MATHEMATICA, Tome 53 (76), N° 2, 2011, pp. 121–130

Φ-LIKE FUNCTIONS IN TWO-DIMENSIONAL FREE BOUNDARY PROBLEMS

PAULA CURT, DENISA FERICEAN, and TEODOR GROŞAN

Abstract. In this paper we apply certain results in the theory of univalent functions to investigate the time evolution of the free boundary of a viscous fluid for a planar flow problem in the Hele-Shaw cell model under injection. More precisely, we prove that the property of strongly Φ-likeness of order $\alpha \in (0, 1]$ (a geometric property which includes strongly starlikeness of order α and strongly spirallikeness of order α) remains invariant in time for two basic cases: the inner problem and the outer problem, under the assumption of zero surface tension. Special cases that are obtained by using numerical computations are also presented.

MSC 2010. 30C45, 76D27.

Key words. Conformal map, free boundary problem, Hele-Shaw flow, Φ-like function, spirallike function, starlike function, strongly Φ-like function.

1. INTRODUCTION

The time evolution of the free boundary of a viscous fluid for planar flows in Hele-Show cells under injection has been intensively investigated in the literature (see e.g. [5]). Various authors (see [6], [9], [14], [16], [17], [18]) proved that a number of geometric properties such as starlikeness, directional convexity, strongly starlikeness of order α are preserved during the time evolution of a moving boundary. The theory of univalent functions provided a powerful tool in the study of these results. In this paper, we continue this study and prove the invariance in time of the property of strongly Φ-likeness of order $\alpha \in (0, 1]$ under the assumption of zero surface tension.

In this section we review certain classical notions that we shall use in the forthcoming sections. Let us consider the flow of a viscous fluid in a planar Hele-Shaw cell under injection through a source of constant strength $Q < 0$ that is located at the origin. Assume that the initial domain $\Omega(0)$, occupied by the fluid at time $t = 0$, contains the origin, is simply connected and is bounded by an analytic and smooth curve $\Gamma(0) = \partial\Omega(0)$. Also assume that $\Omega(t)$ is a simply connected domain occupied by the fluid at the moment t and $0 \in \Omega(t)$.

Paula Curt is supported by the Romanian Ministry of Education and Research, UEFISCSU-CNCSIS Grant PN II-ID-524/2007. Teodor Groşan is supported by the Romanian Ministry of Education and Research, UEFISCSU-CNCSIS Grant PN-II-ID-525/2007. Denisa Fericean is supported by Contract nr: POSDRU/88/1.5/S/60185-"Innovative Doctoral Studies in a Knowledged Based Society". The authors are indebted to Mirela Kohr for valuable discussions during the preparation of this work.

In view of the Riemann mapping theorem, there is a univalent function $f(\cdot,t)$ that maps the unit disk $U=\{\zeta : |\zeta|<1\}$ onto $\Omega(t)$ and is normalized by the conditions $f(0,t)=0$ and $f'(0,t)>0$. Let $\Gamma(t)=\partial\Omega(t)$ be the boundary of $\Omega(t)$. Obviously, the function $f(\cdot,0)=f_0(\cdot)$ produces a parametrization of the boundary $\Gamma_0=\{f_0(\mathrm{e}^{\mathrm{i}\theta}),\ \theta\in[0,2\pi)\}$, while the moving boundary is parameterized by $\Gamma(t)=\{f(\mathrm{e}^{\mathrm{i}\theta},t),\theta\in[0,2\pi)\}$.

The following equation satisfied by the moving boundary $\Gamma(t)$ was obtained by L.A. Galin [3] and P. Polubarinova [11], [12]:

$$\mathrm{Re}\,[\dot f(\zeta,t)\overline{\zeta f'(\zeta,t)}]=-\frac{Q}{2\pi},\quad \zeta=\mathrm{e}^{\mathrm{i}\theta} \tag{1.1}$$

(in the previous equality we used the following notations $f'=\frac{\partial f}{\partial\zeta}$, $\dot f=\frac{\partial f}{\partial t}$).

A classical solution in the interval $[0,T)$ is a function $f(\zeta,t)$, $t\in[0,T)$, that is univalent on $\overline{U}$ and C^1 with respect to t in $[0,T)$. It is known that, starting with an analytic and smooth boundary $\Gamma(0)$, the classical solution exists and is unique locally in time (see [19]; see also [5]). Note that T is called the *blow-up time.*

The case of unbounded domain with bounded complement can be viewed as the dynamics of a contracting bubble in a Hele-Shaw cell since the fluid occupies a neighborhood of infinity and injection (of constant strength $Q<0$) is supposed to take place at infinity. Again, let $\Omega(t)$ be the domain occupied by the fluid at the moment t and let $\Gamma(t)=\partial\Omega(t)$. Taking into account the Riemann mapping theorem, there is a univalent function $F(\cdot,t)$ on the exterior of the unit disk $U^-=\{\zeta\mid|\zeta|>1\}$ such that $F(U^-,t)=\Omega(t)$ and $F(\zeta,t)=a\zeta+a_0+\dfrac{a_1}{\zeta}+\dots,\ a>0$. In this case, the equation satisfied by the free boundary in the case of zero tension surface model is [16], [18]:

$$\mathrm{Re}\,[\dot F(\zeta,t)\overline{\zeta F'(\zeta,t)}]=\frac{Q}{2\pi},\quad \zeta=\mathrm{e}^{\mathrm{i}\theta}. \tag{1.2}$$

2. THE INNER PROBLEM (BOUNDED DOMAINS)

In this section we obtain the invariance in time of strongly Φ-likeness property for the inner problem. Starting with an initial bounded domain $\Omega(0)$ which is strongly Φ-like of order α, we prove that at each moment $t\in[0,T)$ the domain $\Omega(t)$ is strongly Φ-like of order α (for zero surface tension model).

Definition 2.1. Let f be a holomorphic function on the unit disc U such that $f(0)=0$ and $f'(0)\neq0$. Let Φ be a holomorphic function on $f(U)$ such that $\Phi(0)=0$ and $|\arg\Phi'(0)|<\dfrac{\alpha\pi}{2}$, where $\alpha\in(0,1]$. We say that f is *strongly Φ-like of order α on U* if

$$\left|\arg\left(\frac{zf'(z)}{\Phi(f(z))}\right)\right|<\frac{\alpha\pi}{2},\quad z\in U. \tag{2.1}$$

In this case, $f(U)$ is called a *strongly Φ-like domain of order α.*

Remark 2.2. (a) In the case that $\alpha = 1$ in Definition 2.1, we obtain the usual notion of Φ-likeness due to Brickman [1]. This notion is a natural generalization of starlikeness and spirallikness. Applications of Φ-likeness in the study of univalent functions may be found in [4].

(b) If $\Phi(w) \equiv w$ (resp. $\Phi(w) \equiv w$ and $\alpha = 1$) in the above definition, then f is strongly starlike of order α (resp. starlike).

(c) If $\Phi(w) \equiv \lambda w$ and $|\arg \lambda| < \alpha\pi/2$, then f is strongly spirallike of type $-\arg \lambda$ and order α. Clearly, if $\alpha = 1$ and $\operatorname{Re} \lambda > 0$, then f is spirallike of type $-\arg \lambda$. Various properties of starlike and spirallike mappings can be found in [4], [10] and [13].

We recall that a holomorphic function f on U with $f(0) = 0$ and $f'(0) \neq 0$ is strongly spirallike of type $\beta \in (-\alpha\pi/2, \alpha\pi/2)$ and order $\alpha \in (0, 1]$ if (see e.g. [10] and [13])

$$\left|\arg\left(\mathrm{e}^{\mathrm{i}\beta} \frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in U.$$

The following result is a generalization of [16, Theorem 1] (see also [5, Theorem 4.3.2]) to the case of strongly Φ-like functions of order α. The mentioned theorem may be obtained by taking $\Phi(w) \equiv w$ in Theorem 2.3 below. On the other hand, if $\alpha = 1$ in Theorem 2.3, we obtain [2, Theorem 2.3].

Theorem 2.3. *Let $\alpha \in (0, 1]$, $Q < 0$ and let f_0 be a strongly Φ-like function of order α on U and univalent on $\overline{U}$. Let $f(\zeta, t)$ be the classical solution of the Polubarinova-Galin equation* (1.1) *with the initial condition $f(\zeta, 0) = f_0(\zeta)$. Also let $\Omega = \bigcup_{0 \le t < T} \Omega(t) = \bigcup_{0 \le t < T} f(U, t)$ where T is the blow-up time. If Φ is holomorphic on $\overline{\Omega}$ and satisfies the condition*

$$|\arg \Phi'(w)| < \frac{\alpha\pi}{2}, \ \forall \ w \in \overline{\Omega}, \tag{2.2}$$

then $f(\zeta, t)$ is strongly Φ-like of order α for $t \in [0, T)$.

Proof. Taking into account the fact that all the functions $f(\zeta, t)$ have analytic univalent extensions on $\overline{U}$ for each $t \in [0, T)$ and in consequence their derivatives $f'(\zeta, t)$ are continuous and do not vanish in $\overline{U}$, we can replace with " $\le$ " the inequality in Definition 2.1 of a strongly Φ-like function of order α. The equality can be attained for $|\zeta| = 1$.

We suppose by contrary that the conclusion of Theorem 2.3 is not true. Then there exist some $t_0 \ge 0$ and $\zeta_0 = \mathrm{e}^{\mathrm{i}\theta_0}$ such that

$$\arg \frac{\zeta_0 f'(\zeta_0, t_0)}{\Phi(f(\zeta_0, t_0))} = \frac{\alpha\pi}{2} \quad \left(\text{or } -\frac{\alpha\pi}{2}\right), \tag{2.3}$$

and for each $\varepsilon > 0$ there exist $t > t_0$ and $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ such that

$$\arg \frac{\mathrm{e}^{\mathrm{i}\theta} f'(\mathrm{e}^{\mathrm{i}\theta}, t)}{\Phi(f(\mathrm{e}^{\mathrm{i}\theta}, t))} \ge \frac{\alpha\pi}{2} \quad \left(\text{or } \le -\frac{\alpha\pi}{2}\right). \tag{2.4}$$

We consider the sign (+) in (2.3). Let t_0 be the first such point $t_0 \in [0, T)$. Without loss of generality we assume that

$$\operatorname{Im} \frac{\zeta_0 f'(\zeta_0, t_0)}{\Phi(f(\zeta_0, t_0))} > 0. \tag{2.5}$$

(A similar proof can be applied for the case $\operatorname{Im} \dfrac{\zeta_0 f'(\zeta_0, t_0)}{\Phi(f(\zeta_0, t_0))} < 0$.)

Since $(1, \theta_0)$ is a maximum point for the function

$$g(r, \theta) = \arg \frac{r e^{i\theta} f'(r e^{i\theta}, t_0)}{\Phi(f(r e^{i\theta}, t_0))}, \text{ for } r \in [0, 1], \ \theta \in [0, 2\pi],$$

we deduce that $\dfrac{\partial}{\partial \theta} g(1, \theta_0) = 0$ and $\dfrac{\partial}{\partial r} g(1, \theta_0) \geq 0$ (the stationarity condition at an endpoint of an interval). Hence, we obtain the following relations:

$$\operatorname{Re} \left[1 + \frac{\zeta_0 f''(\zeta_0, t_0)}{f'(\zeta_0, t_0)} - \frac{\Phi'(f(\zeta_0, t_0)) \zeta_0 f'(\zeta_0, t_0)}{\Phi(f(\zeta_0, t_0))} \right] = 0 \tag{2.6}$$

$$\operatorname{Im} \left[1 + \frac{\zeta_0 f''(\zeta_0, t_0)}{f'(\zeta_0, t_0)} - \frac{\Phi'(f(\zeta_0, t_0)) \zeta_0 f'(\zeta_0, t_0)}{\Phi(f(\zeta_0, t_0))} \right] \geq 0. \tag{2.7}$$

By straightforward calculations we obtain:

$$\begin{aligned} \frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{\Phi(f(\zeta, t))} &= \operatorname{Im} \frac{\partial}{\partial t} \log \frac{\zeta f'(\zeta, t)}{\Phi(f(\zeta, t))} \\ &= \operatorname{Im} \left(\frac{\frac{\partial}{\partial t} f'(\zeta, t)}{f(\zeta, t)} - \frac{\Phi'(f(\zeta, t)) \frac{\partial}{\partial t} f(\zeta, t)}{\Phi(f(\zeta, t))} \right). \end{aligned} \tag{2.8}$$

By differentiating the Polubarinova-Galin equation (1.1) with respect to θ, we have:

$$\begin{aligned} &|f'(\zeta, t)|^2 \operatorname{Im} \left(\frac{\frac{\partial}{\partial t} f'(\zeta, t)}{f(\zeta, t)} - \frac{\Phi'(f(\zeta, t)) \frac{\partial}{\partial t} f(\zeta, t)}{\Phi(f(\zeta, t))} \right) \\ &= \operatorname{Im} \left(\overline{\zeta f'(\zeta, t)} \dot{f}(\zeta, t) \right) \left(1 + \frac{\overline{\zeta f''(\zeta, t)}}{f'(\zeta, t)} - \frac{\Phi'(f(\zeta, t)) \zeta f'(\zeta, t)}{\Phi(f(\zeta, t))} \right). \end{aligned}$$

If we substitute (2.6), (2.7) and (1.1) in the above expression, replace θ by θ_0 and t by t_0, we obtain:

$$\begin{aligned} &\frac{\partial}{\partial t} \arg \frac{\zeta f'(\zeta, t)}{\Phi(f(\zeta, t))} \Bigg|_{\zeta = \zeta_0, t = t_0} \\ &= \frac{1}{|f'(\zeta_0, t_0)|^2} \frac{Q}{2\pi} \operatorname{Im} \left(\frac{\zeta_0 f''(\zeta_0, t_0)}{f'(\zeta_0, t_0)} + \frac{\Phi'(f(\zeta_0, t_0)) \zeta_0 f'(\zeta_0, t_0)}{\Phi(f(\zeta_0, t_0))} \right) \end{aligned}$$

$$= \frac{Q}{2\pi|f(\zeta_0,t_0)|^2}\operatorname{Im}\left(1+\frac{\zeta_0 f''(\zeta_0,t_0)}{f'(\zeta_0,t_0)}-\frac{\Phi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}\right.$$
$$\left.+2\frac{\Phi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}\right)$$
$$= \frac{Q}{2\pi|f'(\zeta_0,t_0)|^2}\left[\operatorname{Im}\left(1+\frac{\zeta_0 f''(\zeta_0,t_0)}{f'(\zeta_0,t_0)}-\frac{\Phi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}\right)\right.$$
$$\left.+2\operatorname{Im}\frac{\Phi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}\right].$$

Next we shall estimate the term $\operatorname{Im}\dfrac{\Phi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}$.

$$0<\arg\frac{\zeta_0 f'(\zeta_0,t_0)\Phi'(f(\zeta_0,t_0))}{\Phi(f(\zeta_0,t_0))}=\underbrace{\arg\frac{\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}}_{=\frac{\alpha\pi}{2}\ (\text{cf. } (2.3))}+\underbrace{\arg\Phi'(f(\zeta_0,t_0))}_{\in(-\frac{\alpha\pi}{2},\frac{\alpha\pi}{2})\ (\text{cf. } (2.2))}$$
$$<\frac{\alpha\pi}{2}+\frac{\alpha\pi}{2}=\alpha\pi\le\pi.$$

Hence, we proved that $0<\arg\dfrac{\zeta_0 f'(\zeta_0,t_0)\Phi'(f(\zeta_0,t_0))}{\Phi(f(\zeta_0,t_0))}<\pi$, which yields

$$\operatorname{Im}\frac{\zeta_0 f'(\zeta_0,t_0)\Phi'(f(\zeta_0,t_0))}{\Phi(f(\zeta_0,t_0))}>0. \tag{2.9}$$

Using (2.7) and (2.9) in the following relation, we deduce that

$$\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{\Phi(f(\zeta,t))}\bigg|_{\zeta=\zeta_0,t=t_0}$$
$$= \frac{Q}{2\pi|f'(\zeta_0,t_0)|^2}\left[\operatorname{Im}\left(1+\frac{\zeta_0 f''(\zeta_0,t_0)}{f'(\zeta_0,t_0)}-\frac{\Phi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Phi(f(\zeta_0,t_0))}\right)\right.$$
$$\left.+2\operatorname{Im}\left(\frac{\zeta_0 f'(\zeta_0,t_0)\Phi'(f(\zeta_0,t_0))}{\Phi(f(\zeta_0,t_0))}\right)\right]<0.$$

Finally, $\dfrac{\partial}{\partial t}\arg\dfrac{\zeta f'(\zeta,t)}{\Phi(f(\zeta,t))}\bigg|_{\zeta=\zeta_0,t=t_0}<0$. Therefore, $\arg\dfrac{e^{i\theta}f'(e^{i\theta},t)}{\Phi(f(e^{i\theta},t))}<\dfrac{\alpha\pi}{2}$ for $t>t_0$ (close to t_0) in some neighbourhood of θ_0. This contradicts the assumption (2.4) and completes the proof. □

Remark 2.4. Under the assumptions of Theorem 2.3, it follows that if the initial domain $\Omega(0)=f(U,0)$ is strongly Φ-like of order α, then the family of domains $\Omega(t)=f(U,t)$ remain strongly Φ-like of order α for $t\in[0,T)$, where T is the blow-up time.

If in the previous theorem we take $\Phi(w)\equiv e^{-i\beta}w$ where $|\beta|<\alpha\pi/2$ and $\alpha\in(0,1]$, then we obtain the following particular case. The case $\alpha=1$ has been recently considered in [2].

COROLLARY 2.5. *Let $Q < 0$ and let f_0 be a strongly spirallike function of type β and order α on U and univalent on $\overline{U}$, where $\alpha \in (0,1]$ and $\beta \in (-\alpha\pi/2, \alpha\pi/2)$. Then the classical solution of the Polubarinova-Galin equation* (1.1) *with the initial condition $f(\zeta, 0) = f_0(\zeta)$ is strongly spirallike of type β and order α for $t \in [0,T)$, where T is the blow-up time.*

3. THE OUTER PROBLEM (UNBOUNDED DOMAINS WITH BOUNDED COMPLEMENT)

In this section we obtain the invariance in time of the strongly Φ-likeness property for the outer problem.

DEFINITION 3.1. Let F be a holomorphic function on $U^- = \{\zeta : \ |\zeta| > 1\}$ such that $F(\zeta) = a\zeta + a_0 + \dfrac{a_{-1}}{\zeta} + \ldots$, where $a \neq 0$. Let $\alpha \in (0,1]$ and let Φ be a holomorphic function on $F(U^-)$ such that $\lim\limits_{\zeta\to\infty} \Phi(\zeta) = \infty$ and $\lim\limits_{\zeta\to\infty} \Phi'(\zeta) > 0$.

We say that F is *strongly Φ-like of order α* on U^- if

$$\left|\arg \frac{\zeta F'(\zeta)}{\Phi(F(\zeta))}\right| < \frac{\alpha\pi}{2}, \ \forall\, \zeta \in U^-. \tag{3.1}$$

REMARK 3.2. (a) It is obvious to see that if F is a strongly Φ-like function on U^- of order $\alpha \in (0,1]$, then the function $f : U \to \mathbb{C}$ given by $f(z) = \frac{1}{F\left(\frac{1}{z}\right)}$, $z \neq 0$, and $f(0) = 0$, is strongly Ψ-like on U of order α, where $\Psi : f(U) \to \mathbb{C}$, $\Psi(w) = w^2 \Phi\left(\frac{1}{w}\right)$, $\forall\, w \in f(U) \setminus \{0\}$ and $\Psi(0) = 0$.

(b) If f is a strongly Ψ-like function of order α, on U then the function $F : U^- \to \mathbb{C}$, $F(\zeta) = \frac{1}{f\left(\frac{1}{\zeta}\right)}$ is strongly Φ-like of order α on U^-, where $\Phi : F(U^-) \to \mathbb{C}$, $\Phi(\omega) = \omega^2 \Psi\left(\frac{1}{\omega}\right)$, $\forall\, \omega \in F(U^-)$. The proof is immediate and we leave it for the reader.

(c) Any strongly Φ-like function F of order $\alpha \in (0,1]$ on U^- is univalent on U^-. Indeed the corresponding function f is strongly Ψ-like of order α on U, and thus univalent.

We next obtain the analog of Theorem 2.3 to the case of unbounded domains. This result is a generalization of [17, Theorem 3] (see also [5, Theorem 4.3.5]). The mentioned theorem may be obtained by taking $\Phi(w) \equiv w$ and $\alpha = 1$ in Theorem 3.3 below. The case $\alpha = 1$ was considered in [2].

THEOREM 3.3. *Let $\alpha \in (0,1]$ and F_0 be a strongly Φ-like function of order α on U^- and univalent on $\overline{U^-}$. Then the solution $F(\zeta, t)$ of the Polubarinova-Galin equation* (1.2) *with the initial condition $F(\zeta, 0) = F_0(\zeta)$ is strongly Φ-like of order α for $t \in [0,T)$, where T is the blow-up time, $\Omega = \bigcup\limits_{0\le t<T} \Omega(t) =$*

$\bigcup_{0\le t<T} F(U^-, t)$ *and the function* Φ *is a holomorphic function on* $\overline{\Omega}$ *which satisfies the following conditions:*

$$\left|\arg \frac{\Phi(w)}{w}\right| < \frac{\alpha\pi}{2},\ \forall\ w \in \overline{\Omega}, \tag{3.2}$$

and

$$\left|\arg \left(2\frac{\Phi(w)}{w} - \Phi'(w)\right)\right| < \frac{\alpha\pi}{2},\ \forall\ w \in \overline{\Omega}. \tag{3.3}$$

Proof. By considering the function $f(\zeta, t) = \dfrac{1}{F\left(\frac{1}{\zeta}, t\right)}$, the Polubarinova-Galin equation can be rewritten in terms of f as follows:

$$\operatorname{Re}[\dot{f}(\zeta, t)\overline{\zeta f'(\zeta, t)}] = -\frac{Q|f(\zeta, t)|^4}{2\pi}, \quad |\zeta| = 1. \tag{3.4}$$

In view of Remark 3.2, the function $F(\zeta, t)$, $\zeta \in U^-$, is strongly Φ-like of order α if and only if $f(\zeta, t)$, $\zeta \in U$, is strongly Ψ-like of order α, where the relationship between Φ and Ψ is as follows:

$$\Phi(w) = w^2\Psi\left(\frac{1}{w}\right),\ \forall\ w \in F(U^-) \quad \left(\text{or } \Psi(w) = w^2\Phi\left(\frac{1}{w}\right),\ \forall\ w \in f(U)\right).$$

Hence, it suffices to prove that the function $f(\zeta, t)$ is strongly Ψ-like of order α, for all $t \in [0, T)$.

Suppose by contrary that the previous statement is not true. Then there exist $t_0 \ge 0$ and $\zeta_0 = e^{i\theta_0}$ such that

$$\arg \frac{\zeta_0 f'(\zeta_0, t_0)}{\Psi(f(\zeta_0, t_0))} = \frac{\alpha\pi}{2} \quad \left(\text{or } -\frac{\alpha\pi}{2}\right). \tag{3.5}$$

We consider the sign (+) in the previous equality. Let $t_0 \in [0, T)$ be the first such point. Without loss of generality, we assume that

$$\operatorname{Im} \frac{\zeta_0 f'(\zeta_0, t_0)}{\Psi(f(\zeta_0, t_0))} > 0. \tag{3.6}$$

As in the proof of Theorem 2.3, we deduce the following conditions at the critical point ζ_0:

$$\operatorname{Re}\left[1 + \frac{\zeta_0 f''(\zeta_0, t_0)}{f'(\zeta_0, t_0)} - \frac{\Psi'(f(\zeta_0, t_0))\zeta_0 f'(\zeta_0, t_0)}{\Psi(f(\zeta_0, t_0))}\right] = 0 \tag{3.7}$$

$$\operatorname{Im}\left[1 + \frac{\zeta_0 f''(\zeta_0, t_0)}{f'(\zeta_0, t_0)} - \frac{\Psi'(f(\zeta_0, t_0))\zeta_0 f'(\zeta_0, t_0)}{\Psi(f(\zeta_0, t_0))}\right] \ge 0. \tag{3.8}$$

By differentiating (3.4) we get:

$$\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{\Psi(f(\zeta,t))}\Big|_{\zeta=\zeta_0,t=t_0}=\mathrm{Im}\left(\frac{\frac{\partial f'}{\partial t}}{f'}-\frac{\frac{\partial f}{\partial t}\Psi'(f)}{\Psi(f)}\right)\Big|_{\zeta=\zeta_0,t=t_0}$$
$$=\frac{Q|f|^4}{2\pi|f'|^2}\mathrm{Im}\left(\frac{\zeta f''}{f'}+\frac{\Psi'(f)\zeta' f}{\Psi(f)}\right)\Big|_{\zeta=\zeta_0,t=t_0}+4\frac{Q|f|^4}{2\pi|f'|^2}\mathrm{Im}\,\frac{\zeta f'}{f}\Big|_{\zeta=\zeta_0,t=t_0}$$
$$=\frac{Q|f|^4}{2\pi|f'|^2}\mathrm{Im}\left(1+\frac{\zeta f''}{f'}-\frac{\Psi'(f)\zeta f'}{\Psi(f)}\right)\Big|_{\zeta=\zeta_0,t=t_0}+\frac{2Q|f|^4}{2\pi|f'|^2}\mathrm{Im}\,\frac{\Psi'(f)\zeta f'}{\Psi(f)}\Big|_{\zeta=\zeta_0,t=t_0}$$
$$+\frac{4Q|f|^4}{2\pi|f'|^2}\mathrm{Im}\,\frac{\zeta f'}{\Psi(f)}\mathrm{Re}\frac{\Psi(f)}{f}\Big|_{\zeta=\zeta_0,t=t_0}.$$

We evaluate the right-hand sign of the above relation. For this aim, we evaluate the term $\mathrm{Im}\,\frac{\Psi'(f)\zeta f'}{\Psi(f)}$ at $\zeta=\zeta_0$. We have

$$0<\arg\frac{\Psi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Psi(f(\zeta_0,t_0))}=\arg\frac{\zeta_0 f'(\zeta_0,t_0)}{\Psi(f(\zeta_0,t_0))}+\arg\Psi'(f(\zeta_0,t_0))$$
$$<\frac{\alpha\pi}{2}+\frac{\alpha\pi}{2}=\alpha\pi\le\pi.$$

Consequently, we obtain that $0<\arg\frac{\Psi'(f)\zeta f'}{\Psi(f)}\Big|_{\zeta=\zeta_0}<\pi$, and thus

$$\mathrm{Im}\,\frac{\Psi'(f(\zeta_0,t_0))\zeta_0 f'(\zeta_0,t_0)}{\Psi(f(\zeta_0,t_0))}>0. \tag{3.9}$$

Taking into account the above relations,

$$\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{\Psi(f(\zeta,t))}\Big|_{\zeta=\zeta_0,t=t_0}<0.$$

We have used the relations (3.8), (3.9), $|\arg\Psi'(f)|<\alpha\pi/2$ and $|\arg\Psi(f)/f|<\alpha\pi/2$ (the previous inequalities are immediate consequences of (3.2) and (3.3)). Therefore, we deduce that $\arg\frac{\mathrm{e}^{\mathrm{i}\theta}f'(\mathrm{e}^{\mathrm{i}\theta},t)}{\Psi(f(\mathrm{e}^{\mathrm{i}\theta},t))}<\frac{\alpha\pi}{2}$, for $t>t_0$ (close to t_0) in some neighbourhood of θ_0. However, this is a contradiction to the assumption (3.6). The proof is complete. □

4. EXAMPLES

We consider the polynomial function

$$F(\zeta,t)=a_1(t)\zeta+a_2(t)\zeta^2+a_3(t)\zeta^3+a_4(t)\zeta^4.$$

It has to satisfy the Polubarinova-Galin equation (1.4) that leads to the following system of differential equations obtained using Mathematica:

$$
\begin{aligned}
&a_1\frac{\mathrm{d}a_1}{\mathrm{d}t}+2a_2\frac{\mathrm{d}a_2}{\mathrm{d}t}+3a_3\frac{\mathrm{d}a_3}{\mathrm{d}t}+4a_4\frac{\mathrm{d}a_4}{\mathrm{d}t}=-\frac{Q}{2\pi}\\
&2a_2\frac{\mathrm{d}a_1}{\mathrm{d}t}+(a_1+3a_3)\frac{\mathrm{d}a_2}{\mathrm{d}t}+(2a_2+4a_4)\frac{\mathrm{d}a_3}{\mathrm{d}t}+3a_3\frac{\mathrm{d}a_4}{\mathrm{d}t}=0\\
&3a_3\frac{\mathrm{d}a_1}{\mathrm{d}t}+4a_4)\frac{\mathrm{d}a_2}{\mathrm{d}t}+a_1\frac{\mathrm{d}a_3}{\mathrm{d}t}+2a_2\frac{\mathrm{d}a_4}{\mathrm{d}t}=0\\
&4a_4\frac{\mathrm{d}a_1}{\mathrm{d}t}+a_1\frac{\mathrm{d}a_4}{\mathrm{d}t}=0
\end{aligned}
\tag{4.1}
$$

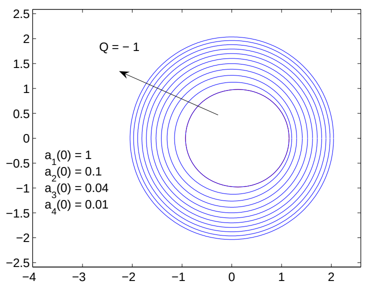
or

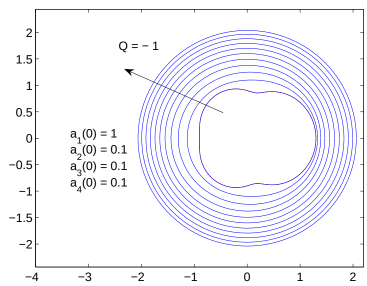
$$
\begin{aligned}
\frac{\mathrm{d}a_1}{\mathrm{d}t}&=-\frac{a_1(a_1^2+3a_1a_3-8a_4(a_2+2a_4))Q}{2(a_1-2a_2+3a_3-4a_4)(a_1+2a_2+3a_3+4a_4)(a_1^2-3a_1a_3+8(a_2-2a_4)a4)\pi}\\
\frac{\mathrm{d}a_2}{\mathrm{d}t}&=-\frac{(a_1^2a_2+8a_2a_4(a_2+2a_4)-3a_1a_3(a_2+4a_4))Q}{(a_1-2a_2+3a_3-4a_4)(a_1+2a_2+3a_3+4a_4)(a_1^2-3a_1a_3+8(a_2-2a_4)a4)\pi}\\
\frac{\mathrm{d}a_3}{\mathrm{d}t}&=-\frac{(3a_1a_3(a1+3a_3)-8a_2a_4(2a_1+3a_3)+48a_3a_4^2)Q}{2(a_1-2a_2+3a_3-4a_4)(a_1+2a_2+3a_3+4a_4)(a_1^2-3a_1a_3+8(a_2-2a_4)a4)\pi}\\
\frac{\mathrm{d}a_4}{\mathrm{d}t}&=-\frac{2a_4(a_1^2+3a_1a_3-8a_4(a_2+2a_4))Q}{(a_1-2a_2+3a_3-4a_4)(a_1+2a_2+3a_3+4a_4)(a_1^2-3a_1a_3+8(a_2-2a_4)a4)\pi}
\end{aligned}
$$

which have to be solved starting from an initial domain given by

$$F(\zeta,0)=a_1(0)\zeta+a_2(0)\zeta^2+a_3(0)\zeta^3+a_4(0)\zeta^4.$$

It is worth to mention that the system of equations (4.1) is similar to those described by [5]. The last system of equations was solved numerically using Matlab for two different initial domains, a convex domain and a starlike domain, respectively. We have also considered a negative value for Q (fluid injection). In the injection case the domains growth infinitely and after some time the domains take a disk shape. In the following figures the domains variations are presented for an injection time $T = 10$.





Convex initial domain, injection. Star-like initial domain, injection

REFERENCES

[1] Brickman, L., *Φ-like analytic functions*, Bull. Amer. Math. Soc., **79** (1973), 555–558.

[2] Curt, P. and Fericean, D., *A special class of univalent functions in Hele-Shaw flow problems*, Abstr. Appl. Anal., **2011** (2011), Art. ID 948236, 10 pp.

[3] Galin, L.A., *Unsteady filtration with a free surface*, Dokl. Akad. Nauk USSR, **47** (1945), 246–249.

[4] Graham, I. and Kohr, G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker Inc., New York, 2003.

[5] Gustafsson, B. and Vasil'ev, A., *Conformal and Potential Analysis in Hele-Shaw Cells*, Birkhäuser Verlag, 2006.

[6] Hohlov, Yu. E., Prokhorov, D.V. and Vasil'ev, A., *On geometrical properties of free boundaries in the Hele-Shaw flows moving boundary problem*, Lobachevskii J. Math., **1** (1998), 3–12.

[7] Howison, S.D., *Complex variable methods in Hele-Shaw moving boundary problems*, Euro. J. Appl. Math., **3** (1992), 209–224.

[8] Huntingford, C. , *An exact solution to the one phase zero surface tension Hele-Shaw free boundary problem*, Comp. Math. Appl., **29** (1995), 45–50.

[9] Kornev, K. and Vasil'ev, A., *Geometric properties of the solutions of a Hele-Shaw type equation*, Proc. Amer. Math. Soc., **128** (2000), 2683–2685.

[10] Mocanu, P.T., Bulboacă, T. and Sălăgean, G., *Geometric Theory of Univalent Functions* (in Romanian), Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006.

[11] Polubarinova-Kochina, P.Ya., *On a problem of the motion of the contour of a petroleum shell* (in Russian), Dokl. Akad. Nauk USSR, **47** (1945), 254–257.

[12] Polubarinova-Kochina, P.Ya. *Concerning unsteady notions in the theory of filtration* (in Russian), Prikl. Matem. Mech., **9** (1945), 79–90.

[13] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[14] Prokhorov, D. and Vasil'ev, A., *Convex Dynamics in Hele-Shaw cells*, Intern. J. Math. Math. Sci., **31** (2002), 639–650.

[15] Reissig. M. and von Wolfersdorf, L., *A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane*, Ark. Mat., **31** (1993), 101–116.

[16] Vasil'ev, A., *Univalent functions in the dynamics of viscous flows*, Comp. Methods and Func. Theory, **1** (2001), 311–337.

[17] Vasil'ev, A., *Univalent functions in two-dimensional free boundary problems*, Acta Applic. Math., **79** (2003), 249–280.

[18] Vasil'ev, A. and Markina, I., *On the geometry of Hele-Shaw flows with small surface tension*, Interfaces and Free Boundaries, **5** (2003), 182–192.

[19] Vinogradov, Yu.P. and Kufarev, P.P., *On a problem of filtration*, Akad. Nauk SSSR, Prikl. Math. Mech. (in Russian), **12** (1948), 181–198.

Received July 8, 2010
Accepted September 20, 2010

"Babeş-Bolyai" University
Faculty of Economics
and Business Administration
Cluj-Napoca, Romania
E-mail: paula.curt@econ.ubbcluj.ro

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science
Cluj-Napoca, Romania
E-mail: denisa.fericean@ubbcluj.ro
E-mail: tgrosan@math.ubbcluj.ro