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PARTIAL SUMS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. Let $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ be the sequence of partial sums of the analytic function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. We determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f'(z)\right\}$ under certain conditions.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. For functions $\Phi, \Psi \in \mathcal{A}$ given by $\Phi(z) = z + \sum_{k=2}^{\infty} \phi_k z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k$, we define the Hadamard product (or convolution) of Φ and Ψ by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \phi_k \psi_k z^k, \quad z \in U. \tag{2}$$

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_l, A_l$ and $\beta_1, B_1, \ldots, \beta_m, B_m$, where $l, m \in \mathbb{N} = \{1, 2, 3, \ldots\}$, such that

$$1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k \geq 0 \quad z \in U, \tag{3}$$

the Wright generalized hypergeometric function ${}_l\Psi_m[(\alpha_k, A_k)_{1,l}(\beta_k, B_k)_{1,m}; z]$ [16], which denotes ${}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z]$, is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \sum_{k=0}^{\infty} \left\{\prod_{t=0}^{l} \Gamma(\alpha_t + kA_t)\right\} \left\{\prod_{t=0}^{m} \Gamma(\beta_t + kB_t)\right\}^{-1} \frac{z^k}{k!},$$

for $z \in U$.

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If $A_t = 1$ $(t = 1, 2, \dots, l)$ and $B_t = 1$ $(t = 1, 2, \dots, m)$ we have the relationship:

$$\begin{aligned}\Omega_l \Psi_m[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z] &\equiv {}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z)\\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_l)_k}{(\beta_1)_k \dots (\beta_m)_k} \frac{z^k}{k!}\end{aligned} \tag{4}$$

$(l \le m + 1;\ l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U)$, which is the generalized hypergeometric function(see for details [16]) where $\mathbb{N}$ denotes the set of all positive integers and $(\lambda)_k$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t)\right). \tag{5}$$

By using the generalized hypergeometric function Dziok and Srivastava [3] introduced a linear operator. In [4] Dziok and Raina extended the linear operator by using Wright generalized hypergeometric function. First we define a function

$${}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] = \Omega z {}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z].$$

Let $\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : \mathcal{A} \to \mathcal{A}$ be a linear operator defined by

$$\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}](f)(z) \quad := \quad z\ {}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z).$$

We observe that, for $f(z)$ of the form (1),we have

$$\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) = z + \sum_{k=2}^{\infty} \Omega \sigma_k(\alpha_1)\ a_k z^k, \tag{6}$$

where Ω is given by (5) and $\sigma_k(\alpha_1)$ is defined by

$$\sigma_k(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(k-1)) \dots \Gamma(\alpha_l + A_l(k-1))}{(k-1)!\Gamma(\beta_1 + B_1(k-1)) \dots \Gamma(\beta_m + B_m(k-1))}. \tag{7}$$

For convenience, we write

$$\mathcal{W}_m^l[\alpha_1]f(z) = \mathcal{W}[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m)]f(z) \tag{8}$$

as introduced by Dziok and Raina [4]. In view of the relationship (4) the linear operator (6) includes the Dziok-Srivastava operator (see [3]), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi [1], Carlson and Shaffer [2], Libera [7], Livingston [8], Ruscheweyh [12] and Srivastava-Owa [15].

For $0 \le \lambda < 1$, $0 \le \gamma < 1$ and $\eta \ge 0$, we let $\mathcal{W}_m^l(\lambda,\gamma,\eta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\begin{aligned}&\operatorname{Re}\left\{\frac{z(\mathcal{W}_m^l[\alpha_1]f(z))'}{(1-\lambda)\mathcal{W}_m^l[\alpha_1]f(z)+\lambda z(\mathcal{W}_m^l[\alpha_1]f(z))'}-\gamma\right\}\\ &> \eta\left|\frac{z(\mathcal{W}_m^l[\alpha_1]f(z))'}{(1-\lambda)\mathcal{W}_m^l[\alpha_1]f(z)+\lambda z(\mathcal{W}_m^l[\alpha_1]f(z))'}-1\right|,\ z\in U,\end{aligned}\tag{9}$$

where $\mathcal{W}_m^l[\alpha_1]f(z)$ is given by (6). By suitably specializing the values of A_t, B_t, l, m, $\alpha_1,\alpha_2,\dots,\alpha_l$, $\beta_1,\beta_2,\dots,\beta_m$, λ, γ and η, the class $\mathcal{W}_m^l(\lambda,\gamma,\eta)$ leads to various new subclasses. As illustrations, we present some examples for the case when $A_t = 1$ $(t = 1,2,\dots,l)$ and $B_t = 1$ $(t = 1,2,\dots,m)$.

Example 1. If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, then $\mathcal{W}_1^2(\lambda,\gamma,\eta) \equiv \mathbb{S}(\lambda,\gamma,\eta)$ is the class of functions $f \in \mathcal{A}$ with the property that $\operatorname{Re}\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\gamma\right\} > \eta\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right|$ $(z\in U)$.

Example 2. If $l = 2$ and $m = 1$ with $\alpha_1 = \delta+1$ $(\delta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, then $\mathcal{W}_1^2(\lambda,\gamma,\eta) \equiv R_\delta(\lambda,\gamma,\eta)$ is the class of functions $f \in \mathcal{A}$ with the property that $\operatorname{Re}\left\{\frac{z(D^\delta f(z))'}{(1-\lambda)D^\delta f(z)+\lambda z(D^\delta f(z))'}-\gamma\right\} > \eta\left|\frac{z(D^\delta f(z))'}{(1-\lambda)D^\delta f(z)+\lambda z(D^\delta f(z))'}-1\right|$ $(z \in U)$, where D^δ is called Ruscheweyh derivative of order δ $(\delta > -1)$ defined by

$$D^\delta f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv H_1^2(\delta+1,1;1)f(z).$$

Example 3. If $l = 2$ and $m = 1$ with $\alpha_1 = \mu+1$ $(\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu+2$, then $\mathcal{W}_1^2(\lambda,\gamma,\eta) \equiv B_\mu(\lambda,\gamma,\eta)$ is the class of functions $f \in \mathcal{A}$ with the property that $\operatorname{Re}\left(\frac{z(J_\mu f(z))'}{(1-\lambda)J_\mu f(z)+\lambda z(J_\mu f(z))'}-\gamma\right) > \eta\left|\frac{z(J_\mu f(z))'}{(1-\lambda)J_\mu f(z)+\lambda z(J_\mu f(z))'}-1\right|$ $(z \in U)$, where J_μ is a Bernardi operator [1] defined by

$$J_\mu f(z) := \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)\mathrm{d}t \equiv H_1^2(\mu+1,1;\mu+2)f(z).$$

Note that the operator J_1 was studied earlier by Libera [7] and Livingston [8].

Example 4. If $l = 2$ and $m = 1$ with $\alpha_1 = a$ $(a > 0)$, $\alpha_2 = 1$, $\beta_1 = c$ $(c > 0)$, then $\mathcal{W}_1^2(\lambda,\gamma,\eta) \equiv L_c^a(\lambda,\gamma,\eta)$ is the class of functions $f \in \mathcal{A}$ with the property $\operatorname{Re}\left(\frac{z(L(a,c)f(z))'}{(1-\lambda)L(a,c)f(z)+\lambda z(L(a,c)f(z))'}-\gamma\right) > \eta\left|\frac{z(L(a,c)f(z))'}{(1-\lambda)L(a,c)f(z)+\lambda z(L(a,c)f(z))'}-1\right|$ $(z \in U)$, where $L(a,c)$ is the Carlson-Shaffer linear operator [2] defined by

$$L(a,c)f(z) := \left(\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1}\right) * f(z) \equiv H_1^2(a,1;c)f(z).$$

The class $\mathcal{W}_m^l(\lambda,\gamma,\eta)$ was introduced and studied by Murugusundaramoorthy and Magesh [9], and they obtained the following sufficient condition for a function $f(z)$ of the form (1) to be in this class.

Lemma 5 ([9]). *A function $f(z)$ of the form (1) is in $\mathcal{W}_m^l(\lambda,\gamma,\eta)$ if*

$$\sum_{k=2}^{\infty}\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)\ |a_k|\leq 1-\gamma, \tag{10}$$

where $\rho_k(\lambda,\gamma,\eta):=k(1+\eta)-(\gamma+\eta)(1+k\lambda-\lambda)$, $0\leq\lambda<1$, $0\leq\gamma<1$, $\eta\geq 0$ and Ω, $\sigma_k(\alpha_1)$ are given by (5) *and* (7)*.*

If $A_t=1$ $(t=1,2,\ldots,l)$, $B_t=1$ $(t=1,2,\ldots,m)$, and in the view of Examples 1 to 4, we state the following sufficient conditions (see [9]). A function $f(z)$ of the form (1) is in $\mathbb{S}(\lambda,\gamma,\eta)$ if

$$\sum_{k=2}^{\infty}\rho_k(\lambda,\gamma,\eta)\ |a_k|\leq 1-\gamma, \tag{11}$$

where $0\leq\lambda<1$, $0\leq\gamma<1$ and $\eta\geq 0$. A function $f(z)$ of the form (1) is in $R_\delta(\lambda,\gamma,\eta)$ if

$$\sum_{k=2}^{\infty}\rho_k(\lambda,\gamma,\eta)\frac{(\delta+1)\ldots(\delta+k-1)}{(k-1)!}|a_k|\leq 1-\gamma, \tag{12}$$

where $0\leq\lambda<1$, $0\leq\gamma<1$, $\eta\geq 0$ and $\delta>-1$. A function $f(z)$ of the form (1) is in $B_\mu(\lambda,\gamma,\eta)$ if

$$\sum_{k=2}^{\infty}\rho_k(\lambda,\gamma,\eta)\left(\frac{\mu+1}{\mu+k}\right)|a_k|\leq 1-\gamma, \tag{13}$$

where $0\leq\lambda<1$, $0\leq\gamma<1$, $\eta\geq 0$ and $\mu>-1$. A function $f(z)$ of the form (1) is in $L_c^a(\lambda,\gamma,\eta)$ if

$$\sum_{k=2}^{\infty}\rho_k(\lambda,\gamma,\eta)\frac{(a)_{k-1}}{(c)_{k-1}}|a_k|\leq 1-\gamma, \tag{14}$$

where $0\leq\lambda<1$, $0\leq\gamma<1$, $\eta\geq 0$ and $a>0$, $c>0$.

Recently, Silverman [11] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. In the present paper, and by following the earlier work by Silverman [11](see [5], [6], [10], [13], [14]) on partial sums of analytic functions, we study the ratio of a function of the form (1) to its sequence of partial sums of the form $f_n(z)=z+\sum\limits_{k=2}^{n}a_kz^k$ when the coefficients of $f(z)$ satisfy the condition (10). Also, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f_n'(z)\}$ and $\operatorname{Re}\left\{f_n'(z)/f'(z)\right\}$. It is seen that this study not only gives as a particular case, the results of Silverman [11], but also gives rise to several new results.

2. MAIN RESULTS

THEOREM 6. *If $f(z)$ of the form* (1) *satisfies condition* (10), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)-1+\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)} \quad (z \in U), \tag{15}$$

where

$$\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1) \geq \begin{cases} 1-\gamma\ , & k=2,3,\dots,n \\ \rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)\ , & k=n+1,n+2,\dots. \end{cases} \tag{16}$$

The result (15) *is sharp with the function given by*

$$f(z) = z + \frac{1-\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)} z^{n+1}. \tag{17}$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma} \\ &\quad \cdot \left[\frac{f(z)}{f_n(z)} - \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)-1+\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}\right] \\ &= \frac{1+\sum_{k=2}^{n} a_k z^{k-1} + \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=2}^{n} a_k z^{k-1}}. \end{aligned} \tag{18}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (18) we can write

$$w(z) = \frac{\left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum_{k=n+1}^{\infty} a_k z^{k-1}}{2+2\sum_{k=2}^{n} a_k z^{k-1} + \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum_{k=n+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|w(z)| \leq \frac{\left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum_{k=n+1}^{\infty} |a_k|}{2-2\sum_{k=2}^{n} |a_k| - \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum_{k=n+1}^{\infty} |a_k|}.$$

Now $|w(z)| \leq 1$ if

$$2\left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum_{k=n+1}^{\infty} |a_k| \leq 2-2\sum_{k=2}^{n} |a_k|$$

or, equivalently,

$$\sum_{k=2}^{n} |a_k| + \sum_{k=n+1}^{\infty} \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\,|a_k| \leq 1.$$

From condition (10), it is sufficient to show that

$$\sum_{k=2}^{n}|a_k| + \sum_{k=n+1}^{\infty}\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}|a_k| \leq \sum_{k=2}^{\infty}\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)}{1-\gamma}|a_k|\ ,$$

which is equivalent to

$$\begin{aligned}(19)\qquad &\sum_{k=2}^{n}\left(\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)-1+\gamma}{1-\gamma}\right)|a_k| \\ &+\sum_{k=n+1}^{\infty}\left(\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)-\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)|a_k| \geq 0.\end{aligned}$$

From condition (10), we obtain (19). To see that the function given by (17) gives the sharp result, we observe that for $z = r\mathrm{e}^{\mathrm{i}\pi/n}$ we have

$$\begin{aligned}\frac{f(z)}{f_n(z)} &= 1+\frac{1-\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}z^n \to 1-\frac{1-\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)} \\ &= \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)-1+\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}\end{aligned}$$

when $r \to 1^-$. □

Taking $A_t = 1$ $(t = 1, 2, \ldots, l)$ and $B_t = 1$ $(t = 1, 2, \ldots, m)$, $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\lambda = \eta = 0$ in Theorem 6, we obtain the following result given by Silverman in [11, Theorem 1, p. 222].

Corollary 7. *If $f(z)$ of the form* (1) *satisfies the condition*

$$(20)\qquad \sum_{k=2}^{\infty}(k-\gamma)|a_k| \leq 1-\gamma,$$

then $\mathrm{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n}{n+1-\gamma}$ *$(z \in U)$. The result is sharp with the function*

$$(21)\qquad f(z) = z+\frac{1-\gamma}{n+1-\gamma}z^{n+1}.$$

Taking $A_t = 1$ $(t = 1, 2, \ldots, l)$ and $B_t = 1$ $(t = 1, 2, \ldots, m)$, $l = 2$ and $m = 1$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\lambda = \eta = 0$ in Theorem 6, we obtain the following result given by Silverman [11, Theorem 2, p. 224].

Corollary 8. *If $f(z)$ of the form* (1) *satisfies the condition*

$$(22)\qquad \sum_{k=2}^{\infty}k(k-\gamma)|a_k| \leq 1-\gamma,$$

then $\mathrm{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n(n+2-\gamma)}{(n+1)(n+1-\gamma)}$ *$(z \in U)$. The result is sharp with the function*

$$(23)\qquad f(z) = z+\frac{1-\gamma}{(n+1)(n+1-\gamma)}z^{n+1}.$$

We next determine bounds for $f_n(z)/f(z)$.

Theorem 9. *If $f(z)$ of the form* (1) *satisfies condition (10), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)+1-\gamma} \quad (z \in U), \tag{24}$$

where

$$\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1) \geq \begin{cases} 1-\gamma\ , & k=2,3,\dots,n,n+1 \\ \rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)\ , & k=n+1,n+2,\dots. \end{cases} \tag{25}$$

The result (24) *is sharp with the function given by* (17).

Proof. Define the function $w(z)$ by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)+1-\gamma}{1-\gamma} \\ &\quad\cdot\left[\frac{f_n(z)}{f(z)} - \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)+1-\gamma}\right] \\ &= \frac{1+\sum\limits_{k=2}^{n} a_k z^{k-1} - \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}\right)\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{\infty} a_k z^{k-1}}.\end{aligned}$$

Then $|w(z)|<1$ if

$$|w(z)| \leq \frac{\left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)+1-\gamma}{1-\gamma}\right)\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{n}|a_k| - \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)-1+\gamma}{1-\gamma}\right)\sum\limits_{k=n+1}^{\infty}|a_k|} \leq 1.$$

This is equivalent to $\sum\limits_{k=2}^{n}|a_k| + \sum\limits_{k=n+1}^{\infty}\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{1-\gamma}|a_k| \leq 1$. Making use of (10) and (24), we get (19). Finally, equality in (24) holds for the extremal function $f(z)$ given by (17). □

Taking $A_t = 1$ $(t = 1,2,\dots,l)$ and $B_t = 1$ $(t = 1,2,\dots,m)$, $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\lambda = \eta = 0$ in Theorem 9, we obtain the following result given by Silverman [11, Theorem 3 (a), p. 225].

Corollary 10. *If $f(z)$ of the form* (1) *satisfies condition* (20)*, then one has* $\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{n+1-\gamma}{n+2-2\gamma}$ *$(z \in U)$. The result is sharp with the function given by* (21)*.*

Taking $A_t = 1$ $(t = 1,2,\dots,l)$ and $B_t = 1$ $(t = 1,2,\dots,m)$, $l = 2$ and $m = 1$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\lambda = \eta = 0$ in Theorem 9, we obtain the following result given by Silverman [11, Theorem 3 (b), p. 225].

Corollary 11. *If $f(z)$ of the form* (1) *satisfies condition* (22), *then one has* $\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)(n+1-\gamma)}{(n+1)[(n+1)-\gamma]+1-\gamma}$ *$(z \in U)$. The results are sharp with the function given by* (23).

We next turn to ratios involving derivatives.

Theorem 12. *If $f(z)$ of the form* (1) *satisfies condition* (10), *then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)-(n+1)(1-\gamma)}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)} \quad (z \in U) \tag{26}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)+(n+1)(1-\gamma)} \quad (z \in U), \tag{27}$$

where

$$\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1) \geq \begin{cases} k(1-\gamma)\ , & k=2,3,\ldots,n,n+1 \\ k\left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)}\right)\ , & k=n+1,n+2,\ldots \end{cases} \tag{28}$$

The results are sharp with the function given by (17).

Proof. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)} \cdot \left[\frac{f'(z)}{f_n'(z)} - \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)-(n+1)(1-\gamma)}{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}\right)\right],$$

where

$$w(z) = \frac{\left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\right)\sum\limits_{k=n+1}^{\infty} ka_k z^{k-1}}{2+2\sum\limits_{k=2}^{n} ka_k z^{k-1} + \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\right)\sum\limits_{k=n+1}^{\infty} ka_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if $\sum\limits_{k=2}^{n} k|a_k| + \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\sum\limits_{k=n+1}^{\infty} k|a_k| \leq 1$. From condition (10), it is sufficient to show that

$$\sum_{k=2}^{n} k|a_k| + \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\sum_{k=n+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)}{1-\gamma}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)-(1-\gamma)k}{1-\gamma}\right)|a_k| + \sum_{k=n+1}^{\infty}\frac{(n+1)\,\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)-k\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}|a_k| \geq 0.$$

Since this condition holds from (28), we obtain (26). To prove the result (27), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{(n+1)(1-\gamma)+\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(1-\gamma)(n+1)} \cdot \left[\frac{f_n'(z)}{f'(z)} - \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)+\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}\right].$$

Then

$$w(z) = \frac{-\left(1+\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\right)\sum\limits_{k=n+1}^{\infty} ka_k z^{k-1}}{2+2\sum\limits_{k=2}^{n} ka_k z^{k-1} + \left(1-\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\right)\sum\limits_{k=n+1}^{\infty} ka_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{n} k\,|a_k| + \left(\frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}\right)\sum_{k=n+1}^{\infty} k\,|a_k| \leq 1. \tag{29}$$

It suffices to show that the left hand side of (29) is bounded above by the condition $\sum\limits_{k=2}^{\infty}[\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)/(1-\gamma)]\,|a_k|$, which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)}{1-\gamma} - k\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{\rho_k(\lambda,\gamma,\eta)\Omega\sigma_k(\alpha_1)}{1-\gamma} - \frac{\rho_{n+1}(\lambda,\gamma,\eta)\Omega\sigma_{n+1}(\alpha_1)}{(n+1)(1-\gamma)}k\right)|a_k| \geq 0.$$

Since this condition holds from (28), we obtain (27). □

Taking $A_t = 1$ $(t = 1,2,\ldots,l)$ and $B_t = 1$ $(t = 1,2,\ldots,m)$, $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\lambda = \eta = 0$ in Theorem 12, we obtain the following result given by Silverman [11, Theorem 4, p. 226].

Corollary 13. *If $f(z)$ of the form* (1) *satisfies condition* (20)*, then one has* $\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{n\gamma}{n+1-\gamma}$ *$(z \in U)$ and* $\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1-\gamma}{(n+1)(2-\gamma)-\gamma}$ *$(z \in U)$. The results are sharp with the function given by* (21)*.*

Taking $A_t = 1$ $(t = 1,2,\ldots,l)$ and $B_t = 1$ $(t = 1,2,\ldots,m)$, $l = 2$ and $m = 1$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\lambda = \eta = 0$ in Theorem 12, we obtain the following result given by Silverman [11, Theorem 5, p. 227].

Corollary 14. *If $f(z)$ of the form* (1) *satisfies condition* (22)*, then one has* $\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{n}{n+1-\gamma}$ *$(z \in U)$ and* $\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1-\gamma}{n+2-2\gamma}$ *$(z \in U)$. The results are sharp with the function given by* (23)*.*

REMARK 15. As special cases of the above theorems, we can determine new sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f'(z)\right\}$ if $f(z)$ satisfies the conditions (11)-(14) by taking $A_t = 1$ $(t = 1, 2, \ldots, l)$, $B_t = 1$ $(t = 1, 2, \ldots, m)$ and by suitably specializing the values of l, m, α_1, α_2 and β_1.

As special cases of Theorem 6, we obtain the following corollaries.

COROLLARY 16. *If $f(z)$ of the form* (1) *satisfies condition* (11), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta) - 1 + \gamma}{\rho_{n+1}(\lambda,\gamma,\eta)} \quad (z \in U),$$

where

$$\rho_k(\lambda,\gamma,\eta) \geq \begin{cases} 1-\gamma\ , & k = 2,3,\ldots,n \\ \rho_{n+1}(\lambda,\gamma,\eta)\ , & k = n+1, n+2, \ldots . \end{cases}$$

and $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $\eta \geq 0$. The result is sharp with the function

$$f(z) = z + \frac{1-\gamma}{\rho_{n+1}(\lambda,\gamma,\eta)} z^{n+1}.$$

COROLLARY 17. *If $f(z)$ of the form* (1) *satisfies condition* (12), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)(\delta+1)\ldots(\delta+n) - (1-\gamma)(n)!}{\rho_{n+1}(\lambda,\gamma,\eta)(\delta+1)\ldots(\delta+n)} \quad (z \in U),$$

where

$$\begin{aligned}&\rho_k(\lambda,\gamma,\eta)\frac{(\delta+1)\ldots(\delta+k-1)}{(k-1)!} \\ &\geq \begin{cases} 1-\gamma\ , & k = 2,3,\ldots,n \\ \rho_{n+1}(\lambda,\gamma,\eta)\frac{(\delta+1)\ldots(\delta+n)}{(n)!}\ , & k = n+1, n+2, \ldots . \end{cases}\end{aligned}$$

and $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $\eta \geq 0$, $\delta > -1$. The result is sharp with the function

$$f(z) = z + \frac{(1-\gamma)(n)!}{\rho_{n+1}(\lambda,\gamma,\eta)(\delta+1)\ldots(\delta+n)} z^{n+1}.$$

COROLLARY 18. *If $f(z)$ of the form* (1) *satisfies condition* (13), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)(\mu+1) - (1-\gamma)(\mu+n+1)}{\rho_{n+1}(\lambda,\gamma,\eta)(\mu+1)} \quad (z \in U),$$

where

$$\rho_k(\lambda,\gamma,\eta)\left(\frac{\mu+1}{\mu+k}\right) \geq \begin{cases} 1-\gamma\ , & k = 2,3,\ldots,n \\ \rho_{n+1}(\lambda,\gamma,\eta)\left(\frac{\mu+1}{\mu+n+1}\right)\ , & k = n+1, n+2, \ldots . \end{cases}$$

and $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $\eta \geq 0$, $\mu > -1$. The result is sharp with the function

$$f(z) = z + \frac{(1-\gamma)(\mu+n+1)}{\rho_{n+1}(\lambda,\gamma,\eta)(\mu+1)} z^{n+1}.$$

COROLLARY 19. *If $f(z)$ of the form* (1) *satisfies condition* (14), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\rho_{n+1}(\lambda,\gamma,\eta)(a)_n-(1-\gamma)(c)_n}{\rho_{n+1}(\lambda,\gamma,\eta)(a)_n} \quad (z\in U),$$

where

$$\rho_k(\lambda,\gamma,\eta)\frac{(a)_{k-1}}{(c)_{k-1}} \geq \begin{cases} 1-\gamma\ , & k=2,3,\ldots,n \\ \rho_{n+1}(\lambda,\gamma,\eta)\frac{(a)_n}{(c)n}\ , & k=n+1,n+2,\ldots. \end{cases}$$

and $0\leq\lambda<1$, $0\leq\gamma<1$, $\eta\geq 0$, $a>0$, $c>0$. The result is sharp with the function

$$f(z)=z+\frac{(1-\gamma)(c)_n}{\rho_{n+1}(\lambda,\gamma,\eta)(a)_n}z^{n+1}.$$

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