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ENTIRE UNBOUNDED FUNCTIONS ON BANACH SPACES WITH A MONOTONE SCHAUDER BASIS

JOSÉ M. ANSEMIL, RICHARD M. ARON and SOCORRO PONTE

Abstract. It is well-known that on any complex infinite dimensional Banach space there is an entire function that is unbounded on some ball. Here we shall give an explicit construction of such entire functions on any infinite dimensional complex normed space with a monotone Schauder basis.

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1. INTRODUCTION

From the 1970's, a property of infinite dimensional holomorphy that has attracted the attention of several people is the fact that in such a situation there are always entire functions which are not bounded on balls (see [1, 4]). This fact deeply depends on the Josefson-Nissenzweig theorem [3], which asserts that in the dual of any infinite dimensional Banach space E there exists a sequence $\{\varphi_n\}$ such that $||\varphi_n\| = 1$ for all n and $\lim_{n\to\infty} \varphi_n(z) = 0$ for every z in E. Then, as it is proved in [5, p. 157], the function

$$f(z) = \sum_{n=1}^{\infty} \varphi_n^n(z)$$

is entire and it is unbounded on some ball of E.

In what follows E will denote an infinite dimensional complex normed space. For a function f defined on a subset B of E we use the notation: $\|f\|_B = \sup\{|f(z)| : z \in B\}$. We recall that a function $f : E \to \mathbb{C}$ is entire if it is Fréchet differentiable at every point in E.

In this paper we restrict our attention to normed spaces with a monotone Schauder basis and construct in a natural way entire functions, which are bounded on a given ball and unbounded on another given ball not included in the first ball.

We recall that a monotone Schauder basis in a normed space E is a sequence $\{e_k\}$ such that for every $z \in E$ there is a unique sequence $\{z_k\}$ in $\mathbb{C}$ such that $z = \sum_{k=1}^{\infty} z_k e_k$ in E and $\| \sum_{k=1}^{n} z_k e_k\| \leq \|z\|$ for every n. It is normalized if $\|e_n\| = 1$ for all n. The coordinate functions $\phi_k(z) = z_k$ are continuous. This kind of Schauder basis has been well studied among others by Singer in his book [6]. For instance the "saw-tooth" functions form a monotone Schauder basis for $C[0, 1]$ (with the sup norm), the Haar basis of $L^p[0, 1]$, $p \geq 1$, is also a monotone Schauder basis. $(C(K), \| \|_\infty)$, with K a compact metric space, has a monotone Schauder basis ([6, p. 215]). The canonical bases of the spaces c_o and $l_p, 1 \leq p < \infty$, are monotone.

Any normalized Schauder basis in a normed space can be monotonized in the sense that there is an equivalent norm $|||\cdot|||$ such that the new normed space has a monotone Schauder basis ([6], p. 250). This norm is defined by:

$$|||z||| = \sup_n \| \sum_{k=1}^{n} z_k e_k \|, \text{ where } z = \sum_{k=1}^{\infty} z_k e_k.$$

However, this is not very convenient for our purposes. We will return to this later on.

2. THE RESULTS

We start by proving a lemma concerning the existence of certain functionals on Banach spaces with a monotone Schauder basis.

Lemma 1. *Let E be a normed space with a monotone Schauder basis $\{e_n\}$. If for a given $k_o \in \mathbb{N}$ the vector $z_o = \sum_{k=1}^{k_0} z_k e_k$ has norm 1 and we define M as the closed subspace of E spanned by the vectors $e_k, k > k_o$, then there exists $\varphi \in E'$ such that*

$$\|\varphi\| = 1,\ \varphi(z_0) = 1 \text{ and } \varphi(z) = 0 \text{ for every } z \text{ in } M.$$

Proof. First note that each element m in M has a unique representation as $m = \sum_{k=k_0+1}^{\infty} m_k e_k$. Indeed, as the coordinate projections are continuous, no linear combinations of the e_k with $k = 1, ..., k_o$ can appear in the sum that represents m. So z_o does not belong to M and so, by a consequence of the Hahn-Banach theorem (see [2, p. 232]), there exists $\varphi \in E'$ such that $\|\varphi\| = 1$, $\varphi(z_0) = \operatorname{dist}(z_0, M)$ and $\varphi(z) = 0$, for every z in M.

Let us compute $\operatorname{dist}(z_0, M)$. Given $m = \sum_{k=k_0+1}^{\infty} m_k e_k$ in M, keeping in mind that our basis is monotone, we have

$$\|z_0 - m\| = \left\| \sum_{k=1}^{k_0} z_k e_k - \sum_{k=k_0+1}^{\infty} m_k e_k \right\| \geq \left\| \sum_{k=1}^{k_0} z_k e_k \right\| = \|z_0\|.$$

On the other hand, $0 \in M$, so

$$\operatorname{dist}(z_0, M) = \inf\{\|z_0 - m\|,\ m \in M\} \leq \|z_0\| = 1.$$

That is, $\varphi(z_0) = 1$. □

Lemma 2. *Let E be a normed space with a Schauder basis $\{e_k\}$. Given $z_o \in E$ with $\|z_o\| = 1$ and a real number $s > 0$, there exists $z_1 \in E$, which is a finite linear combination of the e_k, $\|z_1\| = 1$, and $z_1 \in B(z_0, s)$.*

Proof. Since

$$z_o = \lim_{n\to\infty} \sum_{k=1}^{n} z_k e_k \text{ and } \lim_{n\to\infty} \left\| \sum_{k=1}^{n} z_k e_k \right\| = \|z_o\| = 1,$$

there exists k_0 such that

$$\left\| z_0 - \sum_{k=1}^{k_0} z_k e_k \right\| < \min\left\{\frac{s}{2}, 1\right\} \quad \text{and} \quad \left| 1 - \left\| \sum_{k=1}^{k_0} z_k e_k \right\| \right| < \frac{s}{2}.$$

Let us define

$$z_1^* = \sum_{k=1}^{k_0} z_k e_k \ \text{ and } \ z_1 = \frac{z_1^*}{\|z_1^*\|}.$$

Then z_1 is a finite linear combination of $e_k,\ (k = 1, ..., k_o)$, has norm 1 and

$$\|z_1 - z_0\| \le \|z_1 - z_1^*\| + \|z_1^* - z_0\| < \|z_1^*\| \left| \frac{1}{\|z_1^*\|} - 1 \right| + \frac{s}{2} < s.$$

□

PROPOSITION 3. *Let E be a complex normed space with a monotone normalized Schauder basis $\{e_k\}$. Then, for every $z_0 = \sum_{k=1}^{k_0} \lambda_k e_k$ with norm 1, and for every $r \in (0,1)$ and $s > 0$ there exists an entire function h on E such that*

$$\|h\|_{B(0,r)} < \infty \quad \textit{and } \|h\|_{B(z_0,s)} = \infty.$$

Proof. For every $k \in \mathbb{N}$, the coordinate function ϕ_k satisfies $1 \le \|\phi_k\| \le 2$. Indeed, if we define $T_k(z) = \sum_{j=1}^{k} z_j e_j$, then, since the basis is monotone, $\|T_k(z)\| \le \|z\|$. This implies that

$$|\phi_k(z)| = \|\phi_k(z)e_k\| = \|T_k(z) - T_{k-1}(z)\| \le 2\|z\|$$

and, therefore, $\|\phi_k\| \le 2$. On the other hand, $\phi_k(e_k) = 1$, and so $\|\phi_k\| \ge 1$.

Fix $t \in (1, +\infty)$ such that $tr < 1$ and consider a natural number γ such that $\frac{1}{2} s t^{\gamma+1} > 1$. By Lemma 1 we obtain $\varphi \in E'$ such that $\|\varphi\| = 1$, $\varphi(z_0) = 1$ and $\varphi(z) = 0$ for every z in the closed span of $e_k, k > k_o$. Let us check how the mapping

$$h(z) = \sum_{k=k_0+1}^{\infty} \left([t\varphi(z)]^\gamma \cdot t \frac{\phi_k(z)}{\|\phi_k\|} \right)^k$$

satisfies the required conditions.

Let us see why h is entire on E. For any $z = \sum_{k=1}^{\infty} z_k e_k \in E$, $\phi_k(z) = z_k \to 0$ and then, since the coordinate functions are uniformly bounded (by 2), we get that $\phi_n \to 0$ uniformly on the compact subsets of E. Let K be an arbitrary compact subset of E. Then for every $z \in K$,

$$\sum_{k=k_0+1}^{\infty} \left| [t\varphi(z)]^\gamma \cdot t \frac{\phi_k(z)}{\|\phi_k\|} \right|^k \le \sum_{k=k_0+1}^{\infty} \left((t\, \|\varphi\|_K)^\gamma \, \frac{t}{\|\phi_k\|} \|\phi_k\|_K \right)^k.$$

As

$$\lim_{k\to\infty} (t\, \|\varphi\|_K)^\gamma \, \frac{t}{\|\phi_k\|} \|\phi_k\|_K = 0 \text{ (note that } \|\phi_k\|) \ge 1),$$

the series which defines h is uniformly convergent on K and then h is entire on E.

Now we show that h is bounded on $B(0,r)$. Indeed, for all $z \in B(0,r)$,

$$|h(z)| \leq \sum_{k=k_0+1}^{\infty} \left| [t\varphi(z)]^{\gamma} \cdot t \frac{\phi_k(z)}{\|\phi_k\|} \right|^k \leq \sum_{k=k_0+1}^{\infty} (t\|z\|)^{(\gamma+1)k}$$

$$\leq \sum_{k=k_0+1}^{\infty} \left[(tr)^{(\gamma+1)} \right]^k < \infty$$

since $tr < 1$.

Now we prove that $\|h\|_{B(z_0,s)} = \infty$. For that we consider the vectors,

$$z_0 + \frac{s}{2} \|\phi_m\| \, e_m \in \overline{B}(z_0, s),$$

where m is any fixed natural number bigger than k_0. We have that

$$\begin{aligned} & h\left(z_0 + \frac{s}{2} \|\phi_m\| \, e_m\right) \\ &= \sum_{k=k_0+1}^{\infty} \left[\left(t\varphi\left(z_0 + \frac{s}{2} \|\phi_m\| \, e_m\right)\right)^{\gamma} \cdot t \frac{\phi_k}{\|\phi_k\|} \left(z_0 + \frac{s}{2} \|\phi_m\| \, e_m\right) \right]^k \\ &= \left(t^{\gamma+1} \frac{1}{2} s \right)^m . \end{aligned}$$

Since $t^{\gamma+1}\frac{1}{2}s > 1$, we conclude that

$$\sup_{m>k_0} \left| h\left(z_0 + \frac{s}{2} \|\phi_m\| \, e_m\right) \right| = \infty$$

and thus $\|h\|_{B(z_0,s)} = \|h\|_{\overline{B}(z_0,s)} = \infty$. $\square$

Proposition 4. *Let E be a normed space with a monotone normalized Schauder basis. Then, for every $R > 0$, every $Z_0 \in E$ with $\|Z_o\| > R$, and every $S > 0$ there is an entire function F on E such that*

$$\|F\|_{B(0,R)} < \infty \text{ and } \|F\|_{B(Z_o,S)} = \infty.$$

Proof. Let $r = \frac{R}{\|Z_0\|}$, $s = \frac{S}{\|Z_0\|}$ and $z_0 = \frac{Z_o}{\|Z_0\|}$. If we apply Lemma 2 to z_0 and s we obtain $z_1 \in E$, which is a finite combination of elements of the basis such that $\|z_1\| = 1$ and $z_1 \in B(z_0, s)$. If we apply Proposition 3 with z_1, r and s' such that $B(z_1, s') \subset B(z_0, s)$, we have the existence of an entire function f on E such that

$$\|f\|_{B(0,r)} < \infty \text{ and } \|f\|_{B(z_1,s')} = \infty.$$

If we define $F(z) = f(\frac{z}{\|Z_0\|})$, then F is an entire function on E, $\|F\|_{B(0,R)} = \|f\|_{B(0,r)} < \infty$ and $\|F\|_{B(Z_0,S)} = \|f\|_{B(z_0,s)} \geq \|f\|_{B(z_1,s')} = \infty$. $\square$

Theorem 5. *Let E be a normed space with a monotone Schauder basis. Then, for every $B(z_1, r_1)$, every point $z_2 \in E \backslash \overline{B}(z_1, r_1)$, every $r_2 > 0$ and every $\epsilon > 0$ there is an entire function f on E such that*

$$\|f\|_{B(z_1,r_1)} < \varepsilon$$

and

$$\|f\|_{B(z_2,r_2)} = \infty.$$

Proof. Let us translate z_1 to the origin using the mapping $z \mapsto z - z_1$. The point $z_2' = z_2 - z_1$ does not belong to $\overline{B}(0, r_1)$. If we apply the above proposition, we obtain an entire function F on E such that

$$\|F\|_{B(0,r_1)} < \infty \quad \text{and} \quad \|F\|_{B(z_2',r_2)} = \infty.$$

Then $f(z) = \frac{F(z-z_1)}{2\|F\|_{B(0,r_1)}} \varepsilon$ satisfies the requirements of the theorem. □

Let us construct an example to see how our technique works. Let E be any of the spaces c_o or $l_p, 1 \leq p < \infty$ with the usual basis $\{e_k\}$. Consider $z_1 = 0,\ z_2 = e_2,\ r = \frac{9}{10}$ and $s = \frac{1}{10}$. Then the function h defined by

$$h(z) = \sum_{k=3}^{\infty} \left[\left(\frac{11}{10} z_2 \right)^{31} \frac{11}{10} z_k \right]^k, \quad z = \{z_k\} \in E,$$

is bounded on $B\left(0, \frac{9}{10}\right)$ and unbounded on $B\left(e_2, \frac{1}{10}\right)$.

If we have a non-monotone basis on a Banach space and we renorm the space in such a way that the basis is now monotone, a constant C appears so that

$$\|z\| \leq |||z||| \leq C\|z\| \text{ for every } z \in E.$$

C is bigger than 1 and is obtained using the open mapping theorem. In this situation the following result can be obtained: Given a ball $B(z_1, r_1)$, $s > 0$ and z_2 such that $z_2 \notin E \backslash \overline{B}(z_1, Cr_1)$, there is an entire function f on E such that

$$\|f\|_{B(z_1,r_1)} < \infty \text{ and } \|f\|_{B(z_2,s)} = \infty$$

(all balls are in E with its original norm).

A similar result to the above Theorem 5 can be obtained for Hilbert spaces. If a Hilbert space is separable then it is isometric to l_2, which has a monotone Schauder basis, but this is not the case for non-separable Hilbert spaces.

For a Hilbert space H let us consider a maximal orthonormal system $\{u_\alpha\}_{\alpha \in \Lambda}$. Then every $z \in H$ can be written as a countable sum of $< x, u_\alpha > u_\alpha$. With minor modifications of the proof of the above Lemma 1 we can obtain the following lemma:

Lemma 6. *Let us fix a point $z_0 = \sum_{j=1}^{j_0} \langle z_0, u_{\alpha_j} \rangle u_{\alpha_j} \in H$ with norm 1 and consider a countable set $\{u_{\alpha_n^*}; n \in \mathbb{N}\}$, where none of the u_{α_j}, $j = 1, ..., j_0$, is* *included. Let M be the closed subspace of E spanned by the vectors $u_{\alpha_n^*}$, $n \in \mathbb{N}$. Then there is $\varphi \in H'$ such that*

$$\|\varphi\| = 1,\ \varphi(x_0) = 1 \text{ and } \varphi(z) = 0 \text{ for every } z \in M.$$

From this lemma, using a similar technique to that in the above results, the following result can be obtained:

Theorem 7. *For any complex Hilbert space H and for every balls B_1 and B_2, such that B_2 is not contained in B_1, and every $\epsilon > 0$ there is an entire function f on H such that $\|f\|_{B_1} < \epsilon$ and $\|f\|_{B_2} = \infty$.*

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Universidad Complutense de Madrid
Facultad de Ciencias Matemáticas
Departamento de Análisis Matemático
28040 Madrid, Spain
E-mail: jm_ansemil@mat.ucm.es

Kent State University
Department of Mathematical Sciences
Kent, Ohio, 42242, USA
E-mail: aron@math.kent.edu

Universidad Complutense de Madrid
Facultad de Ciencias Matemáticas
Departamento de Análisis Matemático
28040 Madrid, Spain
E-mail: socorro_ponte@mat.ucm.es