{R}_+, w),$$

with γ_m given by (14), we conclude from (15) that

$$\widetilde{\mathrm{W}} = \sum_{m=1}^{n} \mathrm{W}_m \chi_m + \Big(I - \sum_{m=1}^{n} \chi_m\Big),$$

where the projections $\chi_m = \chi_{J_m}$ can be represented in the form

$$\chi_m = V_1 V_2 \cdots V_{m-1} (I - V_m V_m^{-1}) V_{m-1}^{-1} \cdots V_2^{-1} V_1^{-1}. \tag{16}$$

Taking now $\widehat{\mathrm{W}} = \mathrm{W}(G)$ and $\mathrm{W}_0 = \widetilde{\mathrm{W}}$, we immediately infer the equivalence of the operators $\mathrm{W}(G)$ and $\widetilde{\mathrm{W}}$ (see (12) and (15)) from the following lemma ([3, Lemma 2.3]).

Lemma 3. *Let V_m $(m = 1, 2, \ldots, n)$ be bounded linear operators acting on a Banach space X, invertible from the left with bounded left inverses V_m^{-1}, and let χ_m $(m = 1, 2, \ldots, n)$ be the bounded projections defined by* (16)*. Then for any bounded operators W_m $(m = 1, 2, \ldots, n)$ we have*

$$\widehat{\mathrm{W}} := \begin{bmatrix} V_1^{-1} & 0 & \ldots & 0 & 0 \\ 0 & V_2^{-1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V_n^{-1} & 0 \\ \mathrm{W}_1 & \mathrm{W}_2 V_1 & \ldots & \mathrm{W}_n V_1 V_2 \cdots V_{n-1} & V_1 V_2 \cdots V_n \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \\ \mathrm{W}_1 V_1 & \mathrm{W}_2 V_1 V_2 & \ldots & \mathrm{W}_n V_1 V_2 \cdots V_n & \mathrm{W}_0 \end{bmatrix} Y,$$

where $\quad \mathrm{W}_0 = \sum_{m=1}^{n} \mathrm{W}_m \chi_m + \Big(I - \sum_{m=1}^{n} \chi_m\Big),$

$$Y = \begin{bmatrix} V_1^{-1} & 0 & \ldots & 0 & 0 \\ 0 & V_2^{-1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V_n^{-1} & 0 \\ \chi_1 & \chi_2 V_1 & \ldots & \chi_n V_1 V_2 \cdots V_{n-1} & V_1 V_2 \cdots V_n \end{bmatrix},$$

$$Y^{-1} = \begin{bmatrix} V_1 & 0 & \ldots & 0 & \chi_1 \\ 0 & V_2 & \ldots & 0 & V_1^{-1}\chi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V_n & V_{n-1}^{-1} \cdots V_2^{-1} V_1^{-1} \chi_n \\ 0 & 0 & \ldots & 0 & V_n^{-1} \cdots V_2^{-1} V_1^{-1} \end{bmatrix}.$$

By analogy with [3, Theorem 2.5], Lemmas 2 and 3 imply the following.

Theorem 4. *The operators*

$$\mathrm{W} : L^p(J, w) \to L^p(J, w) \quad \textit{and} \quad \mathrm{W}(G) : L^p_N(\mathbb{R}_+, w) \to L^p_N(\mathbb{R}_+, w)$$

are invertible only simultaneously, and

$$\mathrm{W}^{-1} = \big(\chi_{J_1},\ \chi_{J_2}\mathrm{W}(e_{\varepsilon_1}),\ \ldots,\ \chi_{J_n}\mathrm{W}(e_{\varepsilon_{n-1}}),\ \chi_J \mathrm{W}(e_{\varepsilon_n})\big) \\ \times\ (\mathrm{W}(G))^{-1}\,(0,\ \ldots,\ 0,\ \chi_J I)^T.$$

Theorem 5. *The operators*

$$\mathrm{W} : L^p(J, w) \to L^p(J, w) \quad \textit{and} \quad \mathrm{W}(G) : L^p_N(\mathbb{R}_+, w) \to L^p_N(\mathbb{R}_+, w)$$

are Fredholm only simultaneously, and in that case $\operatorname{Ind}\mathrm{W} = \operatorname{Ind}\mathrm{W}(G)$,

$$\dim\operatorname{Ker}\mathrm{W} = \dim\operatorname{Ker}\mathrm{W}(G), \quad \dim\operatorname{Coker}\mathrm{W} = \dim\operatorname{Coker}\mathrm{W}(G).$$

5. CONVOLUTION OPERATORS WITH OSCILLATING DATA

Given $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$ and $N \in \mathbb{N}$, in this section we study the Fredholmness of Wiener-Hopf operators $W(a)$ with matrix symbols $a \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$ on the space $L_N^p(\mathbb{R}_+, w)$ under the condition that for every $\xi \in \mathcal{M}_\infty(SO)$ the matrix functions $a_{\xi,\pm} = \nu_{\xi,\pm} a$ admit right $AP_{p,w}$ factorizations.

According to [12, Section 5], a matrix function $a \in [AP_{p,w}]_{N\times N}$ is said to admit a *right $AP_{p,w}$ factorization* if it can be represented in the form

$$a = a^- \operatorname{diag}\{e_{\lambda_1}, \dots, e_{\lambda_N}\}\, a^+$$

where $a^\pm \in G[AP^\pm_{p,w}]_{N\times N}$ and $\kappa(a) := (\lambda_1, \dots, \lambda_N) \subset \mathbb{R}^N$. A right $AP_{p,w}$ factorization with $\kappa(a) = (0, \dots, 0)$ is referred to as a *canonical right $AP_{p,w}$ factorization.* If $a \in [AP_{p,w}]_{N\times N}$ admits a canonical right $AP_{p,w}$ factorization, then the *geometric mean* $\mathbf{d}(a) = M(a^-)M(a^+) \in G\mathbb{C}^{N\times N}$ is independent of the particular choice of the canonical right $AP_{p,w}$ factorization of a.

Recall the following theorem ([14, Theorem 7.2])

THEOREM 6. *Let $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$, $N \in \mathbb{N}$, $a \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$. If for every $\xi \in \mathcal{M}_\infty(SO)$ the matrix functions $a_{\xi,\pm} = \nu_{\xi,\pm} a \in [AP_{p,w}]_{N\times N}$ admit right $AP_{p,w}$ factorizations, then the Wiener-Hopf operator $W(a)$ is Fredholm on the space $L_N^p(\mathbb{R}_+, w)$ if and only if the following three conditions are satisfied:*

(i) $\det a(x) \neq 0$ *for all* $x \in \mathbb{R}$;
(ii) *for every* $\xi \in \mathcal{M}_\infty(SO)$, $\kappa(a_{\xi,\pm}) = (0, \dots, 0)$;
(iii) *for every* $\xi \in \mathcal{M}_\infty(SO)$ *and all* $j = 1, 2, \dots, N$, *the eigenvalues* $\eta_{\xi,j}$ *of the matrix* $\mathbf{d}^{-1}(a_{\xi,+})\mathbf{d}(a_{\xi,-})$ *satisfy the condition*

$$\frac{1}{p} + \frac{1}{2\pi} \arg \eta_{\xi,j} \notin \mathbb{Z}. \tag{17}$$

Theorems 5 and 6 immediately imply the following result.

THEOREM 7. *Let $1 < p < \infty$, $w \in A_p^0(\mathbb{R})$, $N \in \mathbb{N}$, let*

$$W := \chi_+ \sum\nolimits_{m=1}^{n} \mathcal{F}^{-1} K_m \mathcal{F} \chi_{J_m} I : \ L^p(J, w) \to L^p(J, w),$$

where $K_m \in [SO_{p,w}, SAP_{p,w}]$ and $J_m = [a_{m-1}, a_m]$ for all $m = 1, 2, \dots, n$ and $0 = a_0 < a_1 < a_2 < \ldots < a_n < \infty$, and let $G \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$ be given by (13)–(14) *where $N = n + 1$. If for every $\xi \in \mathcal{M}_\infty(SO)$ the matrix functions $G_{\xi,\pm} = \nu_{\xi,\pm} G \in [AP_{p,w}]_{N\times N}$ admit right $AP_{p,w}$ factorizations, then the convolution type operator W is Fredholm on the space $L^p(J, w)$ if and only if the following three conditions are satisfied:*

(i) $\det G(x) \neq 0$ *for all* $x \in \mathbb{R}$;
(ii) *for every* $\xi \in \mathcal{M}_\infty(SO)$, $\kappa(G_{\xi,\pm}) = (0, \dots, 0)$;
(iii) *for every* $\xi \in \mathcal{M}_\infty(SO)$ *and all* $j = 1, 2, \dots, N$, *the eigenvalues* $\eta_{\xi,j}$ *of the matrix* $\mathbf{d}^{-1}(G_{\xi,+})\mathbf{d}(G_{\xi,-})$ *satisfy the condition* (17).

We consider now the weights $w \in A_p^0(\mathbb{R})$ satisfying the additional condition (5) and assume that the almost periodic representatives of G at $\pm\infty$ belong to the algebra $[APW_{p,w}]_{N\times N}$. Then we obtain the following result.

THEOREM 8. *Let $1 < p < \infty$, $N \in \mathbb{N}$, $w \in A_p^0(\mathbb{R})$, and let condition* (5) *hold. Suppose the matrix function $G \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$ is given by* (13) *and* (14)*, where $N = n+1$, $K_m \in [SO_{p,w}, SAP_{p,w}]$ for all $m = 1, 2, \dots, n$ and $0 = a_0 < a_1 < a_2 < \dots < a_n < \infty$. If for every $\xi \in \mathcal{M}_\infty(SO)$ the matrix functions $G_{\xi,\pm} = \nu_{\xi,\pm} G$ are in $[APW_{p,w}]_{N\times N}$, then the convolution type operator*

$$\mathrm{W} := \chi_+ \sum\nolimits_{m=1}^{n} \mathcal{F}^{-1} K_m \mathcal{F} \chi_{J_m} I$$

is Fredholm on the space $L^p(J, w)$ if the following three conditions are satisfied:

(i) *$G \in G[SO_{p,w}, SAP_{p,w}]_{N\times N}$ (equivalently, $G^{-1} \in L^\infty_{N\times N}(\mathbb{R})$);*
(ii) *for every $\xi \in \mathcal{M}_\infty(SO)$, the matrix functions $G_{\xi,\pm}$ admit canonical right APW factorizations;*
(iii) *for every $\xi \in \mathcal{M}_\infty(SO)$ and every $j = 1, 2, \dots, N$, the eigenvalues $\eta_{\xi,j}$ of the matrix $\mathbf{d}^{-1}(G_{\xi,+})\mathbf{d}(G_{\xi,-})$ satisfy* (17).

Proof. Suppose all the conditions of Theorem 8 are fulfilled. Then for every $\xi \in \mathcal{M}_\infty(SO)$ we take a matrix function $G_\xi \in [SAP_{p,w}]_{N\times N}$ with almost periodic representatives $G_{\xi,\pm} \in [APW_{p,w}]_{N\times N}$ at $\pm\infty$ and such that $\det G_\xi(x) \neq 0$ for all $x \in \mathbb{R}$. Hence, taking into account the invertibility of the matrix functions $G_{\xi,\pm} \in [APW_{p,w}]_{N\times N}$ in $[APW_{p,w}]_{N\times N}$ due to condition (i) or (ii), we conclude that the matrix function G_ξ is invertible in $L^\infty_{N\times N}(\mathbb{R})$, and therefore $G_\xi \in G[SAP_{p,w}]_{N\times N}$. Applying now Theorem 2, we infer that the Wiener-Hopf operator $\mathrm{W}(G_\xi)$ is Fredholm on the space $L^p_N(\mathbb{R}_+, w)$. Then the coset $\mathrm{W}^\pi_\xi(G_\xi) = \mathrm{W}^\pi(G_\xi) + J^\pi_\xi$ is invertible in the quotient algebra Λ^π_ξ. Since $\mathrm{W}^\pi_\xi(G_\xi) = \mathrm{W}^\pi_\xi(G)$ where $G \in [SO_{p,w}, SAP_{p,w}]_{N\times N}$, we conclude that for every $\xi \in \mathcal{M}_\infty(SO)$ the coset $\mathrm{W}^\pi_\xi(G)$ is invertible in the algebra Λ^π_ξ. As additionally $\det G(\xi) \neq 0$ for $\xi \in \mathbb{R}$ and therefore the coset $\mathrm{W}^\pi_\xi(G) = [G(\xi)I]^\pi + J^\pi_\xi$ is invertible in the quotient algebra Λ^π_ξ for every $\xi \in \mathbb{R}$, we infer from Theorem 3 that the Wiener-Hopf operator $\mathrm{W}(G)$ is Fredholm on the space $L^p_N(\mathbb{R}_+, w)$ because for all $\xi \in \mathcal{M}(SO) = \mathbb{R} \cup \mathcal{M}_\infty(SO)$ the cosets $\mathrm{W}^\pi_\xi(G)$ are invertible in the quotient algebras Λ^π_ξ. Since the operator $\mathrm{W}(G)$ is Fredholm on the space $L^p_N(\mathbb{R}_+, w)$, we infer from Theorem 5 that the convolution type operator W is Fredholm on the space $L^p(J, w)$. □

6. GENERALIZED ALMOST PERIODIC FACTORIZATION AND ITS APPLICATIONS

Consider the set AP^0 of all almost periodic polynomials on $\mathbb{R}$. The *Besicovitch space* B^2 is defined as the completion of AP^0 with respect to the norm

$$\|f\|_{B^2} := \Big(\sum\nolimits_\lambda |f_\lambda|^2\Big)^{1/2} = \big(M(|f|^2)\big)^{1/2},$$

where $f = \sum_\lambda f_\lambda e_\lambda \in AP^0$. As is known (see, e.g., [6, Chapter 7]), AP can be identified with $C(\mathbb{R}_B)$ where $\mathbb{R}_B$ is the Bohr compactification of $\mathbb{R}$. In its turn B^2 can be identified with $L^2(\mathbb{R}_B, \mathrm{d}\mu)$ where $\mathrm{d}\mu$ is the normalized Haar measure on $\mathbb{R}_B$. Thus, B^2 is a (nonseparable) Hilbert space, and the inner product in $B^2 = L^2(\mathbb{R}_B, \mathrm{d}\mu)$ is given by

$$(f, g) = \int_{\mathbb{R}_B} f(\xi)\overline{g(\xi)}\,\mathrm{d}\mu(\xi).$$

Since $\mu(\mathbb{R}_B) = 1$ is finite, we have $AP \subset B^2$. Moreover, AP is a dense subset of B^2. The Cauchy-Schwarz inequality shows that the mean value

$$M(f) := \int_{\mathbb{R}_B} f(\xi)\,\mathrm{d}\mu(\xi)$$

exists and is finite for every $f \in B^2$. The set $\Omega(f) := \big\{\lambda \in \mathbb{R} : M(fe_{-\lambda}) \neq 0\big\}$ called the *Bohr-Fourier spectrum* of f is at most countable. Thus,

$$\|f\|_{B^2} = \sum\nolimits_{\lambda\in\Omega(f)} |M(fe_{-\lambda})|^2 \ \text{ for every } \ f \in B^2.$$

Let $l^2(\mathbb{R})$ denote the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ for which the set $\{\lambda \in \mathbb{C} : f(\lambda) \neq 0\}$ is at most countable and

$$\|f\|^2_{l^2(\mathbb{R})} := \sum |f(\lambda)|^2 < \infty.$$

Note that $l^2(\mathbb{R})$ is a (nonseparable) Hilbert space with pointwise operations and the inner product

$$(f, g) := \sum\nolimits_{\lambda\in\mathbb{R}} f(\lambda)\overline{g(\lambda)}.$$

The map $\mathcal{F}_B : l^2(\mathbb{R}) \to B^2$ which sends a function $f \in l^2(\mathbb{R})$ with a finite support to the function

$$(\mathcal{F}_B f)(x) = \sum_{\lambda\in\mathbb{R}} f(\lambda)\mathrm{e}^{\mathrm{i}\lambda x}, \ \ x \in \mathbb{R},$$

can be extended by continuity to all of $l^2(\mathbb{R})$. It is referred to as the *Bohr-Fourier transform.* The operator $\mathcal{F}_B$ is an isometric isomorphism. The inverse Bohr-Fourier transform acts by the rule

$$\mathcal{F}_B^{-1} : B^2 \to l^2(\mathbb{R}), \quad (\mathcal{F}_B^{-1} f)(\lambda) = M(fe_{-\lambda}), \ \ \lambda \in \mathbb{R}.$$

We also consider the Hilbert subspaces $B^2_\pm := \{f \in B^2 : \Omega(f) \subset \mathbb{R}_\pm\}$ and the projections $\widetilde{P}_\pm := \mathcal{F}_B \chi_\pm \mathcal{F}_B^{-1} : \ B^2 \to B^2_\pm$ where $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x \geq 0\}$.

In order to establish a criterion for the invertibility of $\mathrm{W}(a)$ on $L^2_N(\mathbb{R}_+)$ with $a \in AP_{N\times N}$ and to study the Fredholmness of $\mathrm{W}(a)$ on $L^2_N(\mathbb{R}_+)$ with $a \in SAP_{N\times N}$ it is necessary to generalize the notion of AP factorization.

Definition 1 ([6]). A *canonical generalized right AP factorization* of a matrix function $a \in GAP_{N\times N}$ is a representation $a = a^- a^+$ where

$$a^- \in G[B^2_-]_{N\times N}, \ a^+ \in G[B^2_+]_{N\times N}, \ a^- \widetilde{P}(a^-)^{-1} I \in \mathcal{B}(B^2_N).$$

Recall the following theorem ([6, Theorem 21.7]).

Theorem 9. *Let $a \in AP_{N\times N}$. Then the Wiener-Hopf operator* $\mathrm{W}(a)$ *is invertible on the space $L^2_N(\mathbb{R}_+)$ if and only if a has a canonical generalized right AP factorization.*

Theorems 4 and 9 imply the following corollary for the convolution type operator $\mathrm{W} \in \mathcal{B}(L^2(J))$ with $K_m \in SAP$ $(m = 1, 2, \dots, n)$.

Corollary 3. *Let $K_m \in SAP$ and $J_m = [a_{m-1}, a_m]$ for all $m = 1, 2, \dots, n$, where $0 = a_0 < a_1 < a_2 < \dots < a_n < \infty$. Then the convolution type operator*

$$\mathrm{W} = \chi_+ \sum\nolimits_{m=1}^{n} \mathcal{F}^{-1} K_m \mathcal{F} \chi_{J_m} I$$

is invertible on the space $L^2(J)$ if and only if the matrix function $G \in SAP_{N\times N}$ given by (13)–(14) *with $N = n + 1$ admits a canonical generalized right AP factorization.*

We denote by $\mathcal{G}_{N\times N}$ the open subset of all $a \in AP_{N\times N}$ for which $\mathrm{W}(a)$ is invertible on the space $L^2_N(\mathbb{R}_+)$ or, equivalently, a admits a canonical generalized right AP factorization $a = a^- a^+$. By [6, Corollary 21.8], the map $a \mapsto \mathbf{d}(a)$ given by $\mathbf{d}(a) = M(a^-)M(a^+)$ is continuous from $\mathcal{G}_{N\times N}$ onto $G\mathbb{C}^{N\times N}$.

Recall the following theorem ([6, Theorem 21.7]).

Theorem 10. *Let $a \in SAP_{N\times N}$. Then the Wiener-Hopf operator* $\mathrm{W}(a)$ *is Fredholm on the space $L^2_N(\mathbb{R}_+)$ if and only if*

(i) *$a \in GSAP_{N\times N}$,*
(ii) *the almost periodic representatives a_l and a_r of a have canonical generalized right AP factorizations,*
(iii) *for every $j = 1, 2, \dots, N$ the eigenvalues η_j of the matrix $\mathbf{d}^{-1}(a_r)\mathbf{d}(a_l)$ lie in $\mathbb{C} \setminus \mathbb{R}_-$.*

If $\mathrm{W}(a)$ *is Fredholm, then* $\operatorname{Ind}\mathrm{W}(a)$ *is calculated by* (8) *with p replaced by* 2.

Applying Theorems 9 and 10 we establish now a Fredholm criterion for the convolution type operator $\mathrm{W} \in \mathcal{B}(L^2(J))$ defined by (2), where all functions K_m belong to the C^*-algebra $[SO, SAP]$.

Theorem 11. *Let $0 = a_0 < a_1 < a_2 < \dots < a_n < \infty$, let $K_m \in [SO, SAP]$ and $J_m = [a_{m-1}, a_m]$ for all $m = 1, 2, \dots, n$, and let $G \in [SO, SAP]_{N\times N}$ be given by* (13)–(14)*, where $N = n + 1$. The convolution type operator*

$$\mathrm{W} = \chi_+ \sum\nolimits_{m=1}^{n} \mathcal{F}^{-1} K_m \mathcal{F} \chi_{J_m} I$$

is Fredholm on the space $L^2(J)$ if and only if the following three conditions hold:

(i) $\det G(x) \neq 0$ *for all $x \in \mathbb{R}$;*
(ii) *for every $\xi \in \mathcal{M}_\infty(SO)$ the matrix functions $G_{\xi,\pm} = \nu_{\xi,\pm} G \in AP_{N\times N}$ admit canonical generalized right AP factorizations;*

(iii) *for every* $\xi \in \mathcal{M}_\infty(SO)$ *and all* $j = 1, 2, \dots, N$, *the eigenvalues* $\eta_{\xi,j}$ *of the matrix* $\mathbf{d}^{-1}(G_{\xi,+})\mathbf{d}(G_{\xi,-})$ *lie in* $\mathbb{C} \setminus \mathbb{R}_-$.

Proof. By Theorem 5, the convolution type operator W is Fredholm on the space $L^2(J)$ if and only if the Wiener-Hopf operator $\mathrm{W}(G)$ is Fredholm on the space $L^2_N(\mathbb{R}_+)$. In its turn, the operator $\mathrm{W}(G)$ is Fredholm on the space $L^2_N(\mathbb{R}_+)$ if and only if the Toeplitz operator $T(G) = \mathcal{F}\mathrm{W}(G)\mathcal{F}^{-1}$ is Fredholm on the Hardy space $H^2_N = \mathcal{F}\chi_+\mathcal{F}^{-1}(L^2_N(\mathbb{R}_+))$. Applying now the Fredholm criterion for the Toeplitz operator $T(G)$ with symbol $G \in [SO, SAP]_{N\times N}$ on the space H^2_N, which requires the fulfillment of all conditions (i)–(iii) of Theorem 11 (see [4, Theorem 11]), we complete the proof. □

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