MATHEMATICA, Tome 53 (76), N° 2, 2011, pp. 165–170

ON SUBORDINATION, STARLIKENESS AND CONVEXITY OF CERTAIN INTEGRAL OPERATORS

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Abstract. The aim of this paper is to derive some sufficient conditions for certain integral operators in the open unit disk $\mathcal{U}$ to be subordinate to $\frac{\beta(1-z)}{\beta-z}$ for some real values of β, $z \in \mathcal{U}$ and to be starlike and convex in $\mathcal{U}$.

MSC 2010. 30C45.

Key words. Starlike functions, convex functions, integral operators, subordination.

1. INTRODUCTION

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and let $\mathcal{A}$ denote the class of functions f normalized by $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disk $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be convex function of order ρ, $0 \leq \rho < 1$, if it satisfies the inequality $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \rho,\ \ z \in \mathcal{U}$. We denote the class of convex functions of order ρ by $\mathcal{K}(\rho)$. Similarly, if $f \in \mathcal{A}$ satisfies the inequality $\mathrm{Re}\left(\frac{zf'(z)}{f(z)} + 1\right) > \rho,\ \ z \in \mathcal{U}$ for some ρ, $0 \leq \rho < 1$, then f is said to be starlike of order ρ. We denote the class of starlike functions of order ρ by $\mathcal{S}^*(\rho)$. We note that $f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*$. In particular, the classes $\mathcal{K}(0) = \mathcal{K}$ and $\mathcal{S}^*(0) = \mathcal{S}^*$, are familiar classes of starlike and convex functions in $\mathcal{U}$.

Let f and g be analytic functions in the unit disk $\mathcal{U}$. The function f is said to be subordinate to g and written $f \prec g$ if there exist an analytic function w in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, for $z \in \mathcal{U}$ such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$. If g is univalent on $\mathcal{U}$, these conditions are equivalent to the conditions that $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

We consider the integral operators $F_n(z) = \int\limits_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} \mathrm{d}t$ and $F_{\alpha_1,\ldots,\alpha_n}(z) = \int\limits_0^z (f_1'(t))^{\alpha_1} \cdot \ldots \cdot (f_n'(t))^{\alpha_n} \mathrm{d}t$, where $f_i(z) \in \mathcal{A}$ and $\alpha_i > 0$, for all $i \in \{1, 2, ..., n\}$. These operators were introduced by D. Breaz and N. Breaz [1] and studied by many authors (see [2], [3], [4]).

The work is fully supported by EScienceFund grant: 04-01-02-SF0425, MOSTI, Malaysia.

In the present paper, we obtain some interesting sufficient conditions for $\frac{zF_n'(z)}{F_n(z)}$, $\frac{zF'_{\alpha_1,\dots,\alpha_n}(z)}{F_{\alpha_1,\dots,\alpha_n}(z)}$, $\frac{F_n(z)}{zF_n'(z)}$ and $\frac{F_{\alpha_1,\dots,\alpha_n}(z)}{zF'_{\alpha_1,\dots,\alpha_n}(z)}$ to be subordinate to $\frac{\beta(1-z)}{\beta-z}$ for some real values of β, and the above integral operators F_n and $F_{\alpha_1,\dots,\alpha_n}$ to be starlike and convex of order β in $\mathcal{U}$. In order to derive our main results, we need the following new interesting results due to Shiraishi and Owa [5].

THEOREM 1. [5] *If* $f \in \mathcal{A}$ *satisfies* $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{\beta+1}{2(\beta-1)}$ *for some* β *with* $2 \le \beta < 3$*, or* $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{5\beta-1}{2(\beta+1)}$ *for some* β *with* $1 < \beta \le 2$*, then* $\frac{zf'(z)}{f(z)} \prec \frac{\beta(1-z)}{\beta-z}$ *and* $\left|\frac{zf'(z)}{f(z)} - \frac{\beta}{\beta+1}\right| < \frac{\beta}{\beta+1}$*. This implies that* $f \in \mathcal{S}^*$*, and* $\int_0^z \frac{f(t)}{t}\mathrm{d}t \in \mathcal{K}$.

THEOREM 2. [5] *If* $f \in \mathcal{A}$ *satisfies* $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{\beta+1}{2\beta(\beta-1)}$ *for some* $\beta \le -1$ *or* $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{3\beta+1}{2\beta(\beta+1)}$ *for some* $\beta > 1$*, then* $\frac{f(z)}{zf'(z)} \prec \frac{\beta(1-z)}{\beta-z}$ *and* $f \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$*. This implies that* $\int_0^z \frac{f(t)}{t}\mathrm{d}t \in \mathcal{K}\left(\frac{\beta+1}{2\beta}\right)$.

2. MAIN RESULTS

Our first investigation result is the following:

THEOREM 3. *Let* $\alpha_i > 0$ *be real numbers for* $i \in \{1,2,\dots,n\}$*. If* $f_i \in \mathcal{A}$ *for* $i \in \{1,2,\dots,n\}$ *satisfies*

$$\operatorname{Re}\frac{zf_i'(z)}{f_i(z)} < 1 + \frac{3-\beta}{2(\beta-1)\sum_{i=1}^{n}\alpha_i}, \tag{1}$$

for some β *with* $2 \le \beta < 3$ *or*

$$\operatorname{Re}\frac{zf_i'(z)}{f_i(z)} < 1 + \frac{3}{2}\frac{\beta-1}{(\beta+1)\sum_{i=1}^{n}\alpha_i} \tag{2}$$

for some β *with* $1 < \beta \le 2$*, we obtain* $\frac{zF_n'(z)}{F_n(z)} \prec \frac{\beta(1-z)}{\beta-z}$ *and* $\left|\frac{zF_n'(z)}{F_n(z)} - \frac{\beta}{\beta+1}\right| < \frac{\beta}{\beta+1}$*. This implies that* $F_n(z) \in \mathcal{S}^*$ *and* $\int_0^z \frac{F_n(t)}{t}\mathrm{d}t \in \mathcal{K}$.

Proof. We calculate the derivatives of the first and second order for F_n. Since $F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$, we have $F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$. Differentiating the above expression logarithmically, we have

$\frac{F_n''(z)}{F_n'(z)} = \sum\limits_{i=1}^{n} \alpha_i \left(\frac{f_i'(z)}{f_i(z)} - \frac{1}{z} \right)$. Multiplying the above expression by z we obtain $\frac{zF_n''(z)}{F_n'(z)} = \sum\limits_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right)$. This is equivalent to

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} + 1 - \sum_{i=1}^{n} \alpha_i. \tag{3}$$

Taking real parts in (3) we get

$$\mathrm{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{i=1}^{n} \alpha_i \mathrm{Re}\, \frac{zf_i'(z)}{f_i(z)} + 1 - \sum_{i=1}^{n} \alpha_i. \tag{4}$$

Using (4) and (1) we obtain

$$\mathrm{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) < \sum_{i=1}^{n} \alpha_i \left(1 + \frac{3-\beta}{2\,(\beta-1) \sum\limits_{i=1}^{n} \alpha_i} \right) + 1 - \sum_{i=1}^{n} \alpha_i$$

$$= \sum_{i=1}^{n} \alpha_i + \frac{3-\beta}{2(\beta-1)} + 1 - \sum_{i=1}^{n} \alpha_i = \frac{3-\beta}{2(\beta-1)} + 1 = \frac{\beta+1}{2(\beta-1)}.$$

Therefore $\mathrm{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) < \frac{\beta+1}{2(\beta-1)}$ for some β with $2 \leq \beta < 3$. And using (4) and (2) we obtain

$$\mathrm{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) < \sum_{i=1}^{n} \alpha_i \left(1 + \frac{3}{2} \frac{\beta-1}{(\beta+1) \sum\limits_{i=1}^{n} \alpha_i} \right) + 1 - \sum_{i=1}^{n} \alpha_i$$

$$= \sum_{i=1}^{n} \alpha_i + \frac{3\beta-3}{2(\beta+1)} + 1 - \sum_{i=1}^{n} \alpha_i = \frac{3\beta-3}{2(\beta+1)} + 1 = \frac{5\beta-1}{2(\beta+1)}.$$

Therefore $\mathrm{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) < \frac{5\beta-1}{2(\beta+1)}$ for some β with $1 < \beta \leq 2$. Hence by using Theorem 1 we get $\frac{zF_n'(z)}{F_n(z)} \prec \frac{\beta(1-z)}{\beta-z}$ and $\left| \frac{zF_n'(z)}{F_n(z)} - \frac{\beta}{\beta+1} \right| < \frac{\beta}{\beta+1}$. This implies that $F_n(z) \in \mathcal{S}^*$ and $\int\limits_0^z \frac{F_n(t)}{t} \mathrm{d}t \in \mathcal{K}$. $\square$

Taking $\beta = 2$ in Theorem 3 we have following corollary.

COROLLARY 4. *Let* $\alpha_i > 0$ *be real numbers for* $i \in \{1, 2, ..., n\}$. *If* $f_i \in \mathcal{A}$ *for* $i \in \{1, 2, ..., n\}$ *satisfies* $\mathrm{Re}\, \frac{zf_i'(z)}{f_i(z)} < 1 + \frac{1}{2 \sum\limits_{i=1}^{n} \alpha_i}$, *then* $\frac{zF_n'(z)}{F_n(z)} \prec \frac{2(1-z)}{2-z}$ *and* $\left| \frac{zF_n'(z)}{F_n(z)} - \frac{2}{3} \right| < \frac{2}{3}$.

THEOREM 5. *Let $\alpha_i > 0$ be real numbers for $i \in \{1, 2, ..., n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, 2, ..., n\}$ satisfies*

$$\operatorname{Re}\frac{zf_i'(z)}{f_i(z)} > 1 - \frac{\beta(2\beta-1)+1}{2\beta(\beta-1)\sum\limits_{i=1}^{n}\alpha_i} \tag{5}$$

for some $\beta \leq -1$, or

$$\operatorname{Re}\frac{zf_i'(z)}{f_i(z)} > 1 - \frac{\beta(2\beta-1)-1}{2\beta(\beta+1)\sum\limits_{i=1}^{n}\alpha_i} \tag{6}$$

for some $\beta > 1$, then $\frac{F_n(z)}{zF_n'(z)} \prec \frac{\beta(1-z)}{\beta-z}$ and $F_n(z) \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$. This implies that $\int\limits_0^z \frac{F_n(t)}{t}\mathrm{d}t \in \mathcal{K}\left(\frac{\beta+1}{2\beta}\right)$.

Proof. Proceeding similarly to the proof of Theorem 3, we obtain that

$$\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{i=1}^{n}\alpha_i \operatorname{Re}\frac{zf_i'(z)}{f_i(z)} + 1 - \sum_{i=1}^{n}\alpha_i. \tag{7}$$

Using (7) and (5) we obtain for some $\beta \leq -1$

$$\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) > \sum_{i=1}^{n}\alpha_i\left(1 - \frac{\beta(2\beta-1)+1}{2\beta(\beta-1)\sum\limits_{i=1}^{n}\alpha_i}\right) + 1 - \sum_{i=1}^{n}\alpha_i$$

$$= \sum_{i=1}^{n}\alpha_i - \frac{\beta(2\beta-1)+1}{2\beta(\beta-1)} + 1 - \sum_{i=1}^{n}\alpha_i = -\frac{\beta(2\beta-1)+1}{2\beta(\beta-1)} + 1 = -\frac{\beta+1}{2\beta(\beta-1)}.$$

Therefore $\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) > -\frac{\beta+1}{2\beta(\beta-1)}$. Next, using (7) and (6) we have

$$\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) > \sum_{i=1}^{n}\alpha_i\left(1 - \frac{\beta(2\beta-1)-1}{2\beta(\beta+1)\sum\limits_{i=1}^{n}\alpha_i}\right) + 1 - \sum_{i=1}^{n}\alpha_i$$

$$= \sum_{i=1}^{n}\alpha_i - \frac{\beta(2\beta-1)-1}{2\beta(\beta+1)} + 1 - \sum_{i=1}^{n}\alpha_i = -\frac{\beta(2\beta-1)-1}{2\beta(\beta+1)} + 1 = \frac{3\beta+1}{2\beta(\beta+1)}.$$

for some $\beta > 1$. Therefore $\operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) > \frac{3\beta+1}{2\beta(\beta+1)}$. By Theorem 2 we get $\frac{F_n(z)}{zF_n'(z)} \prec \frac{\beta(1-z)}{\beta-z}$ and $F_n(z) \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$. This implies that $\int\limits_0^z \frac{F_n(t)}{t}\mathrm{d}t \in \mathcal{K}\left(\frac{\beta+1}{2\beta}\right)$. □

THEOREM 6. *Let* $\alpha_i > 0$ *be real numbers for* $i \in \{1, 2, ..., n\}$. *If* $f_i \in \mathcal{A}$ *for* $i \in \{1, 2, ..., n\}$ *satisfies*

$$\operatorname{Re} \frac{z f_i''(z)}{f_i'(z)} < \frac{3-\beta}{2(\beta-1)\sum\limits_{i=1}^{n} \alpha_i} \tag{8}$$

for some $2 \le \beta < 3$, *or*

$$\operatorname{Re} \frac{z f_i''(z)}{f_i'(z)} < \frac{3}{2} \frac{\beta-1}{(\beta+1)\sum\limits_{i=1}^{n} \alpha_i} \tag{9}$$

for some $1 < \beta \le 2$, *then* $\frac{zF'_{\alpha_1,...,\alpha_n}(z)}{F_{\alpha_1,...,\alpha_n}(z)} \prec \frac{\beta(1-z)}{\beta-z}$ *and* $\left|\frac{zF'_{\alpha_1,...,\alpha_n}(z)}{F_{\alpha_1,...,\alpha_n}(z)} - \frac{\beta}{\beta+1}\right| < \frac{\beta}{\beta+1}$. *This implies that* $F'_{\alpha_1,...,\alpha_n}(z) \in \mathcal{S}^*$ *and* $\int_0^z \frac{F'_{\alpha_1,...,\alpha_n}(t)}{t} \in \mathcal{K}$.

Proof. Following the same steps as in the proof of Theorem 3, we have $\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} = \sum\limits_{i=1}^{n} \alpha_i \frac{z f_i''(z)}{f_i'(z)}$. This is equivalent to $1 + \frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} = \sum\limits_{i=1}^{n} \alpha_i \frac{z f_i''(z)}{f_i'(z)} + 1$. Taking real parts, we get

$$\operatorname{Re}\left(1 + \frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right) = \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left(\frac{z f_i''(z)}{f_i'(z)}\right) + 1. \tag{10}$$

Using (10) and (8) we obtain

$$\operatorname{Re}\left(1 + \frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right) = \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left(\frac{z f_i''(z)}{f_i'(z)}\right) + 1$$

$$< \sum_{i=1}^{n} \alpha_i \left(\frac{3-\beta}{2(\beta-1)\sum\limits_{i=1}^{n} \alpha_i}\right) + 1 = \frac{3-\beta}{2(\beta-1)} + 1 = \frac{\beta+1}{2(\beta-1)}$$

for some $2 \le \beta < 3$. Next, using (10) and (9) we obtain

$$\operatorname{Re}\left(1 + \frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right) = \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left(\frac{z f_i''(z)}{f_i'(z)}\right) + 1$$

$$< \sum_{i=1}^{n} \alpha_i \left(\frac{3}{2} \frac{\beta-1}{(\beta+1)\sum\limits_{i=1}^{n} \alpha_i}\right) + 1 = \frac{3\beta-3}{2(\beta+1)} + 1 = \frac{5\beta-1}{2(\beta+1)}$$

for some $1 < \beta \le 2$. By Theorem 1 we get $\frac{zF'_{\alpha_1,...,\alpha_n}(z)}{F_{\alpha_1,...,\alpha_n}(z)} \prec \frac{\beta(1-z)}{\beta-z}$ and $\left|\frac{zF'_{\alpha_1,...,\alpha_n}(z)}{F_{\alpha_1,...,\alpha_n}(z)} - \frac{\beta}{\beta+1}\right| < \frac{\beta}{\beta+1}$. Then $F'_{\alpha_1,...,\alpha_n}(z) \in \mathcal{S}^*$ and $\int_0^z \frac{F'_{\alpha_1,...,\alpha_n}(t)}{t} \in \mathcal{K}$. $\square$

Taking $\beta = 2$ in Theorem 6 we have following corollary.

COROLLARY 7. *Let* $\alpha_i > 0$ *be real numbers for* $i \in \{1, 2, ..., n\}$. *If* $f_i \in \mathcal{A}$ *for* $i \in \{1, 2, ..., n\}$ *satisfies* $\mathrm{Re}\,\frac{zf_i''(z)}{f_i'(z)} < \frac{1}{2\sum_{i=1}^{n}\alpha_i}$, *then* $\frac{zF'_{\alpha_1,...,\alpha_n}(z)}{F_{\alpha_1,...,\alpha_n}(z)} \prec \frac{2(1-z)}{2-z}$ *and* $\left|\frac{zF'_{\alpha_1,...,\alpha_n}(z)}{F_{\alpha_1,...,\alpha_n}(z)} - \frac{2}{3}\right| < \frac{2}{3}$.

THEOREM 8. *Let* $\alpha_i > 0$ *be real numbers for* $i \in \{1, 2, ..., n\}$. *If* $f_i \in \mathcal{A}$ *for* $i \in \{1, 2, ..., n\}$ *satisfies* $\mathrm{Re}\,\frac{zf_i''(z)}{f_i'(z)} > -\frac{\beta(2\beta-1)+1}{2\beta(\beta-1)\sum_{i=1}^{n}\alpha_i}$ *for some* $\beta \leq -1$, *or* $\mathrm{Re}\,\frac{zf_i''(z)}{f_i'(z)} > -\frac{\beta(2\beta-1)-1}{2\beta(\beta+1)\sum_{i=1}^{n}\alpha_i}$ *for some* $\beta > 1$, *then* $\frac{F_{\alpha_1,...,\alpha_n}(z)}{zF'_{\alpha_1,...,\alpha_n}(z)} \prec \frac{\beta(1-z)}{\beta-z}$ *and* $F_{\alpha_1,...,\alpha_n}(z) \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$. *This implies that* $\int_0^z \frac{F_{\alpha_1,...,\alpha_n}(t)}{t}\mathrm{d}t \in \mathcal{K}\left(\frac{\beta+1}{2\beta}\right)$.

Proof. Similar to the proofs of the above theorems. □

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Received October 28, 2009
Accepted January 14, 2011

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