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ON SOME NONZERO RINGEL-HALL NUMBERS IN TAME CASES

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Abstract. Let k be a finite field and consider the finite dimensional path algebra kQ where Q is a quiver of tame type i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. Let $\mathcal{H}(kQ)$ be the corresponding Ringel-Hall algebra. We are going to study the Ringel-Hall numbers of the form $F_{XP}^{P'}$ with P, P' preprojective indecomposables of defect -1 and $F_{IX}^{I'}$ with I, I' preinjective indecomposables of defect 1. More precisely we will give necessary conditions for the module X such that these Ringel-Hall numbers are nonzero.
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1. FACTS ON TAME HEREDITARY ALGEBRAS AND RINGEL-HALL ALGEBRAS

For a detailed description of the forthcoming notions we refer to [1],[2],[3],[4].

Let k be a finite field and consider the path algebra kQ where Q is a quiver of tame type i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. When Q is of type $\tilde{A}_n$ we exclude the cyclic orientation. So kQ is a finite dimensional tame hereditary algebra with the category of finite dimensional (hence finite) right modules denoted by mod-kQ. Let $[M]$ be the isomorphism class of $M \in$ mod-kQ. The category mod-kQ can and will be identified with the category rep-kQ of the finite dimensional k-representations of the quiver $Q = (Q_0 = \{1, 2, ..., n\}, Q_1)$. Here $Q_0 = \{1, 2, ..., n\}$ denotes the set of vertices of the quiver, Q_1 the set of arrows and for an arrow α we denote by $s(\alpha)$ the starting point of the arrow and by $e(\alpha)$ its endpoint. Recall that a k-representation of Q is defined as a set of finite dimensional k-spaces $\{V_i | i = \overline{1, n}\}$ corresponding to the vertices together with k-linear maps $V_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$ corresponding to the arrows. The dimension of a module $M = (V_i, V_\alpha) \in$ mod-kQ = rep-kQ is then $\underline{\dim} M = (\dim_k V_i)_{i=\overline{1,n}} \in \mathbb{Z}^n$. For $a = (a_i), b = (b_i) \in \mathbb{Z}^n$ we say that $a \le b$ iff $b_i - a_i \ge 0$ for all i.

Let $P(i)$ and $I(i)$ be the projective and injective indecomposable corresponding to the vertex i and consider the Cartan matrix C_Q with the j-th column being $\underline{\dim} P(j)$. We have then a biliniar form on $\mathbb{Z}^n$ defined as $\langle a, b \rangle = a C_Q^{-t} b^t$. Then for two modules $X, Y \in$ mod-kQ we have

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim_k \mathrm{Hom}(X, Y) - \dim_k \mathrm{Ext}^1(X, Y).$$

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We denote by q the quadratic form defined by $q(a) = \langle a, a\rangle$. Then q is positive semi-definite with radical $\mathbb{Z}\delta$, that is $\{a \in \mathbb{Z}^n | q(a) = 0\} = \mathbb{Z}\delta$. Here δ is known for each type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ (see [3]). The defect of a module M is $\partial M = \langle \delta, \underline{\dim} M\rangle = -\langle \underline{\dim} M, \delta\rangle$. For a short exact sequence $0 \to X \to Y \to Z \to 0$ we have that $\partial Y = \partial X + \partial Z$.

Consider the Auslander-Reiten translates $\tau = D\operatorname{Ext}^1(-, kQ)$ and $\tau^{-1} = \operatorname{Ext}^1(D(kQ), -)$, where $D = \operatorname{Hom}_k(-, k)$. An indecomposable module M is preprojective (preinjective) if exists a positive integer m such that $\tau^m(M) = 0$ ($\tau^{-m}(M) = 0$). Otherwise M is said to be regular. A module is preprojective (preinjective, regular) if every indecomposable component is preprojective (preinjective, regular). Note that an indecomposable module M is preprojective (preinjective, regular) iff $\partial M < 0$ ($\partial M > 0$, $\partial M = 0$).

The Auslander-Reiten quiver of kQ has as vertices the isomorphism classes of indecomposables and arrows corresponding to so called irreducible maps. It will have a preprojective component (with all the isoclasses of preprojective indecomposables), a preinjective component (with all the isoclasses of preinjective indecomposables). All the other components (containing the isoclasses of regular indecomposables) are "tubes" of the form $\mathbb{Z}A_\infty/m$, where m is the rank of the tube. The tubes are indexed by the points of the scheme $\mathbb{P}^1_k$, the degree of a point $x \in \mathbb{P}^1_k$ being denoted by $\deg x$. A tube of rank 1 is called homogeneous, otherwise is called non-homogeneous. We have at most 3 non-homogeneous tubes indexed by points x of degree $\deg x = 1$. All the other tubes are homogeneous. Indecomposables from different tubes have no nonzero homomorphisms and no non-trivial extensions. So the regulars from a single tube form an extension-closed abelian subcategory of mod kQ, the simple objects in this subcategory being called quasi-simple regulars. An indecomposable regular module is regular uniserial and hence is uniquely determined by its quasi-top and quasi-length. In case of a homogeneous tube τ_x we have a single quasi-simple regular denoted by $R_x[1]$ with $\underline{\dim} R_x[1] = (\deg x)\delta$, which lies on the "mouth" of the tube. In case of a non-homogeneous tube τ_x of rank m on the mouth of the tube we have m quasi-simples denoted by $R^i_x[1]$ $i = \overline{1, m}$ such that $\sum_{i=1}^m \underline{\dim} R^i_x[1] = \delta$.

The following lemma is well known.

Lemma 1.1. a) *For P preprojective, I preinjective, R regular modules we have*

$$\operatorname{Hom}(R, P) = \operatorname{Hom}(I, P) = \operatorname{Hom}(I, R) = 0,$$

$$\operatorname{Ext}^1(P, R) = \operatorname{Ext}^1(P, I) = \operatorname{Ext}^1(R, I) = 0.$$

b) *If $x \neq x'$ and R_x ($R_{x'}$) is a regular with components from the tube τ_x ($\tau_{x'}$), then* $\operatorname{Hom}(R_x, R_{x'}) = \operatorname{Ext}^1(R_x, R_{x'}) = 0$.

c) *For τ_x homogeneous, $R_x[t]$ an indecomposable from τ_x and $R_x[1]$ the quasi-simple on the mouth of τ_x we have* $\dim_k \operatorname{Hom}(R_x[t], R_x[1]) = \deg x$.

We consider now the rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ. Its $\mathbb{Q}$-basis is formed by the isomorphism classes $[M]$ from mod-kQ and the multiplication is defined by

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M].$$

The structure constants $F^M_{N_1 N_2} = |\{M \supseteq U|\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers.

2. SOME NONZERO RINGEL-HALL NUMBERS

Consider the Ringel-Hall numbers of the form $F^{P'}_{XP}$ with P, P' preprojetive indecomposables of defect -1 and $F^{I'}_{IX}$ with I, I' preinjective indecomposables of defect 1. We are going to give necessary conditions for the module X such that these Ringel-Hall numbers are nonzero.

We start with the preprojective case by formulating some lemmas. (The first lemma can be also found in [5]).

Lemma 2.1. *Let P be a preprojective indecomposable with defect $\partial P = -1$, P' a preprojective module and R a regular indecomposable. Then we have*

a) *Every nonzero morphism $f : P \to P'$ is a monomorphism.*

b) *For every nonzero morphism $f : P \to R$, f is either a monomorphism or $\operatorname{Im} f$ is regular. In particular if R is quasi-simple and $\operatorname{Im} f$ is regular then f is an epimorphism.*

Proof. a) Consider the short exact sequence $0 \to \operatorname{Ker} f \to P \to \operatorname{Im} f \to 0$. Since $\operatorname{Ker} f \subseteq P$ and $\operatorname{Im} f \subseteq P'$ we have that $\operatorname{Ker} f$ and $\operatorname{Im} f$ are preprojective (so with negative defect) or 0. Moreover we have that $\partial \operatorname{Ker} f + \partial \operatorname{Im} f = \partial P = -1$ and we know that $\operatorname{Im} f \neq 0$ (since f is nonzero). It follows that $\operatorname{Ker} f = 0$

b) Consider the short exact sequence $0 \to \operatorname{Ker} f \to P \to \operatorname{Im} f \to 0$. Since $\operatorname{Ker} f \subseteq P$ we have that $\operatorname{Ker} f$ is preprojective (so with negative defect) or 0. On the other hand $\operatorname{Im} f \subseteq R$ implies that $\operatorname{Im} f$ can contain preprojectives and regulars as direct summands (and it is nonzero since f is nonzero). The equality $\partial \operatorname{Ker} f + \partial \operatorname{Im} f = \partial P = -1$ gives us two cases. When $\partial \operatorname{Ker} f = 0$ then $\operatorname{Ker} f$ is 0 so f is monomorphism. In the second case, when $\partial \operatorname{Ker} f = -1$ then $\partial \operatorname{Im} f = 0$, so $\operatorname{Im} f$ can contain just regular direct summands. □

Lemma 2.2. *Let P be a preprojective indecomposable with defect $\partial P = -1$.*

a) *Suppose that $\underline{\dim} P > \delta$. Then P projects to the quasi-simple regular $R_x[1]$ from each homogeneous tube τ_x with $(\deg x)\delta < \dim P$. Also P projects to a unique quasi-simple regular from the mouth of each non-homogeneous tube τ_x. We will denote these quasi-simple regulars by $R^P_x[1]$ where for τ_x homogeneous with $(\deg x)\delta < \dim P$ we have $R^P_x[1] = R_x[1]$.*

b) *Suppose that $\underline{\dim} P \not> \delta$. Then P projects at most to a single quasi-simple regular from each non-homogeneous tube τ_x denoted by $R^P_x[1]$.*

Proof. a) Suppose that $R_x[1]$ denotes the quasi-simple regular from the mouth of the homogeneous tube τ_x with $\underline{\dim} R_x[1] = (\deg x)\delta < \underline{\dim} P$. Then we have $\mathrm{Ext}^1(P, R_x[1]) = 0$ (see Lemma 1.1.) so

$$\begin{aligned}\dim_k \mathrm{Hom}(P, R_x[1]) &= \langle \underline{\dim} P, \underline{\dim} R_x[1]\rangle = \langle \underline{\dim} P, (\deg x)\delta\rangle \\ &= (\deg x)(-\partial P) = \deg x \neq 0.\end{aligned}$$

This means that we have a nonzero morphism $f : P \to R_x[1]$ with $\underline{\dim} P > \underline{\dim} R_x[1]$. Using Lemma 2.1. we deduce that f is not a monomorphism, so $\mathrm{Im}\, f$ is regular and $R_x[1]$ is quasi-simple, which means that f is an epimorphism.

Denote by $R_x^i[1]$, $i = \overline{1, m}$ the i-th quasi-simple regular from the mouth of the non-homogeneous tube τ_x of rank $m \geq 2$. Notice that this time $\deg x = 1$, $\sum_{i=1}^m \underline{\dim} R_x^i[1] = \delta$ and $\mathrm{Ext}^1(P, R_x^i[1]) = 0$ so we have

$$\sum_{i=1}^m \dim_k \mathrm{Hom}(P, R_x^i[1]) = \sum_{i=1}^m \langle \underline{\dim} P, \underline{\dim} R_x^i[1]\rangle$$

$$= \langle \underline{\dim} P, \sum_{i=1}^m \underline{\dim} R_x^i[1]\rangle = \langle \underline{\dim} P, \delta\rangle = -\partial P = 1.$$

It follows that $\exists! i_0$ such that $\mathrm{Hom}(P, R_x^{i_0}[1]) \neq 0$, so we have a nonzero morphism $f : P \to R_x^{i_0}[1]$ with $\underline{\dim} P > \delta > \underline{\dim} R_x^{i_0}[1]$. Using Lemma 2.1. we deduce that f is not a monomorphism, so $\mathrm{Im}\, f$ is regular and $R_x^{i_0}[1]$ is quasi-simple, which means that f is an epimorphism. Let $R_x^P[1] := R_x^{i_0}[1]$.

b) Since $\underline{\dim} P \not> \delta$ clearly P could project only on quasi-simple regulars from non-homogeneous tubes. Denote again by $R_x^i[1]$, $i = \overline{1, m}$ the i-th quasi-simple regular on the mouth of the non-homogeneous tube τ_x of rank $m \geq 2$. As above we can deduce that $\exists! i_0$ such that $\mathrm{Hom}(P, R_x^{i_0}[1]) \neq 0$, so we have a nonzero morphism $f : P \to R_x^{i_0}[1]$. But if $\underline{\dim} P \not> \underline{\dim} R_x^{i_0}[1]$ then f is a monomorphism and not an epimorphism. □

Remark 2.3. Notice that $\dim_k \mathrm{Hom}(P, R_x^P[1]) = \deg x$.

Theorem 2.4. *Let $P \not\cong P'$ be preprojective indecomposables with defect -1 and suppose $F_{XP}^{P'} \neq 0$ for some module X. Then X satisfies the following conditions:*

i) *it is a regular module with $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$,*

ii) *if it has an indecomposable component from a tube τ_x then the quasi-top of this component is the quasi-simple regular $R_x^{P'}[1]$,*

iii) *its indecomposable components are taken from pairwise different tubes.*

Proof. We will check the conditions i),ii),iii).

Condition i). Since $F_{XP}^{P'} \neq 0$ we have a short exact sequence $0 \to P \to P' \to X \to 0$. Then $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$ and $\partial P' = \partial P + \partial X$, but $\partial P' = \partial P = -1$, so $\partial X = 0$. Notice that X can't have preprojective components, since if P'' would be such a component then $P' \twoheadrightarrow P'' \ncong P'$ which is impossible due to Lemma 2.1. a). So X is regular.

Condition ii). Let R be an indecomposable component of X taken from the tube τ_x. Denote by topR its quasi-top which must be quasi-simple due to uniseriality. Then $P' \twoheadrightarrow X \twoheadrightarrow R \twoheadrightarrow \mathrm{top}R$ so using Lemma 2.2. $\mathrm{top}R \cong R_x^{P'}[1]$.

Condition iii). Suppose $X = X' \oplus R_1 \oplus ... \oplus R_l$, where $R_1, ..., R_l$ are taken from the same tube τ_x. Then by Condition ii) they have the same quasi-top $R_x^{P'}[1]$ and we have the monomorphism $0 \to \mathrm{Hom}(X, R_x^{P'}[1]) \to \mathrm{Hom}(P', R_x^{P'}[1])$.

It follows that $\dim_k \mathrm{Hom}(X, R_x^{P'}[1]) \leq \dim_k \mathrm{Hom}(P', R_x^{P'}[1]) = \deg x$. Then $\dim_k \mathrm{Hom}(X, R_x^{P'}[1]) = \dim_k \mathrm{Hom}(X', R_x^{P'}[1]) + \sum_{i=1}^{l} \dim_k \mathrm{Hom}(R_i, R_x^{P'}[1]) \leq \deg x$. Hence we have $\dim_k \mathrm{Hom}(R_i, R_x^{P'}[1]) = \deg x$ for τ_x homogeneous and $\dim_k \mathrm{Hom}(R_i, R_x^{P'}[1]) \geq 1 = \deg x$ for τ_x non-homogeneous. It follows that $l = 1$. □

We move on to the preinjective case.

LEMMA 2.5. *Let I be preinjective indecomposable with defect $\partial I = 1$ and I' a preinjective. If $f : I' \to I$ is a monomorphism then its an isomorphism.*

Proof. Consider the short exact sequence $0 \to I' \to I \to I/I' \to 0$. Since $I \twoheadrightarrow I/I'$ then I/I' is either preinjective or 0. But if I/I' is preinjective then $1 = \partial I = \partial I' + \partial I/I' > 1$ a contradiction, so I/I' is 0 and f is an isomorphism. □

LEMMA 2.6. *Let R be a quasi-simple regular and I a preinjective indecomposable with defect $\partial I = 1$. Suppose that $\underline{\dim} R < \underline{\dim} I$. Then a nonzero morphism $f : R \to I$ is a monomorphism.*

Proof. Consider the short exact sequence $0 \to \mathrm{Ker}\, f \to R \to \mathrm{Im}\, f \to 0$. Since $\mathrm{Ker}\, f \hookrightarrow R$, R is quasi-simple regular and f is nonzero $\mathrm{Ker}\, f$ could be preprojective or 0. But if $\mathrm{Ker}\, f$ is preprojective then from $0 = \partial R = \partial \,\mathrm{Ker}\, f + \partial \,\mathrm{Im}\, f$ results that $\mathrm{Im}\, f$ has a preinjective component which embeds into I. This would imply that $\mathrm{Im}\, f = I$ so $R \twoheadrightarrow I$, a contradiction due to $\underline{\dim} R < \underline{\dim} I$. So $\mathrm{Ker}\, f$ is 0. □

LEMMA 2.7. *Let I be a preinjective indecomposable with defect $\partial I = 1$.*

a) *Suppose that $\underline{\dim} I > \delta$. Then the quasi-simple regular $R_x[1]$ from each homogeneous tube τ_x with $(\deg x)\delta < \dim I$ embeds into I. Also a unique quasi-simple regular from the mouth of each non-homogeneous tube τ_x embeds into I. We will denote these quasi-simple regulars by $R_x^I[1]$ where for τ_x homogeneous with $(\deg x)\delta < \dim I$ we have $R_x^I[1] = R_x[1]$.*

b) *Suppose that $\underline{\dim} I \not> \delta$. Then at most a single quasi-simple regular from each non-homogeneous tube τ_x embeds into I. We denote this quasi-simple regular by $R_x^I[1]$.*

Proof. a) Suppose that $R_x[1]$ denotes the quasi-simple regular from the mouth of the homogeneous tube τ_x with $\underline{\dim} R_x[1] = (\deg x)\delta < \underline{\dim} I$. Then we have $\mathrm{Ext}^1(R_x[1], I) = 0$ (see Lemma 1.1.) so

$$\begin{aligned}\dim_k \mathrm{Hom}(R_x[1], I) &= \langle \underline{\dim} R_x[1], \underline{\dim} I\rangle = \langle (\deg x)\delta, \underline{\dim} I\rangle \\ &= (\deg x)(\partial I) = \deg x \neq 0.\end{aligned}$$

This means that we have a nonzero morphism $f : R_x[1] \to I$ with $\underline{\dim} I > \underline{\dim} R_x[1]$. Using Lemma 2.6. we deduce that f is a monomorphism.

Denote by $R_x^i[1]$, $i = \overline{1, m}$ the i-th quasi-simple regular from the mouth of the non-homogeneous tube τ_x of rank $m \geq 2$. Notice that this time $\deg x = 1$, $\sum_{i=1}^m \underline{\dim} R_x^i[1] = \delta$ and $\mathrm{Ext}^1(R_x^i[1], I) = 0$ so we have

$$\sum_{i=1}^m \dim_k \mathrm{Hom}(R_x^i[1], I) = \sum_{i=1}^m \langle \underline{\dim} R_x^i[1], \underline{\dim} I\rangle$$

$$= \langle \sum_{i=1}^m \underline{\dim} R_x^i[1], \underline{\dim} I\rangle = \langle \delta, \underline{\dim} I\rangle = \partial I = 1.$$

It follows that $\exists ! i_0$ such that $\mathrm{Hom}(R_x^{i_0}[1], I) \neq 0$, so we have a nonzero morphism $f : R_x^{i_0}[1] \to I$ with $\underline{\dim} I > \delta > \underline{\dim} R_x^{i_0}[1]$. Using Lemma 2.6. we deduce that f is a monomorphism. Let $R_x^I[1] := R_x^{i_0}[1]$.

b) Since $\underline{\dim} I < \delta$ clearly only quasi-simple regulars from non-homogeneous tubes could embed into I. Denote again by $R_x^i[1]$, $i = \overline{1, m}$ the i-th quasi-simple regular on the mouth of the non-homogeneous tube τ_x of rank $m \geq 2$. As above we can deduce that $\exists ! i_0$ such that $\mathrm{Hom}(R_x^{i_0}[1], I) \neq 0$, so we have a nonzero morphism $f : R_x^{i_0}[1] \to I$. But if $\underline{\dim} I \not> \underline{\dim} R_x^{i_0}[1]$ then f is not a monomorphism. □

Remark 2.8. Notice that $\dim_k \mathrm{Hom}(R_x^I[1], I) = \deg x$.

Theorem 2.9. *Let $I \ncong I'$ be preinjective indecomposables with defect 1 and suppose $F_{IX}^{I'} \neq 0$ for some module X. Then X satisfies the following conditions:*

i) *it is a regular module with $\underline{\dim} X = \underline{\dim} I' - \underline{\dim} I$,*

ii) *if it has an indecomposable component from a tube τ_x then the quasi-socle of this component is the quasi-simple regular $R_x^{I'}[1]$,*

iii) *its indecomposable components are taken from pairwise different tubes.*

Proof. We will check the conditions i),ii),iii).

Condition i). Since $F_{IX}^{I'} \neq 0$ we have a short exact sequence $0 \to X \to I' \to I \to 0$. Then $\underline{\dim} X = \underline{\dim} I' - \underline{\dim} I$ and $\partial I' = \partial I + \partial X$, but $\partial I' = \partial I = 1$, so $\partial X = 0$. Notice that X can't have preinjective components, since if I'' would be such a component then $I'' \hookrightarrow I'$ so $I'' \cong I'$ due to Lemma 2.5. which is a contradiction. It follows that X is regular

Condition ii). Let R be an indecomposable component of X taken from the tube τ_x. Denote by soc$R$ its quasi-socle which must be quasi-simple due to uniseriality. Then $\mathrm{soc}R \hookrightarrow R \hookrightarrow X \hookrightarrow I'$ so using Lemma 2.7. $\mathrm{soc}R \cong R_x^{I'}[1]$.

Condition iii). Suppose $X = X' \oplus R_1 \oplus ... \oplus R_l$, where $R_1, ..., R_l$ are taken from the same tube τ_x. Then by Condition ii) they have the same quasi-socle $R_x^{I'}[1]$ and we have the monomorphism $0 \to \mathrm{Hom}(R_x^{I'}[1], X) \to \mathrm{Hom}(R_x^{I'}[1], I')$.

It follows that $\dim_k \mathrm{Hom}(R_x^{I'}[1], X) \leq \dim_k \mathrm{Hom}(R_x^{I'}[1], I') = \deg x$. Then $\dim_k \mathrm{Hom}(R_x^{X'}[1], X) = \dim_k \mathrm{Hom}(R_x^{I'}[1], X') + \sum_{i=1}^{l} \dim_k \mathrm{Hom}(R_x^{I'}[1], R_i) \leq \deg x$. Hence we have $\dim_k \mathrm{Hom}(R_x^{I'}[1], R_i) = \deg x$ for τ_x homogeneous and $\dim_k \mathrm{Hom}(R_x^{I'}[1], R_i) \geq 1 = \deg x$ for τ_x non-homogeneous. It follows that $l = 1$. □

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