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INTEGRAL INEQUALITIES INVOLVING MAXIMA OF THE UNKNOWN FUNCTION

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Abstract. Some new nonlinear integral inequalities that involve the maximum of the unknown scalar function of one variable are solved. The considered inequalities are generalizations of the classical nonlinear integral inequality of Bihari. The importance of these integral inequalities is given by their wide applications in qualitative investigations of differential equations with "maxima" and it is illustrated by some direct applications.

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1. INTRODUCTION

The integral inequalities that provide explicit bounds for unknown functions play an important role in the development of the theory of differential and integral equations. For instance, the explicit bounds given by the well-known Gronwall−Bellman inequality and its nonlinear generalization due to Bihari ([2], [3], [5], [7]) are used to a considerable extent in the literature. However, in the situations of qualitative investigations of differential equations with "maxima" ([1], [4], [6], [8]) totally different types of integral inequalities are required.

The main purpose of this paper is to solve some nonlinear Bihari-like inequalities that can be used to study the qualitative behavior of the solutions of differential equations with "maxima". Some applications of the obtained results are also given.

2. MAIN RESULTS

Let $t_0,\ T$ be fixed points such that $0 \leq t_0 < T \leq \infty$.

THEOREM 1. *Let the following conditions be satisfied:*

1) *The function* $\alpha \in C^1([t_0,T),\mathbb{R}_+)$ *is nondecreasing and* $\alpha(t) \leq t$.
2) *Assume that* $p,\ q \in C([t_0,T),\mathbb{R}_+)$ *and* $a,\ b \in C([\alpha(t_0),T),\mathbb{R}_+)$.
3) *The function* $\phi \in C([t_0-h,T),\mathbb{R}_+)$ *is such that* $\phi(t) \geq k$*, where* $k = const \geq 0$.
4) *The function* $g \in C(\mathbb{R}_+,(0,\infty))$ *is increasing.*

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5) *The function* $u \in C([\alpha(t_0)-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$u(t) \le k + \int_{t_0}^{t}\Big[p(s)g\Big(u(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s \tag{1}$$
$$+\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s, \quad \textit{for } t\in[t_0,T),$$

$$u(t) \le \phi(t), \quad \textit{for} \quad t\in[t_0-h,t_0], \ \textit{where } h = \textit{const} \ge 0. \tag{2}$$

Then the following inequality holds for every $t_0 \le t \le t_1$

$$u(t) \le G^{-1}\left(G(k) + \int_{t_0}^{t}\Big[p(s)+q(s)\Big]\mathrm{d}s + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)+b(s)\Big]\mathrm{d}s\right), \tag{3}$$

where G^{-1} *is the inverse function of*

$$G(r) = \int_{r_0}^{r}\frac{\mathrm{d}s}{g(s)}, \quad r_0 > 0, \tag{4}$$

and

$$t_1 = \sup\left\{\tau \ge t_0 \ : \ G(k) + \int_{t_0}^{t}\Big[p(s)+q(s)\Big]\mathrm{d}s + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)+b(s)\Big]\mathrm{d}s \ \in \mathrm{Dom}\left(G^{-1}\right) \ \textit{for} \ \ t\in[t_0,\tau]\right\}.$$

Proof. Define the function $z\colon [\alpha(t_0)-h,T)\to\mathbb{R}_+$ by

$$z(t) = \begin{cases} k, \text{ if } t\in[\alpha(t_0)-h,t_0] \\ k + \int_{t_0}^{t}\Big[p(s)g\Big(u(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s, \text{ if } t\in[t_0,T). \end{cases}$$

The function z is nondecreasing and the inequality $u(t)\le z(t)$ holds for $t\in[\alpha(t_0)-h,T)$. Note that

$$\max_{s\in[t-h,t]} z(s) = z(t) \text{ for } t\in[\alpha(t_0),T).$$

Inequality (1) and the definition of the function z yield for $t\in[t_0,T)$

$$\begin{aligned} z(t) &\le k + \int_{t_0}^{t}\Big[p(s)g\Big(z(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\ &\le k + \int_{t_0}^{t}\Big[p(s)+q(s)\Big]g\Big(z(s)\Big)\mathrm{d}s + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)+b(s)\Big]g\Big(z(s)\Big)\mathrm{d}s. \end{aligned} \tag{5}$$

After differentiation we get from inequality (5)

$$
\begin{aligned}
\big(z(t)\big)' &\le \Big[p(t)+q(t)\Big]g\Big(z(t)\Big) \\
&\quad + \Big[a\big(\alpha(t)\big)+b\big(\alpha(t)\big)\Big]g\Big(z\big(\alpha(t)\big)\Big)\big(\alpha(t)\big)' \\
&\le\ g\Big(z(t)\Big)\Big[p(t)+q(t)+\Big(a\big(\alpha(t)\big)+b\big(\alpha(t)\big)\Big)\big(\alpha(t)\big)'\Big].
\end{aligned} \tag{6}
$$

Relations (4) and (6) imply

$$
\frac{\mathrm{d}}{\mathrm{d}t}G\Big(z(t)\Big)=\frac{\big(z(t)\big)'}{g\Big(z(t)\Big)}\le p(t)+q(t)+\Big(a\big(\alpha(t)\big)+b\big(\alpha(t)\big)\Big)\big(\alpha(t)\big)'. \tag{7}
$$

We integrate inequality (7) from t_0 to t for $t \in [t_0, T)$, change the variable $\eta = \alpha(s)$ and we obtain

$$
G\Big(z(t)\Big)\le G(k)+\int_{t_0}^{t}\Big[p(\eta)+q(\eta)\Big]\mathrm{d}\eta+\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\eta)+b(\eta)\Big]\mathrm{d}\eta. \tag{8}
$$

Since G^{-1} is an increasing function, we obtain from (8) and $u(t) \le z(t)$ the required inequality (3). □

Remark 2. If $h = 0$ and $\alpha(t) \equiv t$, the statement of Theorem 1 reduces to the classical Bihari inequality.

If the constant k on the right-hand side of inequality (1) is replaced by a function, we will obtain a bound for $u(t)$, using the class of functions defined below.

Definition 3. We say that a function $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is from the class* Ω if the following conditions are satisfied:

(i) g is a nondecreasing function;
(ii) $g(x) > 0$ for $x > 0$;
(iii) $g(tx) \ge tg(x)$ for $0 \le t \le 1$, $x \ge 0$;
(iv) $g(x) + g(y) \ge g(x + y)$ for $x, y \ge 0$;
(v) $\int_1^\infty \frac{\mathrm{d}x}{g(x)} = \infty$.

Remark 4. Note that the functions $f(x) = \sqrt{x}$ and $g(x) = x$ are from the class Ω.

Theorem 5. *Let the following conditions be satisfied:*

1) *The function* $\alpha \in C^1([t_0, T), \mathbb{R}_+)$ *is nondecreasing and* $\alpha(t) \le t$.
2) *Assume that* $p,\ q \in C([t_0, T), \mathbb{R}_+)$ *and* $a,\ b \in C([\alpha(t_0), T), \mathbb{R}_+)$.
3) *Let* $k \in C([t_0 - h, T), \mathbb{R}_+)$.
4) *Let* $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* $g \in \Omega$.

5) *The function* $u \in C([\alpha(t_0) - h, T), \mathbb{R}_+)$ *satisfies the inequalities*

(9)
$$\begin{aligned} u(t) \le k(t) &+ \int_{t_0}^{t} \Big[p(s) g\Big(u(s)\Big) + q(s) g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big] \mathrm{d}s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) g\Big(u(s)\Big) + b(s) g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big] \mathrm{d}s \quad \text{for } t \in [t_0, T), \end{aligned}$$

(10)
$$u(t) \le k(t) \ \text{for } t \in [t_0 - h, t_0], \ \text{where } h = const \ge 0.$$

Then the following inequality holds for every $t_0 \le t \le t_2$

(11)
$$u(t) \le k(t) + e(t) G^{-1}\Bigg(G(1) + \int_{t_0}^{t} \Big[p(s) + q(s)\Big] \mathrm{d}s + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s)\Big] \mathrm{d}s\Bigg),$$

where $e\colon [t_0, T) \to \mathbb{R}_+$ *is defined by*

(12)
$$\begin{aligned} e(t) = 1 &+ \int_{t_0}^{t} \Big[p(s) g\Big(k(s)\Big) + q(s) g\Big(\max_{\xi\in[s-h,s]} k(\xi)\Big)\Big] \mathrm{d}s \\ &+ \int_{t_0}^{\max\left(\alpha(t), t_0\right)} \Big[a(s) g\Big(k(s)\Big) + b(s) g\Big(\max_{\xi\in[s-h,s]} k(\xi)\Big)\Big] \mathrm{d}s, \end{aligned}$$

G^{-1} *is the inverse of the function* G *defined by equality* (4), *and*

(13)
$$\begin{aligned} t_2 = \sup\Bigg\{\tau \ge t_0 \ : \ G(1) &+ \int_{t_0}^{t} \Big[p(s) + q(s)\Big] \mathrm{d}s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s)\Big] \mathrm{d}s \ \in \mathrm{Dom}\left(G^{-1}\right) \ \text{ for } \ t \in [t_0, \tau]\Bigg\}. \end{aligned}$$

Proof. Define the function $z\colon [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by

$$z(t) = \begin{cases} u(t), \ \text{if } t \in [\alpha(t_0) - h, t_0] \\ \int_{t_0}^{t} \Big[p(s) g\Big(u(s)\Big) + q(s) g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big] \mathrm{d}s \\ \ + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) g\Big(u(s)\Big) + b(s) g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big] \mathrm{d}s, \ \text{if } t \in (t_0, T). \end{cases}$$

From (9) and the definition of $z(t)$ we have for $t \in [t_0, T)$

(14)
$$u(t) \le k(t) + z(t).$$

Let $t \in [t_0, T)$ be such that $\alpha(t) \geq t_0$. Inequality (14), the definition of $z(t)$, and condition 4 of Theorem 5 imply that

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s \\
&\leq \int_{\alpha(t_0)}^{t_0} a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{t_0}^{\alpha(t)} \Big[a(s)g\Big(k(s) + z(s)\Big) \\
&+ b(s)g\Big(\max_{s\in[s-h,s]} k(\xi) + \max_{s\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&\leq \int_{t_0}^{\max(\alpha(t),t_0)} \Big[a(s)g\Big(k(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} k(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s.
\end{aligned} \tag{15}$$

Let $t \in [t_0, T)$ be such that $\alpha(t) < t_0$. Then, using the definition of $z(t)$, we get

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s \\
&= \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&= \int_{t_0}^{\max(\alpha(t),t_0)} \Big[a(s)g\Big(k(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} k(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{s\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s.
\end{aligned} \tag{16}$$

By the definition of $z(t)$ and by (10), (15), (16) we obtain that

$$\begin{aligned}
z(t) \leq e(t) &+ \int_{t_0}^{t} \Big[p(s)g\Big(z(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s, \text{ for } t \in [t_0, T),
\end{aligned} \tag{17}$$

$$z(t) \leq k(t), \text{ for } t \in [t_0 - h, t_0), \tag{18}$$

where the function e is defined by (12). Note that the function e is nondecreasing on $[t_0, T)$ and that $e(t_0) = 1$.

From (17), (18), condition 4 of Theorem 5, and $\frac{1}{e(t)} \leq 1$ we obtain for $t \in [t_0, T)$ the inequality

$$(19)\quad \begin{aligned} \frac{z(t)}{e(t)} \leq 1 &+ \int_{t_0}^{t} \Big[p(s) g\Big(\frac{z(s)}{e(s)}\Big) + q(s) g\Big(\frac{\max_{\xi\in[s-h,s]} z(\xi)}{e(t)}\Big)\Big] \mathrm{d}s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) g\Big(\frac{z(s)}{e(s)}\Big) + b(s) g\Big(\frac{\max_{\xi\in[s-h,s]} z(\xi)}{e(t)}\Big)\Big] \mathrm{d}s. \end{aligned}$$

The monotonicity of e yields for $t \in [t_0, T)$ and $s \in [\alpha(t_0), t]$ the relations

$$(20)\quad \frac{\max_{\xi\in[s-h,s]} z(\xi)}{e(t)} \leq \frac{\max_{\xi\in[s-h,s]} z(\xi)}{\hat{e}(s)} = \max_{\xi\in[s-h,s]} \frac{z(\xi)}{\hat{e}(s)} \leq \max_{\xi\in[s-h,s]} \frac{z(\xi)}{\hat{e}(\xi)},$$

where the continuous and nondecreasing function $\hat{e} : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ is defined by

$$\hat{e}(t) = \begin{cases} e(t), & \text{for } t \in [t_0, T), \\ e(t_0), & \text{for } t \in [\alpha(t_0) - h, t_0]. \end{cases}$$

From (19) and (20) it follows that for $t \in [t_0, T)$ the inequality

$$(21)\quad \begin{aligned} \frac{z(t)}{\hat{e}(t)} \leq 1 &+ \int_{t_0}^{t} \Big[p(s) g\Big(\frac{z(s)}{\hat{e}(s)}\Big) + q(s) g\Big(\max_{\xi\in[s-h,s]} \frac{z(\xi)}{\hat{e}(\xi)}\Big)\Big] \mathrm{d}s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) g\Big(\frac{z(s)}{\hat{e}(s)}\Big) + b(s) g\Big(\max_{\xi\in[s-h,s]} \frac{z(\xi)}{\hat{e}(\xi)}\Big)\Big] \mathrm{d}s \end{aligned}$$

holds.

Define now the function $u_1 \colon [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by $u_1(t) = \frac{z(t)}{\hat{e}(t)}$, and denote the right-hand side of inequality (21) by $z_1 : [t_0, T) \to \mathbb{R}_+$. Note that the function z_1 is increasing, $z_1(t_0) = 1$, and for $t \in [t_0, T)$ the inequality $u_1(t) \leq z_1(t)$ holds. Therefore,

$$(22)\quad z_1(t) \leq 1 + \int_{t_0}^{t} \Big[p(s) + q(s)\Big] g(z_1(s)) \mathrm{d}s + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s)\Big] g(z_1(s)) \mathrm{d}s.$$

After differentiation we obtain from (22)

$$(23)\quad \begin{aligned} \big(z_1(t)\big)' \leq &\Big[p(t) + q(t)\Big] g\Big(z_1(t)\Big) \\ &+ \Big[a(\alpha(t)) + b(\alpha(t))\Big] g\Big(z_1(\alpha(t))\Big) \big(\alpha(t)\big)'. \end{aligned}$$

Inequality (23) and condition 1 of Theorem 5 yield

$$(24)\quad \big(z_1(t)\big)' \leq g\Big(z_1(t)\Big) \Big[p(t) + q(t) + \Big(a(\alpha(t)) + b(\alpha(t))\Big) \big(\alpha(t)\big)'\Big].$$

Relations (4) and (24) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}G\Big(z_1(t)\Big) = \frac{\big(z_1(t)\big)'}{g\Big(z_1(t)\Big)} \le p(t) + q(t) + \Big(a(\alpha(t)) + b(\alpha(t))\Big)\big(\alpha(t)\big)'. \tag{25}$$

Integrating inequality (25) from t_0 to t, we get for $t \in [t_0, T)$ the inequality:

$$\begin{aligned} G\Big(z_1(t)\Big) &\le G(1) + \int_{t_0}^{t} \Big[p(s) + q(s)\Big]\mathrm{d}s \\ &\quad + \int_{t_0}^{t} \Big[a(\alpha(s)) + b(\alpha(s))\Big]\big(\alpha(s)\big)'\mathrm{d}s \\ &\le G(1) + \int_{t_0}^{t} \Big[p(\eta) + q(\eta)\Big]\mathrm{d}\eta + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\eta) + b(\eta)\Big]\mathrm{d}\eta. \end{aligned} \tag{26}$$

Since G^{-1} is an increasing function, we obtain from (26) and $u_1(t) \le z_1(t)$ that for $t \in [t_0, T)$ the inequality

$$\frac{z(t)}{\hat{e}(t)} \le G^{-1}\left(G(1) + \int_{t_0}^{t} \Big[p(s) + q(s)\Big]\mathrm{d}s + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s)\Big]\mathrm{d}s\right) \tag{27}$$

holds. The inequalities (14), (27), and the definition of $\hat{e}(t)$ imply (11). □

In the nonlinear case, when the unknown function is defined with the aid of a power, the following result is valid.

Theorem 6. *Let the following conditions be satisfied:*

1) *The function $\alpha \in C^1([t_0, T), \mathbb{R}_+)$ is nondecreasing and $\alpha(t) \le t$.*
2) *Assume that $p,\ q \in C([t_0, T), \mathbb{R}_+)$ and $a,\ b \in C([\alpha(t_0), T), \mathbb{R}_+)$.*
3) *Let $\phi \in C([t_0 - h, t_0), \mathbb{R}_+)$.*
4) *The function $k \in C([t_0, T), (0, \infty))$ is nondecreasing and the inequality $M = \max_{s\in[t_0-h,t_0]} \phi(s) \ge \sqrt[n]{k(t_0)}$ holds.*
5) *Let $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $g \in \Omega$.*
6) *The function $u \in C([\alpha(t_0) - h, T), \mathbb{R}_+)$ satisfies the inequalities*

$$\begin{aligned} \Big(u(t)\Big)^n &\le k(t) + \int_{t_0}^{t} \left[p(s)g\Big(u(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\right]\mathrm{d}s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\right]\mathrm{d}s \ \textit{for}\ t \in [t_0, T), \end{aligned} \tag{28}$$

$$u(t) \le \phi(t)\ \textit{for}\ \ t \in [t_0 - h, t_0],\ \textit{where}\ h = \textit{const} \ge 0,\ \ n = \textit{const} > 1. \tag{29}$$

Then the following inequality holds for every $t_0 \le t \le t_3$

$$u(t) \le \sqrt[n]{k(t)} + e_1(t)\left\{\frac{1}{n}\Big(k(t)\Big)^{\frac{1-n}{n}} + G^{-1}\Big(G(1) + A_1(t) + B_1(t)\Big)\right\}, \tag{30}$$

where

$$
\begin{aligned}
e_1(t) = 1 &+ \int_{t_0}^{t} \Big[p(s)g\Big(\psi(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]}\psi(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{t_0}^{\max(\alpha(t),t_0)} \Big[a(s)g\Big(\psi(s)\Big)\mathrm{d}s + b(s)g\Big(\max_{\xi\in[s-h,s]}\psi(s)\Big)\Big]\mathrm{d}s,
\end{aligned} \tag{31}
$$

$$
A_1(t) = \frac{1}{n}\int_{t_0}^{t}\Big[p(s)\Big(k(s)\Big)^{\frac{1-n}{n}} + q(s)\max_{\xi\in[s-h,s]}\Big(k(\xi)\Big)^{\frac{1-n}{n}}\Big]\mathrm{d}s, \tag{32}
$$

$$
B_1(t) = \frac{1}{n}\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)\Big(K(s)\Big)^{\frac{1-n}{n}} + b(s)\max_{\xi\in[s-h,s]}\Big(K(\xi)\Big)^{\frac{1-n}{n}}\Big]\mathrm{d}s, \tag{33}
$$

$$
K(t) = \begin{cases} k(t), & t\in[t_0,T) \\ k(t_0), & t\in[\alpha(t_0),t_0), \end{cases} \qquad \psi(t) = \begin{cases} \sqrt[n]{k(t)}, & t\in(t_0,T) \\ M, & t\in[t_0-h,t_0], \end{cases}
$$

G^{-1} *is the inverse of the function* G *defined by* (4), *and*

$$
t_3 = \sup\Big\{\tau:\ G(1) + A_1(t) + B_1(t) \in \mathrm{Dom}\left(G^{-1}\right) \textit{ for } t\in[t_0,\tau]\Big\}.
$$

Proof. Define the function $z\colon [\alpha(t_0)-h,T)\to\mathbb{R}_+$ by

$$
z(t) = \begin{cases} u(t), \text{ if } t\in[\alpha(t_0)-h,t_0) \\ \frac{\sqrt[n]{k(t)}}{n\ k(t)}\Big(\int_{t_0}^{t}\Big[p(s)g\Big(u(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]}u(\xi)\Big)\Big]\mathrm{d}s \\ + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]}u(\xi)\Big)\Big]\mathrm{d}s\Big), \text{ if } t\in[t_0,T). \end{cases}
$$

From (28) and the definition of $z(t)$ we get for $t\in[t_0,T)$

$$
\Big(u(t)\Big)^n \le\ k(t)\left(1 + n\frac{z(t)}{\sqrt[n]{k(t)}}\right)
$$

or, equivalently,

$$
u(t)\ \le\ \sqrt[n]{k(t)}\left(1 + n\frac{z(t)}{\sqrt[n]{k(t)}}\right)^{\frac{1}{n}}.
$$

Applying Bernoulli's inequality $(1+x)^a \le 1 + ax$, where $0 < a < 1$ and $-1 < x$, we observe that

$$
u(t) \le \sqrt[n]{k(t)}\left(1 + \frac{z(t)}{\sqrt[n]{k(t)}}\right) = \sqrt[n]{k(t)} + z(t) = \psi(t) + z(t), \text{ if } t\in[t_0,T), \tag{34}
$$

and

$$
u(t) \le \phi(t) \le \phi(t) + z(t) = \psi(t) + z(t), \text{ if } t\in[t_0-h,t_0]. \tag{35}
$$

Therefore,

$$\max_{\xi\in[s-h,s]} u(\xi) \le \max_{\xi\in[s-h,s]} \psi(\xi) + \max_{\xi\in[s-h,s]} z(\xi), \qquad s \in [t_0, T). \tag{36}$$

Let $t \in [t_0, T)$ be such that $\alpha(t) \ge t_0$. Then we get from (34) and (35)

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s \\
&\le \int_{\alpha(t_0)}^{t_0} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{t_0}^{\alpha(t)} \Big[a(s)g\Big(\psi(s) + z(s)\Big) \\
&+ b(s)g\Big(\max_{\xi\in[s-h,s]} \psi(\xi) + \max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&= \int_{t_0}^{\max(\alpha(t),t_0)} \Big[a(s)g\Big(\psi(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} \psi(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s.
\end{aligned} \tag{37}$$

Let $t \in [t_0, T)$ be such that $\alpha(t) < t_0$. The definition of $z(t)$ and the inequalities (34), (35) imply then that

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(u(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big]\mathrm{d}s \\
&= \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&= \int_{t_0}^{\max(\alpha(t),t_0)} \Big[a(s)g\Big(\sqrt[n]{\psi(s)}\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} \sqrt[n]{\psi(\xi)}\Big)\Big]\mathrm{d}s \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s.
\end{aligned} \tag{38}$$

It follows from the definition of $z(t)$ and the inequalities (37), (38) that

$$\begin{aligned}
z(t) \le \frac{1}{n\Big(k(t)\Big)^{\frac{n-1}{n}}} \Bigg(&e_1(t) + \int_{t_0}^{t} \Big[p(s)g\Big(z(s)\Big) + q(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(z(s)\Big) + b(s)g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s\Bigg)
\end{aligned}$$

It follows that

$$z(t) \le \frac{\sqrt[n]{k(t)}}{n\, k(t)} e_1(t) + \frac{1}{n}\int_{t_0}^{t}\Big[p(s)\Big(k(s)\Big)^{\frac{1-n}{n}} g\Big(z(s)\Big)$$

$$+ q(s)\Big(k(s)\Big)^{\frac{1-n}{n}} g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s \tag{39}$$

$$+ \frac{1}{n}\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)\Big(K(s)\Big)^{\frac{1-n}{n}} g\Big(z(s)\Big)$$

$$+ b(s)\Big(K(s)\Big)^{\frac{1-n}{n}} g\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big]\mathrm{d}s, \text{ if } t\in[t_0,T),$$

$$z(t) \le \phi(t), \text{ if } t\in[t_0-h,t_0]. \tag{40}$$

According to Theorem 5, we get from inequalities (39), (40) that

$$z(t) \le e_1(t)\left\{\frac{\sqrt[n]{k(t)}}{n\, k(t)} + G^{-1}\Big(G(1)+A_1(t)+B_1(t)\Big)\right\}, \tag{41}$$

where A_1 and B_1 are defined by (32) and (33), respectively. From (34) and (41) we finally obtain the required inequality (30). □

3. APPLICATIONS

We will apply some of the previously obtained results to the following system of differential equations with "maximum"

$$x' = f\Big(t,\ x(t),\ \max_{s\in[\beta(t),\alpha(t)]} x(s)\Big), \text{ for } t\ge t_0, \tag{42}$$

with initial condition

$$x(t) = \varphi(t), \text{ for } t\in[t_0-h,t_0], \tag{43}$$

where x takes values in $\mathbb{R}^n$, $\phi\colon [t_0-h,t_0]\to\mathbb{R}^n$, $f\colon [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, and $h>0$ is a constant.

THEOREM 7 (Bounds). *Let the following conditions be satisfied:*

1) *The functions $\alpha,\beta\in C^1([t_0,\infty),\mathbb{R}_+)$ are so that α is nondecreasing, $\beta(t)\le\alpha(t)\le t$, and there exists a constant $h>0$ such that $0<\alpha(t)-\beta(t)\le h$ for $t\ge t_0$.*
2) *The function $f\in C([t_0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ satisfies for $t\ge t_0$ and $x,y\in\mathbb{R}^n$ the condition*

$$\Big|\Big|f(t,\ x,\ y)\Big|\Big| \le P(t)\sqrt{||x||}+Q(t)\sqrt{||y||},$$

where $P,\ B\in C([t_0,\infty),\mathbb{R}_+)$.

3) *Let $\varphi\in C([t_0-h,t_0],\mathbb{R}_+)$.*
4) *The function $x(\cdot;t_0,\varphi)$ is a solution of the initial value problem* (42), (43), *defined for $t\ge t_0-h$.*

Then the solution $x(\cdot; t_0, \varphi)$ satisfies for $t \geq t_0$ the inequality

$$||x(t; t_0, \varphi)|| \leq \frac{1}{4}\left(2\sqrt{||\varphi(t_0)||} + \int_{t_0}^{t} \Big[P(s) + Q(s)\Big] \mathrm{d}s\right)^2. \tag{44}$$

Proof. The function $x = x(\cdot; t_0, \varphi)$ satisfies the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f\Big(s,\ x(s), \max_{s\in[\beta(t),\alpha(t)]} x(s)\Big)\mathrm{d}s, \text{ for } t \geq t_0,$$

with

$$x(t) = \varphi(t), \text{ for } t \in [t_0 - h, t_0].$$

Then we obtain the following relations for the norm of the vector $x(t)$

$$\begin{aligned} ||x(t)|| &\leq ||\varphi(t_0)|| + \int_{t_0}^{t} \Big|\Big|f\Big(s,\ x(s), \max_{s\in[\beta(t),\alpha(t)]} x(s)\Big)\Big|\Big|\mathrm{d}s \\ &\leq ||\varphi(t_0)|| + \int_{t_0}^{t} \Bigg(P(s)\sqrt{||x(s)||} \\ &+ Q(s)\sqrt{\Big|\Big|\max_{s\in[\beta(t),\alpha(t)]} x(s)\Big|\Big|}\ \Bigg)\mathrm{d}s \\ &\leq ||\phi(t_0)|| + \int_{t_0}^{t} P(s)\sqrt{||x(s)||}\mathrm{d}s \\ &+ \int_{t_0}^{t} Q(s)\sqrt{\max_{s\in[\beta(t),\alpha(t)]} ||x(s)||}\mathrm{d}s, \text{ for } \ t \geq t_0, \end{aligned} \tag{45}$$

and

$$||x(t)|| \ = \ ||\varphi(t)||, \text{ for } t \in [t_0 - h, t_0]. \tag{46}$$

Put $u(t) = ||x(t)||$ for $t \in [t_0 - h, \infty)$. Then we get for $t \geq t_0$

$$u(t) \leq ||\varphi(t_0)|| + \int_{t_0}^{t} P(s)\sqrt{\varphi(s)}\mathrm{d}s + \int_{t_0}^{t} Q(s)\sqrt{\max_{s\in[\beta(t),\alpha(t)]} \varphi(s)}\mathrm{d}s. \tag{47}$$

Changing the variable $s = \alpha^{-1}(\eta)$ in the second integral of (47) and using the inequality $\max_{\xi\in[\beta(t),\, \alpha(t))]} u(\xi) \leq \max_{\xi\in[\alpha(t)-h,\, \alpha(t)]} u(\xi)$ (that follows from condition 1 of Theorem 7), we obtain

$$\begin{aligned} u(t) \leq& ||\varphi(t_0)|| + \int_{t_0}^{t} P(\eta)\sqrt{u(\eta)}\mathrm{d}\eta \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} Q(\alpha^{-1}(\eta))\Big(\alpha^{-1}(\eta)\Big)'\sqrt{\max_{\xi\in[\eta-h,\eta]} u(\xi)}\mathrm{d}\eta. \end{aligned} \tag{48}$$

Note that the conditions of Theorem 1 are satisfied for the constant $k = ||\varphi(t_0)||$ and the functions $p(t) \equiv P(t)$, $q(t) \equiv 0$ for $t \in [t_0, \infty)$, $a(t) \equiv 0$,

$b(s) \equiv Q(\alpha^{-1}(s))\big(\alpha^{-1}(s)\big)'$ for $t \in [\alpha(t_0), \infty)$, $g(u) = \sqrt{u}$, $G(u) = 2\sqrt{u}$, $G^{-1}(u) = \frac{1}{4}u^2$, $\mathrm{Dom}\left(G^{-1}\right) = \mathbb{R}_+$, and $t_1 = \infty$. Applying now Theorem 1, we obtain from (48) that the following inequality holds for $t \geq t_0$

$$u(t) \leq \frac{1}{4}\left(2\sqrt{\big|\big|\varphi(t_0)\big|\big|} + \int_{t_0}^{t} \Big[P(s) + Q(s)\Big] \mathrm{d}s\right)^2. \tag{49}$$

Inequality (49) implies the validity of (44). □

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