MATHEMATICA, Tome 54 (77), N° Spécial, 2012, pp. 16–24

ONE TOPIC ON WAVELET ALGORITHM BY USING ONE DIMENSIONAL HAAR WAVELETS

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Abstract. In this paper we obtain an algorithm to compute a fast wavelet transform and use this algorithm to analyze and synthesize a signal or function f. We consider a sample point (t_j, s_j) that includes a value $s_j = f(t_j)$ at height s_j and abscissa (time or location) t_j, and apply wavelet decomposition by using shifts and dilations of the basic Haar transform. Some relationship between wavelet coefficients are investigated.

MSC 2010. 42C40, 65T60.

Key words. Haar wavelets, fast wavelets, wavelet algorithm, estimation, discreet wavelet, multiresolution analysis.

1. INTRODUCTION

Wavelets are regarded by many as primarily a new subject in pure and applied mathematics. Indeed, many papers published on wavelets contain esoteric-looking theorems with complicated proofs. Wavelet analysis was led by Ingrid Daubechies [4], and many colleagues contributed in different ways: Meyer [7], Walter [10], Vidakovic [9], Cohen et al. [3], Antoniadis et al. [1], Clyed et al. [2].

Perhaps one of the most common application of wavelets is in signal processing. A signal, broadly defined, is a sequence of numerical measurements, typically obtained electronically.

To analyze and synthesize a signal, which can be any array of data in terms of simple wavelets, we employ shifts and dilations of a mathematical function, but we do not involve either calculus or linear algebra. The first step in applying wavelets to any signal consists is representing the signal under consideration by a mathematical function f. For example, a sound, the values $s = f(t)$ measure the sound at each time t at a fixed location.

The first step in the analysis of a signal with wavelets consists in approximating its function by means of sample alone. One of the simplest methods of approximation uses a horizontal stair step extended through each sample point. The resulting steps form a new function denote by $\tilde{f}$ and called a step function, which approximates the sampled function $s = f(t)$. The analysis of the approximating function $\tilde{f}$ in terms of wavelets requires a precise labeling

The support of Research Committee of Persian Gulf University is greatly acknowledged.

of each step, by means of shifts and dilations of the basic unit step function, denoted by $\varphi_{[0,1)}$.

If a sample point (t_j, s_j) includes a value $s_j = f(t_j)$ at height s_j and abscissa (time or location) t_j, then the sample point corresponds to the step function $s_j \cdot \varphi_{[t_j, t_{[j+1)})}(t)$, which approximates f at height s_j on the interval $[t_j, t_{[j+1)})$, where φ_A denote the indicator function of set A.

Adding all the step functions approximating corresponding to all the points in the sample, yields the simple step function below:

$$\tilde{f} = \sum_{j=0}^{n-1} s_j \cdot \varphi_{[t_j, t_{[j+1)})}(t). \tag{1}$$

To facilitate comparisons between different signals, and to allow for the use of common algorithms, simple wavelets pertain to the interval $t \in [0,1)$, so that one unit corresponds to the entire length of the signal. Thus, $t = \frac{1}{2}$ denotes the middle of the signal, and $t = \frac{7}{8}$ denotes the location at the seventh eighth of the signal.

Haar basic transformation expresses the approximating function $\tilde{f}$ with wavelets by replacing and adjacent pair of steps by one wider step and one wavelet. The sum of two adjacent steps with width $\frac{1}{2}$ produces the basic unit step function $\varphi_{[0,1)}$ and the difference of two such narrower steps gives the corresponding basic wavelet as following:

$$\varphi_{[0,1)} = \varphi_{[0,\frac{1}{2})} + \varphi_{[\frac{1}{2},1)} \quad \text{and} \quad \psi_{[0,1)} = \varphi_{[0,\frac{1}{2})} - \varphi_{[\frac{1}{2},1)}.$$

It is clear that we have

$$\frac{1}{2}(\varphi_{[0,1)} + \psi_{[0,1)}) = \varphi_{[0,\frac{1}{2})} \quad \text{and} \quad \frac{1}{2}(\varphi_{[0,1)} - \psi_{[0,1)}) = \varphi_{[\frac{1}{2},1)}. \tag{2}$$

2. FAST WAVELETS TRANSFORM AND ALGORITHM

To analyze a signal or function in term of wavelets, the Fast Haar wavelet transform begins with initialization of an array with 2^n entries, and then proceeds with n iterations of the basic transform explained in (1).

For each index $j \in \{1, 2, ..., n\}$, before iteration number J, the array will consist of $2^{n-(j-1)}$ coefficients of $2^{n-(j-1)}$ step function $\varphi_{k,n-(j-1)}$, defined below. After iteration number j, the array will consist of half as many, 2^{n-j} coefficient of 2^{n-j} step function $\varphi_{k,n-j}$ and 2^{n-j} coefficient $\psi_{k,n-j}$, such as

$$\varphi_{k,n-j}(t) = \varphi_{[0,1)}(2^{n-j}[t - k2^{j-n}]), \tag{3}$$

$$\psi_{k,n-j}(t) = \psi_{[0,1)}(2^{n-j}[t - k2^{j-n}]). \tag{4}$$

ALGORITHM. For Haar wavelets the initialization consists only on establishing a one dimensional array

$$\begin{aligned} \vec{\mathbf{v}}_{(n)} &= (v_{0,n}, v_{1,n}, ..., v_{2^n-2,n}, v_{2^n-1,n}) \\ &= (s_0, s_1, ..., s_j, ..., s_{2^n-2}, s_{2^n-1}) \\ &= \vec{s}. \end{aligned} \tag{5}$$

With the total number of sample values equal to an integral power of two, say 2^n. Though indices ranging from 1 through 2^n would also serve the same purpose, indices ranging from 0 through $2^n - 1$ will accommodate a binary encoding with only n binary digits. The array corresponds to the sampled step function

$$\hat{f}_n = \sum_{k=0}^{2^n-1} v_{k,n}\varphi_{k,n}(t). \tag{6}$$

In general, the jth sweep of the basic transform begins with an array of $2^{n-(j-1)}$ values

$$\vec{\mathbf{v}}_{n-(j-1)} = (v_{0,n-(j-1)}, \cdots, v_{2^{n-(j-1)}-1,n-(j-1)}). \tag{7}$$

It means that lists the values $v_{k,n-(j-1)}$ of a simple step function $\tilde{f}_{(n-(j-1))}$ that approximates f with $2^{(n-(j-1)}$ steps of narrower width $2^{(n-(j-1))}$ as following:

$$\tilde{f}_{n-(j-1)} = \sum_{k=0}^{2^{n-(j-1)}-1} v_{k,n-(j-1)}\varphi_{k,n-(j-1)}(t). \tag{8}$$

We apply the basic transform to each pair $(v_{2n,n-(j-1)}, v_{2n+1,n-(j-1)})$, which gives two new wavelets coefficients

$$\begin{aligned} v_{k,(n-j)} &= \frac{v_{2k,n-(j-1)} + v_{2k+1,n-(j-1)}}{2}, \\ c_{k,(n-j)} &= \frac{v_{2k,n-(j-1)} - v_{2k+1,n-(j-1)}}{2}. \end{aligned}$$

These $2^{(n-j)}$ pairs of new coefficients represented the result of the jth sweep

$$\begin{aligned} \vec{\mathbf{v}}_{n-j} &= (v_{0,n-j}, v_{1,n-j}, \cdots, v_{k,n-j}, \cdots, v_{2^{n-j}-1,n-j}), \\ \vec{\mathbf{c}}_{n-j} &= (c_{0,n-j}, c_{1,n-j}, \cdots, c_{k,n-j}, \cdots, c_{2^{n-j}-1,n-j}). \end{aligned}$$

It means that lists the values $v_{k,(n-j)}$ of a simple step function $\tilde{f}_{(n-j))}$ that approximates f with $2^{(n-j)}$ steps of narrower width $2^{(n-j)}$:

$$\hat{f}_{n-j} = \sum_{k=0}^{2^{n-j}-1} v_{k,n-j}\varphi_{k,n-j}(t), \tag{9}$$

$$\check{f}_{n-j} = \sum_{k=0}^{2^{n-j}-1} c_{k,n-j}\psi_{k,n-j}(t). \tag{10}$$

The wavelets given by second new array, $\vec{\mathbf{c}}_{n-j}$, represent the difference between the finer steps of the initial estimation $\check{f}_{n-(j-1)}$ and the coarser steps of $\check{f}n-j$. so the initial approximation $\hat{f}_{n-(j-1)}$ still equals the sum of two new approximations, $\hat{f}_{n-j}$ and $\check{f}_{n-j}$, so we have

$$\hat{f}_{n-(j-1)} = \check{f}_{n-j} + \hat{f}_{n-j}. \tag{11}$$

Repeating these sweeps, the approximation of function$f$ is complete.

EXAMPLE 1. Let the signal or physical phenomena consist in representing the signal under consideration by mathematical function $f(t) = s$, where $t \in [0, 1)$. For the approximation of f, suppose we choose sample as following:

j	:	0	1	2	3	4	5	6	7
s_j	:	1	5	4	6	9	5	3	7
t_j	:	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$

According to this algorithm, our sample size is $8 = 2^n$. Thus,

$$n = 3 \Rightarrow \vec{\mathbf{v}}_3 = \vec{\mathbf{s}} = (1, 5, 4, 6, 9, 5, 3, 7).$$

The first sweep:

$$\begin{aligned} \vec{\mathbf{v}}_{3-1} &= \left(\frac{1+5}{2}, \frac{4+6}{2}, \frac{9+5}{2}, \frac{3+7}{2}\right) = (3, 5, 7, 5), \\ \vec{\mathbf{c}}_{3-1} &= \left(\frac{1-5}{2}, \frac{4-6}{2}, \frac{9-5}{2}, \frac{3-7}{2}\right) = (-2, -1, 2, -2). \end{aligned}$$

So we have

$$\vec{\mathbf{s}}_{3-1} = (\vec{\mathbf{v}}_{3-1}; \vec{\mathbf{c}}_{3-1}) = (\mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{5}, -2, -1, 2, -2).$$

The second sweep:

$$\begin{aligned} \bar{\mathbf{v}}_{3-1} &= (\mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{5}), \\ \vec{\mathbf{v}}_{3-2} &= \left(\frac{3+5}{2}, \frac{7+5}{2}\right) = (\mathbf{4}, \mathbf{6}), \\ \vec{\mathbf{c}}_{3-2} &= \left(\frac{3-5}{2}, \frac{7-5}{2}\right) = (-1, 1). \end{aligned}$$

Now we have

$$\vec{\mathbf{s}}_{3-2} = (\bar{\mathbf{v}}_{3-1}; \vec{\mathbf{c}}_{3-2}; \vec{\mathbf{c}}_{3-1}) = (\mathbf{4}, \mathbf{6}, -1, 1, -2, -1, 2, -1).$$

The third sweep:

$$\begin{aligned} \vec{\mathbf{v}}_{3-2} &= (\mathbf{4}, \mathbf{6}), \\ \vec{\mathbf{v}}_{3-3} &= \left(\frac{4+6}{2}\right) = (\mathbf{5}), \\ \vec{\mathbf{c}}_{3-3} &= \left(\frac{2-8}{2}\right) = (-2) \end{aligned}$$

can be written as follows:

$$\vec{\mathbf{s}}_{3-3} = (\bar{\mathbf{v}}_{3-3}; \vec{\mathbf{c}}_{3-3}; \vec{\mathbf{c}}_{3-2}; \vec{\mathbf{c}}_{3-1}) = (\mathbf{5}; -2; -1, 1; -2, -1, 2, -1). \tag{12}$$

3. RESULTS

1. According to equation (9), the initial array $\vec{\mathbf{v}}_3 = (1, 5, 4, 6, 9, 5, 3, 7)$ represents the approximation function $\hat{f}$ by its sample values,

$$\begin{aligned} \hat{f} = 1\varphi_{[0,\frac{1}{8})} + 5\varphi_{[\frac{1}{8},\frac{2}{8})} + 4\varphi_{[\frac{2}{8},\frac{3}{8})} + 6\varphi_{[\frac{3}{8},\frac{4}{8})} \\ + 9\varphi_{[\frac{4}{8},\frac{5}{8})} + 5\varphi_{[\frac{5}{8},\frac{6}{8})} + 3\varphi_{[\frac{6}{8},\frac{7}{8})} + 7\varphi_{[\frac{7}{8},1)}. \end{aligned} \tag{13}$$

2. In contrast, the wavelet coefficient $\vec{\mathbf{c}}_{3-j}$ produced by the consecutive sweeps of basic transforms expresses the same approximating function $\hat{f}$ in terms of consecutively lower frequencies, ending with a constant step across entire interval. According to equation (10), we estimate f by sample size 8 as following:

$$\begin{aligned} \hat{f} &= (-2)\psi_{[0,\frac{1}{4})} + (-1)\psi_{[\frac{1}{4},\frac{2}{4})} + 2\psi_{[\frac{2}{4},\frac{3}{4})} + (-1)\psi_{[\frac{3}{4},1)} \\ &+ (-1)\psi_{[0,\frac{1}{2})} + 1\psi_{[\frac{1}{2},1)} \\ &+ (-2)\psi_{[0,1)} + 5\varphi_{[0,1)}. \end{aligned} \tag{14}$$

3. Coefficient 5 of $5\varphi_{[0,1)}$ means that the sample has average value equal to 5. Coefficient -2 of $-2\psi_{[0,1)}$ means that the sample undergoes a jump 3 times the size of and in the opposite direction from the wavelet $\psi_{[0,1)}$ with jump of size equal 4. The other coefficients are explained similarly.

4. For each pair $(v_{2k,n-(j-1)}, v_{2k+1,n-(j-1)})$, instead of placing its results in two additional arrays, the jth sweep can replace the pair

$$(v_{2k,n-(j-1)}, v_{2k+1,n-(j-1)})$$

by the new entries $(v_{k,n-j}, v_{k,n-j})$ as Example 1, so we have

$$\vec{\mathbf{v}}_3 = \vec{\mathbf{s}} = (1, 5, 4, 6, 9, 5, 3, 7),$$

then

$$\begin{aligned} \vec{\mathbf{s}}_{3-1} &= (v_{0,3-1}, c_{0,3-1}, v_{1,3-1}, c_{1,3-1}, v_{2,3-1}, c_{2,3-1}, v_{3,3-1}, c_{3,3-1}) \\ &= \left(\frac{1+5}{2}, \frac{1-5}{2}, \frac{4+6}{2}, \frac{4-6}{2}, \frac{9+5}{2}, \frac{9-5}{2}, \frac{3+7}{2}, \frac{3-7}{2}\right) \\ &= (\mathbf{3}, -2, \mathbf{5}, -1, \mathbf{7}, 2, \mathbf{5}, -2), \end{aligned}$$

$$\begin{aligned} \vec{\mathbf{s}}_{3-2} &= \left(\frac{3+5}{2}, -2, \frac{3-5}{2}, -1, \frac{7+5}{2}, 2, \frac{7-5}{2}, -2\right) \\ &= (\mathbf{4}, -2, -1, -1, \mathbf{6}, 2, 1, -2), \end{aligned}$$

$$\vec{\mathbf{s}}_{3-3} = \left(\frac{4+6}{2}, -2, -1, -1, \frac{4-6}{2}, 2, 1-2\right) \tag{15}$$
$$= (\mathbf{5}, -2, -1, -1, -1, 2, 1, -2).$$

We can see that equation (12) and (15) give the same approximation for f.

4. TWO SCALE RELATIONSHIP

In this section we define a function space, $\mathbf{v}_j$, $j \in Z$ to be $\{\mathbf{v}_j = f \in L^2(R) : f$ is piecewise constant on $[k2^{-j}, (k+1)2^{-j}], k \in Z\}$. If this sequence of subspaces has the following properties:

1. $\cdots \subset \mathbf{V}_{-2} \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \cdots$.
2. $\bigcap_{j\in Z} \mathbf{V}_j = 0, \overline{\bigcup_{j\in Z} \mathbf{V}_j} = L^2(R)$.
3. $f(x) \in \mathbf{V}_j \Longleftrightarrow f(2x) \in \mathbf{V}_{j+1}$.
4. $f(x) \in \mathbf{V}_0 \Longrightarrow f(x-k) \in \mathbf{V}_0 \ \forall k \in Z$.
5. There is a function $\varphi(x) \in \mathbf{V}_0$ such that $\{\varphi_{0,k}(x) = \varphi(x-k), k \in Z\}$ constitutes an *orthonormal basis* for $\mathbf{V}_0$.

then we say that $(\mathbf{v}_j)_{j\in Z}$ form a multiresolution analysis (**MRA**) of $L^2(R)$, which is $\mathbf{v}_j = \text{span}\{\varphi_{j,k}, k \in Z\}$, $\mathbf{W}_j = \text{span}\{\psi_{j,k}, k \in Z\}$. For any function $f \in L^2(R)$ we can write (see [4]):

$$f = \sum_{k\in Z} v_{m,k}\varphi_{m,k} + \sum_{j=m}^{\infty}\sum_{k\in Z} c_{j,k}\psi_{j,k}, \tag{16}$$

where the functions

$$\varphi_{m,k}(x) = 2^{\frac{m}{2}}\phi(2^m x - k), \ \psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) \tag{17}$$

constitute an (inhomogeneous) *orthonormal* basis of $L^2(R)$. Here $\varphi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet, respectively.

It is clear that for Haar wavelet

$$\varphi(x) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) + \frac{1}{\sqrt{2}}\varphi_{1,1}(x), \ \psi(x) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) - \frac{1}{\sqrt{2}}\varphi_{1,1}(x). \tag{18}$$

So, we can write

$$\varphi_{m,k}(x) = \frac{\varphi_{j+1,2k}(x) + \varphi_{j+1,2k+1}(x)}{\sqrt{2}}, \ \psi_{j,k}(x) = \frac{\varphi_{j+1,2k}(x) - \varphi_{j+1,2k+1}(x)}{\sqrt{2}}.$$

Note that $\varphi \in \mathbf{v}_0$, therefore $\varphi \in \mathbf{v}_1$ since $\mathbf{v}_0 \subset \mathbf{v}_1$. Since $\{\varphi_{1,k}(x), k \in Z\}$ is an *orthonormal* basis for $\mathbf{V}_1$, there exists a sequence b_k such that

$$\varphi(x) = \sum_{k\in Z} b_k\varphi_{1,k}(x) \tag{19}$$

THEOREM 1. *For equation* (19) *we have*

(I) $\sum_k b_k = \sqrt{2}$,

(II) $\sum_k b_k^2 = 1$.

Proof. According to equation (17) we can write $\varphi_{m,k}(x) = \sqrt{2}\varphi(2x-k)$, which implies

$$\int \varphi(x)\mathrm{d}x = \sqrt{2}\sum_k b_k \int \varphi(2x-k)\mathrm{d}x = \frac{\sqrt{2}}{2}\sum_k b_k \int \varphi(x)\mathrm{d}x.$$

Since $\int \varphi(x)\mathrm{d}x \neq 0$, we have $\sum_k b_k = \sqrt{2}$.

For the proof of (II), we know that

$$\int \varphi(x)\varphi(x-l)\mathrm{d}x = 1 \text{ for } l = 0. \tag{20}$$

By using equations (20) and (17), we obtain

$$\begin{aligned}
\int \varphi(x)\varphi(x-l)\mathrm{d}x &= \int \sqrt{2}\sum_k b_k\varphi(2x-k)\varphi(x-l)\mathrm{d}x \\
&= \int \sqrt{2}\sum_k b_k\varphi(2x-k)\sqrt{2}\sum_m b_k\varphi(2(x-l)-m)\mathrm{d}x \\
&= 2\sum_k b_k\left[\sum_m b_m\frac{1}{2}\int \varphi(2x-k)\varphi(2x-2l-m)\mathrm{d}(2x)\right] \qquad (21)\\
&= \sum_k\sum_m b_k b_m\int \varphi(2x-k)\varphi(2x-2l-m)\mathrm{d}(2x) \\
&= \sum_k b_k b_{k-2l}.
\end{aligned}$$

The last line is obtained by taking $k = 2l+m$. By replacing $l = 0$ in equation (21), the proof of the second part is complete. □

Remark 1. The coefficient b_k may be written

$$b_k = \int \varphi(x)\varphi_{1,k}(x)\mathrm{d}x. \tag{22}$$

$\{b_k\}$ is a square-summable sequence, that is, we say $\{b_k\} \in l^2 Z$, if $\sum_{k\in Z} b_k^2 < \infty$. For the Haar basis, it was seen that $b_k = 2^{-1/2}$ for $k = 0, 1$ and it is zero otherwise. In this multiresolution context, this same sequence that relates scaling function at two levels of b_k can be used to define the mother wavelet:

$$\psi(x) = \sum_{k\in Z}(-1)^k b_{-k+1}\varphi_{1,k}(x). \tag{23}$$

A special case of this construction was seen in (18).

Theorem 2. *The wavelet spaces* $\{\mathbf{W}_j, j \in Z\}$ *and scale space* $\{\mathbf{v}_j, j \in Z\}$ *are mutually orthogonal.*

Proof. First we prove that the scaling function and wavelet are orthogonal.

$$\begin{aligned}\prec \psi, \varphi \succ &= \int \psi(x)\varphi(x)\mathrm{d}x = \int \left(\sum_k (-1)^k b_{-k+1}\varphi_{1,k}(x)\right)\varphi(x)\mathrm{d}x \\ &= \sum_k (-1)^k b_{-k+1}\int \varphi_{1,k}(x)\varphi(x)\mathrm{d}x = \sum_k (-1)^k b_{-k+1}b_k = 0.\end{aligned}$$

The last step follows since the summand for k is the opposite of the summand for $1-k$, so each term is negated. It can be seen similarly that each integer translation of the mother wavelet ψ is also orthogonal to φ:

$$\begin{aligned}\prec \psi_{0,l}, \varphi \succ &= \int \psi(x-l)\varphi(x)\mathrm{d}x = \int \left(\sum_k (-1)^k b_{-k+1}\varphi_{1,k}(x-l)\right)\varphi(x)\mathrm{d}x \\ &= \sum_k (-1)^k b_{-k+1}\int \varphi_{1,2l+k}(x)\varphi(x)\mathrm{d}x = \sum_k (-1)^k b_{-k+1}b_{2l+k} \\ &= 0.\end{aligned}$$

The last step follows since the summand for k is the opposite of the summand for $1-k-2l$, so each term is negated, because of the square sumability of the sequence b_k.

A straightforward extension of this argument will show that $\psi_{0,l}\perp\varphi_{0,k}$, for all $k, l \in Z$, and similarly it can be seen that $\psi_{j,l}\perp\varphi_{j,k}$, for all $k, l \in Z$. Thus we completed the proof. □

THEOREM 3. *Suppose that $v_{m,k}$ are scaling coefficients and $c_{j,k}$ are wavelet coefficients. Then $v_{m,k} = \sum_{k\in Z} b_{k-2l}\, v_{m+1,k}$ and $c_{j,k} = \sum_{k\in Z}(-1)^k b_{-k+2l-1}\, v_{m+1,k}$.*

Proof.

$$\begin{aligned}v_{m,l} &= \int \varphi_{m,l}(x)f(x)\mathrm{d}x = \int \sum_{k\in\mathbf{Z}} b_k 2^{\frac{m}{2}}\varphi_{1,k}(2^m x - l)f(x)\mathrm{d}x \\ &= \int \sum_{k\in\mathbf{Z}} b_k 2^{\frac{m+1}{2}}\varphi(2^{m+1}x - 2l - k)f(x)\mathrm{d}x \\ &= \int \sum_{k\in\mathbf{Z}} b_k \varphi_{m+1,k+2l}(x)f(x)\mathrm{d}x = \int \sum_{k\in\mathbf{Z}} b_{k-2l}\varphi_{m+1,k}(x)f(x)\mathrm{d}x \\ &= \sum_{k\in\mathbf{Z}} b_{k-2l}\int \varphi_{m+1,k}(x)f(x)\mathrm{d}x = \sum_{k\in\mathbf{Z}} b_{k-2l}v_{m+1,k}.\end{aligned}$$

Similarly, from two-scale relationship for any $\psi_{j,k}$ to the $\varphi_{j+1,l}$ in equation (23), we can write the wavelet coefficient $c_{i,j}$ as follows:

$$c_{j,k} = \sum_{k\in Z}(-1)^k b_{-k+2l-1}v_{m+1,k}. \tag{24}$$

□

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