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APPROXIMATION OF FUNCTIONS OF WEIGHTED LEBESGUE AND SMIRNOV SPACES

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Abstract. In this work we investigate the inverse approximation problems in the Lebesgue and Smirnov spaces with weights satisfying the so-called Muckenhoupt's A_p condition in terms of the α-th mean modulus of smoothness, $\alpha > 0$. We obtain a converse theorem of trigonometric approximation in the weighted Lebesgue spaces and obtain some converse theorems of algebraic polynomial approximation in the weighted Smirnov spaces.

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1. INTRODUCTION

Let $L^p(\mathsf{T})$ be the *Lebesgue space* of 2π-periodic real valued functions defined on $\mathsf{T} := [-\pi, \pi]$ such that

$$\|f\|_p := \begin{cases} \left(\int_{\mathsf{T}} |f(x)|^p \, \mathrm{d}x\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x \in \mathsf{T}} |f(x)|, & p = \infty, \end{cases}$$

is finite.

A function $\omega : \mathsf{T} \to [0, \infty]$ will be called a *weight* if ω is measurable and almost everywhere (a.e.) positive.

For a weight ω we denote by $L^p(\mathsf{T}, \omega)$ the class of measurable functions $f : \mathsf{T} \to \mathbb{R}$ such that $\omega f \in L^p(\mathsf{T})$. We set $\|f\|_{p,\omega} := \|\omega f\|_p$.

If $p^{-1} + q^{-1} = 1$, $1 < p < \infty$, $\omega \in L^p(\mathsf{T})$, and $1/\omega \in L^q(\mathsf{T})$ then

$$L^\infty(\mathsf{T}) \subset L^p(\mathsf{T}, \omega) \subset L^1(\mathsf{T}).$$

A 2π-periodic weight function ω belongs to the *Muckenhoupt class* A_p, if

$$\left(\frac{1}{|J|} \int_J \omega^p(x) \, \mathrm{d}x\right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-q}(x) \, \mathrm{d}x\right)^{1/q} \le C$$

with a finite constant C independent of J, where J is any subinterval of T and $|J|$ denotes the length of J.

Let

$$(1) \qquad S[f] := \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}$$

be the *Fourier series* of a function $f \in L^1(\mathsf{T})$ with $\int_{\mathsf{T}} f(x)\,\mathrm{d}x = 0$; so $c_0 = 0$ in (1).

For $\alpha > 0$, the *α-th integral* of f is defined by

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k (\mathrm{i}k)^{-\alpha} \mathrm{e}^{\mathrm{i}kx},$$

where

$$(\mathrm{i}k)^{-\alpha} := |k|^{-\alpha} \mathrm{e}^{(-1/2)\pi \mathrm{i}\alpha \operatorname{sign} k} \text{ and } \mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \dots\}.$$

It is known [9, V. 2, p. 134] that

$$f_\alpha(x) := I_\alpha(x, f)$$

exists a.e. on T and $f_\alpha \in L^1(\mathsf{T})$.

For $\alpha \in (0,1)$ we set

$$f^{(\alpha)}(x) := \frac{\mathrm{d}}{\mathrm{dx}} I_{1-\alpha}(x, f)$$

if the right-hand side exists. Then we define

$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{\mathrm{d}^{r+1}}{\mathrm{d}x^{r+1}} I_{1-\alpha}(x, f),$$

where $r \in \mathbb{Z}^+ := \{1, 2, 3, \dots\}$.

Throughout this work by $C(\alpha)$, c_1, c_2, $\dots$, $c_i(\alpha, \dots)$, $c_j(\beta, \dots)$, $\dots$ we denote the constants (which can be different in different places) such that they are absolute or depend only on the parameters given in the corresponding brackets.

Let $x, t \in \mathbb{R}$, $\alpha \in \mathbb{R}^+ := (0, \infty)$, $1 < p < \infty$. We set

$$\Delta_t^\alpha f(x) := \sum_{k=0}^{\infty} (-1)^\alpha [C_k^\alpha] f(x + (\alpha - k)t), \quad f \in L^p(\mathsf{T}, \omega), \tag{2}$$

where $[C_k^\alpha] := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$ for $k > 1$, $[C_k^\alpha] := \alpha$ for $k = 1$ and $[C_k^\alpha] := 1$ for $k = 0$.

Since

$$|[C_k^\alpha]| \le \frac{c_1(\alpha)}{k^{\alpha+1}}, \text{ for } k \in \mathbb{Z}^+,$$

we have

$$C(\alpha) := \sum_{k=0}^{\infty} |[C_k^\alpha]| < \infty,$$

and $\Delta_t^\alpha f(x)$ is defined a.e. If $\alpha \in \mathbb{Z}^+$, then the fractional difference $\Delta_t^\alpha f(x)$ coincides with usual forward difference, namely,

$$\begin{aligned} \Delta_t^\alpha f(x) &= \sum_{k=0}^{\alpha} (-1)^\alpha [C_k^\alpha] f(x + (\alpha - k)t) \\ &= \sum_{k=0}^{\infty} (-1)^{\alpha-k} [C_k^\alpha] f(x + kt), \quad \alpha \in \mathbb{Z}^+. \end{aligned}$$

We define

$$\sigma_\delta^\alpha f(x) := \frac{1}{\delta}\int_0^\delta \left|\Delta_t^\alpha f(x)\right| \mathrm{d}t, \quad f \in L^p(\mathsf{T},\omega), \quad 1<p<\infty.$$

Using the boundedness of the Hardy-Littlewood Maximal function in $L^p(\mathsf{T},\omega)$, $1<p<\infty$, $\omega\in A_p$, we get

$$\left\|\sigma_\delta^\alpha f(x)\right\|_{p,\omega} \le C(\alpha)\, c_1(p)\, \|f\|_{p,\omega} < \infty. \tag{3}$$

Now, if $\alpha\in\mathbb{R}^+$, we define the *α-th mean modulus of smoothness* of a function $f\in L^p(\mathsf{T},\omega)$, where $1<p<\infty$ and $\omega\in A_p$, as

$$\Omega_\alpha(f,h)_{p,\omega} := \sup_{|\delta|\le h} \left\|\sigma_\delta^\alpha f(x)\right\|_{p,\omega}.$$

Remark 1. The α-th mean modulus of smoothness $\Omega_\alpha(f,h)_{p,\omega}$, $\alpha\in\mathbb{R}^+$, has the following properties:
(i) $\Omega_\alpha(f,h)_{p,\omega}$ is a non-negative and non-decreasing function of $h\ge 0$.
(ii) $\Omega_\alpha(f_1+f_2,\cdot)_{p,\omega} \le \Omega_\alpha(f_1,\cdot)_{p,\omega} + \Omega_\alpha(f_2,\cdot)_{p,\omega}$.
(iii) $\lim\limits_{h\to 0}\Omega_\alpha(f,h)_{p,\omega}=0$.

In what follows let

$$E_n(f)_{p,\omega} := \inf_{T\in\mathcal{T}_n}\|f-T\|_{p,\omega}, \quad f\in L^p(\mathsf{T},\omega), \quad 1<p<\infty, \quad n=0,1,2,\ldots,$$

where $\mathcal{T}_n$ is the class of trigonometrical polynomials of degree not greater than n.

We denote by $W_p^\alpha(\mathsf{T},\omega)$, $\alpha>0$, $1<p<\infty$, the linear space of 2π-periodic real valued functions $f\in L^p(\mathsf{T},\omega)$ such that $f^{(\alpha)}\in L^p(\mathsf{T},\omega)$ a.e.

The next theorem is new for positive values of the integer α. For $\omega\equiv 1$ the result was proved in [7].

Theorem 2. *Let $f\in W_p^\alpha(\mathsf{T},\omega)$, $\alpha>0$, $\omega\in A_p$, $1<p<\infty$. If, for some $T_n\in\mathcal{T}_n$*

$$\|f-T_n\|_{p,\omega} \le c(p)\, E_n(f)_{p,\omega}, \quad n=0,1,2,\ldots,$$

then

$$\left\|f^{(\alpha)}-T_n^{(\alpha)}\right\|_{p,\omega} \le c(\alpha,p)\, E_n\left(f^{(\alpha)}\right)_{p,\omega}, \quad n=0,1,2,\ldots.$$

Proof. We put $S_\nu f(x) := S_\nu(x,f) := \sum\limits_{k=-\nu}^{\nu} c_k \mathrm{e}^{\mathrm{i}kx}$ for the ν-th partial sum of the Fourier series (1) of $f\in W_p^\alpha(\mathsf{T},\omega)$ and $W_n(f) := W_n(x,f) := \frac{1}{n+1}\sum\limits_{\nu=n}^{2n} S_\nu(x,f)$, $n=0,1,2,\ldots$.

Hence $W_n(x,f^{(\alpha)}) = W_n^{(\alpha)}(x,f)$.

Consequently

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\omega} \leq \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)})\right\|_{p,\omega}$$

$$+ \left\| T_n^{(\alpha)}(\cdot, W_n(f)) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\omega} + \left\| W_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}(\cdot, W_n(f))\right\|_{p,\omega}.$$

We denote by $T_n^{*}(x, f)$ the best approximating trigonometric polynomial of degree at most n to f in $L^p(\mathbb{T}, \omega)$. In this case, using the boundedness of W_n in $L^p(\mathsf{T}, \omega)$, we obtain

$$\begin{aligned}
&\left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)})\right\|_{p,\omega} \\
&\leq \left\| f^{(\alpha)}(\cdot) - T_n^{*}(\cdot, f^{(\alpha)})\right\|_{p,\omega} + \left\| T_n^{*}(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)})\right\|_{p,\omega} \\
&\leq c(p) E_n \left(f^{(\alpha)}\right)_{p,\omega} + \left\| W_n(\cdot, T_n^{*}(f^{(\alpha)}) - f^{(\alpha)})\right\|_{p,\omega} \\
&\leq c_1(\alpha, p) E_n \left(f^{(\alpha)}\right)_{p,\omega}.
\end{aligned}$$

From [5] we get

$$\left\| T_n^{(\alpha)}(\cdot, W_n(f)) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\omega} \leq c_2(\alpha, p) n^{\alpha} \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f)\right\|_{p,\omega}$$

and

$$\begin{aligned}
\left\| W_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}(\cdot, W_n(f))\right\|_{p,\omega} &\leq c_3(\alpha, p) (2n)^{\alpha} \left\| W_n(\cdot, f) - T_n(\cdot, W_n(f))\right\|_{p,\omega} \\
&\leq c_4(\alpha, p) (2n)^{\alpha} E_n (W_n(f))_{p,\omega}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f)\right\|_{p,\omega} \leq \left\| T_n(\cdot, W_n(f)) - W_n(\cdot, f)\right\|_{p,\omega} \\
&+ \left\| W_n(\cdot, f) - f(\cdot)\right\|_{p,\omega} + \left\| f(\cdot) - T_n(\cdot, f)\right\|_{p,\omega} \\
&\leq c(p) E_n (W_n(f))_{p,\omega} + c_5(p) E_n(f)_{p,\omega} + c(p) E_n(f)_{p,\omega}.
\end{aligned}$$

Since $E_n(W_n(f))_{p,\omega} \leq c_6(p) E_n(f)_{p,\omega}$ we get

$$\begin{aligned}
&\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\omega} \\
&\leq c_1(\alpha, p) E_n \left(f^{(\alpha)}\right)_{p,\omega} + n^{\alpha} \left\{ c_6(\alpha, p) E_n (W_n(f))_{p,\omega} + c_7(\alpha, p) E_n(f)_{p,\omega} \right\} \\
&+ c_8(\alpha, p) (2n)^{\alpha} E_n (W_n(f))_{p,\omega} \\
&\leq c_1(\alpha, p) E_n \left(f^{(\alpha)}\right)_{p,\omega} + c_9(\alpha, p) n^{\alpha} E_n(f)_{p,\omega}.
\end{aligned}$$

By [1, Th. 1.1] we have

$$E_n(f)_{p,\omega} \leq \frac{c(\alpha, p)}{(n+1)^{\alpha}} E_n \left(f^{(\alpha)}\right)_{p,\omega}, \tag{4}$$

so we finally obtain

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{p,\omega} \leq c(\alpha, p)\, E_n\left(f^{(\alpha)}\right)_{p,\omega}.$$

□

The next result was proved in [8] for $\omega \equiv 1$.

THEOREM 3. *Let* $0 < \alpha \leq 1$, $r = 0, 1, 2, 3, \ldots$, $\omega \in A_p$, $1 < p < \infty$, *and* $T_n \in \mathcal{T}_n$, $n \geq 1$. *Then*

$$\Omega_{r+\alpha}(T_n, h)_{p,\omega} \leq c(p, r)\, h^{\alpha+r} \left\| T_n^{(\alpha+r)} \right\|_{p,\omega}, \quad 0 < h \leq \pi/n. \tag{5}$$

Proof. Let

$$F(x) := \Delta_t^{\alpha+r} T_n\left(x - \frac{\alpha+r}{2} t\right) = \sum_{\nu \in \mathbb{Z}_n^*} (2\mathrm{i}\sin \nu t/2)^{\alpha+r} c_\nu \mathrm{e}^{\mathrm{i}\nu x}$$

and

$$f(x) := \Delta_t^r T_n^{(\alpha)}\left(x - \frac{r}{2} t\right) = \sum_{\nu \in \mathbb{Z}_n^*} (2\mathrm{i}\sin \nu t/2)^r (\mathrm{i}\nu)^{(\alpha)} c_\nu \mathrm{e}^{\mathrm{i}\nu x}.$$

If we put

$$\varphi(z) := (2\mathrm{i}\sin zt/2)^r (\mathrm{i}z)^{(\alpha)}, \quad g(z) := \left(\frac{2}{z}\sin tz/2\right)^\alpha, \quad |z| \leq n,\ g(0) := t^\alpha,$$

we find that

$$f(x) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu)\, c_\nu \mathrm{e}^{\mathrm{i}\nu x}, \quad F(x) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu)\, g(\nu)\, c_\nu \mathrm{e}^{\mathrm{i}\nu x}.$$

The function g is positive, even and satifies $g'(z) \leq 0$, $g''(z) \leq 0$ for $z \in [0, n]$, $0 < t \leq \pi/n$. Hence

$$g(z) = \sum_{k=-\infty}^{\infty} d_k \mathrm{e}^{\mathrm{i}k\pi z/n}$$

uniformly on $[-n, n]$, with $d_0 > 0$, $(-1)^{k+1} d_k \geq 0$, $d_{-k} = d_k$ $(k = 1, 2, \ldots)$ (see, [8]). We get that

$$F(x) = \sum_{k=-\infty}^{\infty} d_k f\left(x + \frac{k\pi}{n}\right)$$

and therefore

$$\Delta_t^{\alpha+r} T_n(\cdot) = \sum_{k=-\infty}^{\infty} d_k \Delta_t^r T_n^{(\alpha)}\left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t\right).$$

Consequently, we obtain

$$\left\| \frac{1}{\delta} \int\limits_0^\delta \left| \Delta_t^{\alpha+r} T_n(\cdot) \right| \mathrm{d}t \right\|_{p,\omega} = \left\| \frac{1}{\delta} \int\limits_0^\delta \left| \sum_{k=-\infty}^{\infty} d_k \Delta_t^r T_n^{(\alpha)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t \right) \right| \mathrm{d}t \right\|_{p,\omega}$$

$$\leq \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta} \int\limits_0^\delta \left| \Delta_t^r T_n^{(\alpha)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t \right) \right| \mathrm{d}t \right\|_{p,\omega}.$$

Since

$$\Delta_t^r T_n^{(\alpha)}(\cdot) = \int\limits_0^t \cdots \int\limits_0^t T_n^{(\alpha+r)} (\cdot + t_1 + \ldots + t_r) \, \mathrm{d}t_1 \ldots \mathrm{d}t_r,$$

we find

$$\Omega_{r+\alpha}(T_n, h)_{p,\omega} \leq \sum_{k=-\infty}^{\infty} |d_k| \sup_{|\delta| \leq h} \left\| \frac{1}{\delta} \int\limits_0^\delta \left| \sum_{k=-\infty}^{\infty} d_k \Delta_t^r T_n^{(\alpha)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t \right) \right| \mathrm{d}t \right\|_{p,\omega}$$

$$= \sum_{k=-\infty}^{\infty} |d_k|,$$

$$\sup_{|\delta| \leq h} \left\| \frac{1}{\delta} \int\limits_0^\delta \left| \int\limits_0^t \cdots \int\limits_0^t T_n^{(\alpha+r)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t + t_1 + \ldots + t_r \right) \mathrm{d}t_1 \ldots \mathrm{d}t_r \right| \mathrm{d}t \right\|_{p,\omega}$$

$$\leq h^r \sum_{k=-\infty}^{\infty} |d_k|,$$

$$\sup_{|\delta| \leq h} \left\| \frac{1}{\delta} \int\limits_0^\delta \left| \frac{1}{\delta^r} \int\limits_0^\delta \cdots \int\limits_0^\delta T_n^{(\alpha+r)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t + t_1 + \ldots + t_r \right) \mathrm{d}t_1 \ldots \mathrm{d}t_r \right| \mathrm{d}t \right\|_{p,\omega}$$

$$\leq h^r \sum_{k=-\infty}^{\infty} |d_k| \sup_{|\delta| \leq h},$$

$$\left\| \frac{1}{\delta^r} \int\limits_0^\delta \cdots \int\limits_0^\delta \left\{ \frac{1}{\delta} \int\limits_0^\delta \left| T_n^{(\alpha+r)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t + t_1 + \ldots + t_r \right) \right| \mathrm{d}t \right\} \mathrm{d}t_1 \ldots \mathrm{d}t_r \right\|_{p,\omega}$$

$$\leq c_{10}(r,p) \, h^r \sum_{k=-\infty}^{\infty} |d_k| \sup_{|\delta| \leq h} \left\| \frac{1}{\delta} \int\limits_0^\delta \left| T_n^{(\alpha+r)} \left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2} t \right) \right| \mathrm{d}t \right\|_{p,\omega}$$

$$\leq c_2(r,p)\,h^r \sup_{|\delta|\leq h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\frac{\alpha}{2}\delta} \int_{\cdot+\frac{k\pi}{n}}^{\cdot+\frac{k\pi}{n}+\frac{\alpha}{2}\delta} \left| T_n^{(\alpha+r)}(u) \right| \mathrm{d}u \right\|_{p,\omega}.$$

By [8] we have

$$\sum_{k=-\infty}^{\infty} |d_k| < 2g(0) = 2t^{\alpha}, \quad 0 < t \leq \pi/n,$$

so

$$\sum_{k=-\infty}^{\infty} |d_k| < 2h^{\alpha}$$

for $0 < t \leq \delta \leq h \leq \pi/n$. Hence

$$\Omega_{\alpha+r}(T_n, h)_{p,\omega} \leq c_{11}(r,p)\,h^{\alpha+r} \left\| T_n^{(\alpha+r)} \right\|_{p,\omega}.$$

On the other hand, we get, by a similar argument, that the same inequality holds also if $0 < -h \leq \pi/n$. Thus the proof of the theorem is completed. □

The next result is a generalization of Theorem 2 of [4] to the fractional case.

THEOREM 4. *Let $\alpha > 0$, $\omega \in A_p$, $1 < p < \infty$. Then the following inequality holds for $f \in L^p(\mathsf{T}, \omega)$*

$$\Omega_{\alpha}(f, \pi/(n+1))_{p,\omega} \leq \frac{c(\alpha,p)}{(n+1)^{\alpha}} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_{\nu}(f)_{p,\omega}, \quad n = 0, 1, 2, \ldots.$$

Proof. Let $T_n \in \mathcal{T}_n$ be the best approximating polynomial of $f \in L^p(\mathsf{T}, \omega)$ and let $m \in \mathbb{Z}^+$. Then by assertion (ii) of Remark 1 and by (3) we have

$$\begin{aligned}\Omega_{\alpha}(f, \pi/(n+1))_{p,\omega} &\leq \Omega_{\alpha}(f - T_{2^m}, \pi/(n+1))_{p,\omega} + \Omega_{\alpha}(T_{2^m}, \pi/(n+1))_{p,\omega} \\ &\leq c_{12}(\alpha,p)\,E_{2^m}(f)_{p,\omega} + \Omega_{\alpha}(T_{2^m}, \pi/(n+1))_{p,\omega}.\end{aligned}$$

Using Theorem 2, we get

$$\Omega_{\alpha}(T_{2^m}, \pi/(n+1))_{p,\omega} \leq c_{13}(\alpha,p) \left(\frac{\pi}{n+1}\right)^{\alpha} \left\| T_{2^m}^{(\alpha)} \right\|_{p,\omega}, \quad n+1 \geq 2^m.$$

Since

$$T_{2^m}^{(\alpha)}(x) = T_1^{(\alpha)}(x) + \sum_{\nu=0}^{m-1} \left\{ T_{2^{\nu+1}}^{(\alpha)}(x) - T_{2^{\nu}}^{(\alpha)}(x) \right\},$$

we obtain

$$\begin{aligned}&\Omega_{\alpha}(T_{2^m}, \pi/(n+1))_{p,\omega} \\ &\leq c_{13}(\alpha,p) \left(\frac{\pi}{n+1}\right)^{\alpha} \left\{ \left\| T_1^{(\alpha)} \right\|_{p,\omega} + \sum_{\nu=0}^{m-1} \left\| T_{2^{\nu+1}}^{(\alpha)} - T_{2^{\nu}}^{(\alpha)} \right\|_{p,\omega} \right\}.\end{aligned}$$

From Bernstein's inequality (see [5]) for fractional derivatives in $L^p(\mathsf{T},\omega)$, where $\omega \in A_p$ and $1 < p < \infty$, we have

$$\left\| T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)} \right\|_{p,\omega} \leq c_{14}(\alpha,p)\, 2^{\nu\alpha} \left\| T_{2^{\nu+1}} - T_{2^\nu} \right\|_{p,\omega} \leq c_{15}(\alpha,p)\, 2^{\nu\alpha+1} E_{2^\nu}(f)_{p,\omega}$$

and

$$\left\| T_1^{(\alpha)} \right\|_{p,\omega} = \left\| T_1^{(\alpha)} - T_0^{(\alpha)} \right\|_{p,\omega} \leq c_{16}(\alpha,p)\, E_0(f)_{p,\omega}.$$

Hence

$$\begin{aligned} &\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p,\omega} \\ &\leq c_{17}(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \left\{ E_0(f)_{p,\omega} + \sum_{\nu=0}^{m-1} 2^{(\nu+1)\alpha} E_{2^\nu}(f)_{p,\omega} \right\}. \end{aligned}$$

It is easily seen that

$$2^{(\nu+1)\alpha} E_{2^\nu}(f)_{p,\omega} \leq c_{18}(\alpha) \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1} E_\mu(f)_{p,\omega}, \quad \nu = 1,2,3,\ldots.$$

Therefore

$$\begin{aligned} &\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p,\omega} \\ &\leq c_{17}(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \\ &\quad \left\{ E_0(f)_{p,\omega} + 2^\alpha E_1(f)_{p,\omega} + c_{18}(\alpha) \sum_{\nu=1}^{m} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1} E_\mu(f)_{p,\omega} \right\} \\ &\leq c_{19}(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \left\{ E_0(f)_{p,\omega} + \sum_{\mu=1}^{2^m} \mu^{\alpha-1} E_\mu(f)_{p,\omega} \right\} \\ &\leq c_{20}(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \sum_{\nu=0}^{2^m-1} (\nu+1)^{\alpha-1} E_\nu(f)_{p,\omega}. \end{aligned}$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p,\omega} \leq \frac{c_{21}(\alpha,p)}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p,\omega}$$

and

$$E_{2^m}(f)_{p,\omega} \leq E_{2^{m-1}}(f)_{p,\omega} \leq \frac{c_{22}(\alpha,p)}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p,\omega}.$$

This finishes the proof. □

The next result was proved for $\alpha = 1$ in [4].

Theorem 5. *If $f\in W_p^{\alpha+r}(\mathsf{T},\omega)$, $0<\alpha\leq 1$, $r=0,1,2,3,\ldots$, $\omega\in A_p$, $1<p<\infty$, then*

$$\Omega_{r+\alpha}(f,h)_{p,\omega}\leq c(\alpha,r,p)\,h^{\alpha+r}\left\|f^{(\alpha+r)}\right\|_{p,\omega},\quad 0<h\leq\pi.$$

Proof. Let $T_n\in\mathcal{T}_n$ be the trigonometric polynomial of best approximation of f in $L^p(\mathbb{T},\omega)$ metric. By Remark 1 (ii), Theorem 2, and (3) we get

$$\begin{aligned}&\Omega_{\alpha+r}(f,h)_{p,\omega}\leq\Omega_{\alpha+r}(T_n,h)_{p,\omega}+\Omega_{\alpha+r}(f-T_n,h)_{p,\omega}\\&\leq c(p,r)\,h^{\alpha+r}\left\|T_n^{(\alpha+r)}\right\|_{p,\omega}+c_{22}(p,\alpha,r)\,E_n(f)_{p,\omega},\quad 0<h\leq\pi/n.\end{aligned}$$

Then, using inequality (10) of [4], (4), and Theorem 2 of [4], we have

$$\begin{aligned}E_n(f)_{p,\omega}&\leq\frac{c(p,\alpha,r)}{(n+1)^{\alpha}}E_n\left(f^{(\alpha)}\right)_{p,\omega}\leq\frac{c_{18}(p,\alpha,r)}{(n+1)^{\alpha}}\Omega_r\left(f^{(\alpha)},\frac{2\pi}{n+1}\right)_{p,\omega}\\&\leq\frac{c_{23}(p,\alpha,r)}{(n+1)^{\alpha}}\left(\frac{2\pi}{n+1}\right)^r\left\|f^{(\alpha+r)}\right\|_{p,\omega}.\end{aligned}$$

By Theorem 2 we find

$$\begin{aligned}&\left\|T_n^{(\alpha+r)}\right\|_{p,\omega}\leq\left\|T_n^{(\alpha+r)}-f^{(\alpha+r)}\right\|_{p,\omega}+\left\|f^{(\alpha+r)}\right\|_{p,\omega}\\&\leq c(p,\alpha,r)\,E_n\left(f^{(\alpha+r)}\right)_{p,\omega}+\left\|f^{(\alpha+r)}\right\|_{p,\omega}\leq c_{24}(p,\alpha,r)\left\|f^{(\alpha+r)}\right\|_{p,\omega}.\end{aligned}$$

Choosing h with $\pi/(n+1)<h\leq\pi/n$, $n=1,2,3,\ldots$, we obtain

$$\Omega_{\alpha+r}(f,h)_{p,\omega}\leq c(p,\alpha,r)\,h^{\alpha+r}\left\|f^{(\alpha+r)}\right\|_{p,\omega}$$

and we are done. □

Theorem 6. *Let $f\in L^p(\mathsf{T},\omega)$, $1<p<\infty$, $\omega\in A_p$. If $\beta\in(0,\infty)$ and*

$$\sum_{\nu=1}^{\infty}\nu^{\beta-1}E_\nu(f)_{p,\omega}<\infty,$$

then

$$E_n\left(f^{(\beta)}\right)_{p,\omega}\leq c(p,\beta)\left((n+1)^{\beta}E_n(f)_{p,\omega}+\sum_{\nu=n+1}^{\infty}\nu^{\beta-1}E_\nu(f)_{p,\omega}\right).$$

Proof. Since

$$\begin{aligned}&\left\|f^{(\beta)}-S_nf^{(\beta)}\right\|_{p,\omega}\\&\leq\left\|S_{2^{m+2}}f^{(\beta)}-S_nf^{(\beta)}\right\|_{p,\omega}+\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}f^{(\beta)}-S_{2^k}f^{(\beta)}\right\|_{p,\omega},\end{aligned}$$

we have for $2^m < n < 2^{m+1}$

$$\left\| S_{2^{m+2}} f^{(\beta)} - S_n f^{(\beta)} \right\|_{p,\omega}$$
$$\leq c_{25}(p,\beta) 2^{(m+2)\beta} E_n(f)_{p,\omega} \leq c_{26}(p,\beta)(n+1)^\beta E_n(f)_{p,\omega}.$$

On the other hand, we find

$$\sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}} f^{(\beta)} - S_{2^k} f^{(\beta)} \right\|_{p,\omega} \leq c_{27}(p,\beta) \sum_{k=m+2}^{\infty} 2^{(k+1)\beta} E_{2^k}(f)_{p,\omega}$$

$$= c_{29}(p,\beta) \sum_{\nu=2^{m+1}+1}^{\infty} \nu^{\beta-1} E_\nu(f)_{p,\omega} \leq c_{29}(p,\beta) \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} E_\nu(f)_{p,\omega}$$

which finishes the proof. □

COROLLARY 7. *Let* $f \in W_p^\alpha(\mathsf{T},\omega)$, $(1 < p < \infty)$, $\omega \in A_p$, $\beta \in (0,\infty)$ *and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p,\omega} < \infty$$

for some $\alpha > 0$. *If* $n = 0,1,2,\ldots$, *then*

$$\Omega_\beta\left(f^{(\alpha)}, \frac{\pi}{n+1}\right)_{p,\omega}$$
$$\leq \frac{c_{43}(\alpha,p,\beta)}{(n+1)^\beta} \sum_{\nu=0}^{n} (\nu+1)^{\alpha+\beta-1} E_\nu(f)_{p,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p,\omega}.$$

2. APPLICATIONS TO WEIGHTED SMIRNOV SPACES

Let Γ be a rectifiable Jordan curve and let $G := \operatorname{int}\Gamma$, $G^- := \operatorname{ext}\Gamma$, $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, $\mathbb{T} := \partial\mathbb{D}$, $\mathbb{D}^- := \operatorname{ext}\mathbb{T}$. Without loss of generality we may assume $0 \in G$. We denote by $L^p(\Gamma)$, $1 \leq p < \infty$, the set of all measurable complex valued functions f on Γ such that $|f|^p$ is Lebesgue integrable with respect to arclength. By $E_p(G)$ and $E_p(G^-)$, $0 < p < \infty$, we denote the *Smirnov classes* of analytic functions in G and G^-, respectively. Let $w = \varphi(z)$ and $w = \varphi_1(z)$ be the conformal mappings of G^- and G onto $\mathbb{D}^-$ normalized by the conditions

$$\varphi(\infty) = \infty,\ \lim_{z\to\infty} \varphi(z)/z > 0 \text{ and } \varphi_1(0) = \infty,\ \lim_{z\to 0} z\varphi_1(z) > 0,$$

respectively. Let $f \in L^1(\Gamma)$. Then

$$f^+(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \qquad z \in G,$$

is analytic on G.

Let ω be a weight function on Γ and let $L^p(\Gamma,\omega)$ be the *weighted Lebesgue space* on Γ, i.e., the space of measurable functions on Γ for which

$$\|f\|_{L^p(\Gamma,\omega)} := \left(\int_\Gamma |f(z)|^p \omega^p(z) |\mathrm{d}z| \right)^{1/p} < \infty.$$

The *weighted Smirnov spaces* $E_p(G,\omega)$ and $E_p(G^-,\omega)$ are defined as

$$E_p(G,\omega) := \{f \in E_1(G) : f \in L^p(\Gamma,\omega)\},$$

$$E_p\left(G^-,\omega\right) := \left\{f \in E_1\left(G^-\right) : f \in L^p(\Gamma,\omega)\right\}.$$

We also define the following subspace of $E_p(G^-,\omega)$

$$\tilde{E}_p\left(G^-,\omega\right) := \left\{f \in E_p\left(G^-,\omega\right) : f(\infty) = 0\right\}.$$

Let $1 < p < \infty$, $z \in \Gamma$, $\varepsilon > 0$, and $\Gamma(z,\varepsilon) := \{t \in \Gamma : |t-z| < \varepsilon\}$, $\frac{1}{p} + \frac{1}{q} = 1$. A weight function ω belongs to the *Muckenhoupt class* $A_p(\Gamma)$ if the condition

$$\sup_{z\in\Gamma} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} \omega^p(\tau) |d\tau| \right)^{\frac{1}{p}} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} \omega^{-q}(\tau) |d\tau| \right)^{\frac{1}{q}} < \infty,$$

holds.

With every weight function ω on Γ, we associate the other weights on $\mathbb{T}$ by setting $\omega_0 := \omega \circ \psi$, $\omega_1 := \omega \circ \psi_1$. For an arbitrary $f \in L^p(\Gamma,\omega)$ we set

$$f_0(w) := f(\psi(w)), \quad f_1(w) := f(\psi_1(w)), \quad w \in \mathbb{T}.$$

If Γ is a Dini-smooth curve, then the condition $f \in L^p(\Gamma,\omega)$ implies that $f_0 \in L^p(\mathbb{T},\omega_0)$ and $f_1 \in L^p(\mathbb{T},\omega_1)$. Using the nontangential boundary values of f_0^+ and f_1^+ on $\mathbb{T}$ we define for a function $f \in L^p(\Gamma,\omega)$ and $\alpha \in \mathbb{R}^+$

$$\begin{aligned} \Omega_k(f,\delta)_{\Gamma,p,\omega} &:= \Omega_k\left(f_0^+,\delta\right)_{p,\omega_0}, \quad \delta > 0, \\ \tilde{\Omega}_k(f,\delta)_{\Gamma,p,\omega} &:= \Omega_k\left(f_1^+,\delta\right)_{p,\omega_1}, \quad \delta > 0. \end{aligned} \tag{6}$$

We set

$$E_n(f,G)_{p,\omega} := \inf_{P\in\mathcal{P}_n} \|f-P\|_{L^p(\Gamma,\omega)}, \quad \tilde{E}_n\left(g,G^-\right)_{p,\omega} := \inf_{R\in\mathcal{R}_n} \|g-R\|_{L^p(\Gamma,\omega)},$$

where $f \in E_p(G,\omega)$, $g \in E_p(G^-,\omega)$, $\mathcal{P}_n$ is the set of algebraic polynomials of degree not greater than n, and $\mathcal{R}_n$ is the set of rational functions of the form $\sum_{k=0}^{n} \frac{a_k}{z^k}$.

Some converse approximation theorems in the weighted Lebesgue spaces $L^p(\mathsf{T},\omega)$, $1 < p < \infty$, $\omega \in A_p$ were proved in [1] and [4]. In the weighted Smirnov spaces $E_p(G,\omega)$, $\omega \in A_p(\Gamma)$, $1 < p < \infty$, the converse approximation theorems were proved in [3] for Butzer-Wehrens modulus of smoothness.

In the following we investigate the approximation problems in the weighted Smirnov spaces in terms of the α-th mean modulus of smoothness.

The following converse theorems can be proved by the method given in [2] and [3].

THEOREM 8. *Let G be a finite, simply connected domain with a Dini-smooth boundary Γ. If $\alpha > 0$ and $f \in E_p(G,\omega)$, $\omega \in A_p(\Gamma)$, $1 < p < \infty$, then*

$$\Omega_\alpha (f, 1/n)_{\Gamma,p,\omega} \leq \frac{c(\Gamma,p,\alpha)}{n^\alpha} \sum_{k=0}^{n} (k+1)^{\alpha-1} E_k(f,G)_{p,\omega}, \quad n = 1, 2, \ldots .$$

If $\alpha = 2r$, $r = 1, 2, \ldots$, this result was proved in [3] for a different but equivalent modulus of smoothness.

The converse theorem for an unbounded domain G^- is also true.

THEOREM 9. *Let Γ be a Dini-smooth curve. If $\alpha > 0$, $f \in \tilde{E}_p(G^-,\omega)$, and $\omega \in A_p(\Gamma)$, $1 < p < \infty$, then*

$$\tilde{\Omega}_\alpha (f, 1/n)_{\Gamma,p,\omega} \leq \frac{c(\Gamma,p,\alpha)}{n^\alpha} \sum_{k=0}^{n} (k+1)^{\alpha-1} \tilde{E}_k\left(f,G^-\right)_{p,\omega}, \quad n = 1, 2, 3, \ldots .$$

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