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GENERALIZED ALMOST STARLIKENESS ASSOCIATED WITH EXTENSION OPERATORS FOR BIHOLOMORPHIC MAPPINGS

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Abstract. In this paper we consider an operator introduced by Pfaltzgraff and Suffridge which provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$. Using the Loewner chains we prove that if f is an generalized almost starlike mapping on B^n then F is also generalized almost starlike mapping on B^{n+1}.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, ..., z_n)'$ with respect to the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the norm $\|z\| = \sqrt{\langle z, z\rangle}$. The symbol $'$ means the transpose of vectors and matrices. For $n \geq 2$, let $\widetilde{z} = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$, so that $z = (z_1, \widetilde{z}) \in \mathbb{C}^n$.

Let $B_r^n = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B^n = B_1^n$. In the case of one complex variable B_r^n is denoted by U_r and U_1 by U. If G is an open set in $\mathbb{C}^n$, let $H(G)$ be the set of holomorphic maps from G into $\mathbb{C}^n$. A holomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exist and is holomorphic on the open set $f(B^n)$. A mapping $f \in H(B^n)$ is said to be locally biholomorphic if the Frechet derivative $Df(z)$ has a bounded inverse for each $z \in B^n$. If $f \in H(B^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. Let $S(B^n)$ be the set of all normalized biholomorphic mappings on B^n.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of complex linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm $\|A\| = \sup\{\|Az\| : \|z\| = 1\}$ and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n, and let $S(B^n)$ denote the set of normalized biholomorphic mappings on B^n. In the case of one variable, the set $S(B^1)$ is denoted by S, and $\mathcal{L}S(B^1)$ is denoted by $\mathcal{L}S$. A mapping $f \in S(B^n)$ is called starlike (respectiv convex) if its image is a starlike domain with respect to the origin (respectively convex domain). The classes of normalized starlike (respectively convex) mappings on B^n will be denoted by $S^*(B^n)$ (respectively $K(B^n)$). In the case of one variable, $S^*(B^1)$ (respectively $K(B^1)$) is denoted by S^* (respectively K).

Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e., $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|, z \in B^n$) such that $f(z) = g(v(z)), z \in B^n$. If g is biholomorphic on B^n, this is equivalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

Definition 1.1. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a *Loewner chain* if it satisfies the following conditions:
(i)$f(\cdot, t)$ is biholomorphic on $B^n, f(0, t) = 0, Df(0, t) = \mathrm{e}^t I_n$ for $t \geq 0$;
(ii) $f(z, s) \prec f(z, t)$ whenever $0 \leq s \leq t < \infty$ and $z \in B^n$.

We note that condition *(ii)* implies that there is a unique univalent Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), z \in B^n, 0 \leq s \leq t < \infty, z \in B^n. \tag{1.1}$$

Further, the normalization of $f(z, t)$ implies the normalization $Dv(0, s, t) = \mathrm{e}^{s-t} I_n$, $0 \leq s \leq t < \infty$, for the transition mapping.

On the other hand, (1.1) and the univalence of $f(\cdot, t), t \geq 0$, imply the important semigroup property of the transition mapping $v(z, s, t)$, i.e.,

$$v(z, s, u) = v(v(z, s, t), t, u), \quad z \in B^n, 0 \leq s \leq t < \infty. \tag{1.2}$$

The following class of holomorphic mappings of B^n plays the role of Carathéodory class in n dimensions:

$$\mathcal{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \mathrm{Re}\,\langle h(z), z\rangle > 0, z \in B^n \setminus \{0\}\}.$$

In this paper we shall prove that the Pfaltzgraff-Suffridge operator Φ_n preserves the notion of generalized almost starlikeness.

Certain subclasses of $S(B^n)$ can be characterized in terms of Loewner chains. We recall the definition of a spirallike mapping of α type on B^n.

Definition 1.2. Suppose $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. A biholomorphic function $f : B^n \to \mathbb{C}^n$ is said to be *spirallike mapping of α type* if $\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\alpha}\langle [Df(z)]^{-1} f(z), z\rangle\right] > 0,\ z \in B^n \setminus \{0\}$.

This subclass of biholomorphic mappings can be characterized in terms of Loewner chains (see [9]).

Lemma 1.3. *Suppose f is a normalized locally biholomorphic mapping on $B^n, \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $a = \tan\alpha$, then f is a spirallike mapping of α type if and only if $F(z, t) = \mathrm{e}^{(1-\mathrm{i}a)t} f(\mathrm{e}^{\mathrm{i}at} z),\ z \in B^n,\ t > 0$, is a Loewner chain. In particular, f is a starlike mapping if and only if $F(z, t) = \mathrm{e}^t f(z)$ is a Loewner chain.*

The following definition was introduced by G. Kohr ([10]) in the case $\alpha = \frac{1}{2}$ and by Feng [3] in the case of $\alpha \in [0, 1)$ and on the unit ball in a complex Banach space X. For our purpose, we present this notion only in the Euclidean setting.

Definition 1.4. Suppose $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be an *almost starlike mapping of order* α if $\mathrm{Re}\,\langle [Df(z)]^{-1} f(z), z\rangle > \alpha \|z\|^2$, $z \in B^n \setminus \{0\}$.

It is clear that if f is almost starlike of order α, then f is also starlike.

Q.H. Xu and T.S. Liu [16] proved the following characterization of almost starlikeness of order α in terms of Loewner chains.

Lemma 1.5. *Suppose f is a normalized locally biholomorphic mapping in $B^n, 0 \leq \alpha < 1$. Then f is an almost starlike mapping of order α if and only if $F(z,t) = \mathrm{e}^{\frac{1}{1-\alpha}} f(\mathrm{e}^{\frac{\alpha}{\alpha-1}t} z)$, $z \in B^n$, $t \geq 0$, is a Loewner chain. In particular, f is a starlike mapping (i.e., $\alpha = 0$) if and only if $F(z,t) = \mathrm{e}^t f(z)$ is a Loewner chain.*

Next we recall the definition for generalized almost starlikeness [2].

Definition 1.6. Let $a : [0,\infty) \to \mathbb{C}$ of class C^∞ with $\eta \leq \mathrm{Re}\ a(t) \leq 0, t \in [0,\infty), \eta < 0$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *generalized almost starlike* if

$$\mathrm{Re}\,\{(1 - a'(t))\mathrm{e}^{-a(t)} \langle [Df(\mathrm{e}^{a(t)} z)]^{-1} f(\mathrm{e}^{a(t)} z), z\rangle\} \geq -\mathrm{Re}\,(a'(t))\|z\|^2, \tag{1.3}$$

for $z \in B^n$, $t \geq 0$.

We denote with $S_g^*(B^n)$ the set of generalized almost starlike mapping.

Remark 1.7. In the case of $a'(t) = \frac{\alpha}{\alpha-1}, \alpha \in [0,1)$, in Definition 1.6, we obtain the usual notion of almost starlikeness of order α. In the case of $a'(t) = -1$ in Definition 1.6, we obtain the notion of almost starlikeness of order $\frac{1}{2}$. If $a'(t) = \lambda$, $\lambda \in \mathbb{C}^n$, $\mathrm{Re}\,\lambda \leq 0$, in Definition 1.6, we obtain the notion of almost starlike mapping of complex order λ.

The following result provides a necessary and sufficient condition for generalized almost starlikeness in terms of Loewner chains.

Lemma 1.8. [2] *Let $f : U \to \mathbb{C}$ be a normalized holomorphic function and let $a : [0,\infty) \to \mathbb{C}$, be a function of class C^∞, such that $\eta \leq \mathrm{Re}\, a(t) \leq 0$, $t \in [0,\infty)$, for some $\eta < 0$. Then f is a generalized almost starlike mapping if and only if $g(z,t) = \mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)} z)$, $z \in U$, $t \geq 0$, is a Loewner chain. In particular, f is a starlike function (i.e., $a(t) = 0$) if and only if $g(z,t) = \mathrm{e}^t f(z)$ is a Loewner chain.*

The next definition refers to the notion of parametric representation for biholomorphic mappings on B^n. Note that this notion was studied in [6], [4], [12], [13]. Here we present an equivalent definition to the original one given in [8] and [12].

Definition 1.9. Let $f \in H(B^n)$ be a normalized mapping. We say that f *has parametric representation* if there exists a Loewner chain $f(z,t)$ such that $\{\mathrm{e}^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f = f(\cdot, 0)$.

Let $S^0(B^n)$ be the set of mappings which have parametric representation on B^n. A key role in our discussion is played by the following Schwarz- type lemma for the Jacobian determinant of a holomorphic mapping from B^n.

LEMMA 1.10. [15] *Let $\psi \in H(B^n)$ be such that $\psi(B^n) \subseteq B^n$. Then*

$$|J_\psi(z)| \leq \left[\frac{1-\|\psi(z)\|^2}{1-\|z\|^2}\right]^{\frac{n+1}{2}}, \quad z \in B^n. \tag{1.4}$$

This inequality is sharp and equality at a given point $z \in B^n$ holds if and only if $\psi \in Aut(B^n)$, where $Aut(B^n)$ is the set of holomorphic automorphism of B^n.

For $n \geq 1$, set $z' = (z_1, ..., z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

DEFINITION 1.11 ([11]). The Pfaltzgraff-Suffridge extension operator $\Phi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ is defined by $\Phi_n(f)(z) = F(z) = (f(z'), z_{n+1}[J_f(z')]^{\frac{1}{n+1}})$, $z = (z', z_{n+1}) \in B^{n+1}$. We choose the branch of the power function such that $[J_f(z')]^{\frac{1}{n+1}}|_{z'=0} = 1$.

Then $F = \Phi_n(f) \in \mathcal{L}S_{n+1}$ whenever $f \in \mathcal{L}S_n$. It is easy to see that if $f \in S(B^n)$ then $F = \Phi_n(f) \in S(B^{n+1})$.
If $n = 1$ then Φ_1 reduces to the well-known Roper-Suffridge extension operator. For general $n \geq 2$ we have:

DEFINITION 1.12. [14] The Roper-Suffride extension operator $\Psi_n : \mathcal{L}S \to \mathcal{L}S_n$ is defined by $\Psi_n(f)(z) = (f(z_1), \widetilde{z}\sqrt{f'(z_1)})$, $z = (z_1, \widetilde{z}) \in B^n$. We choose the branch of the power function such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

Various properties of the Pfaltzgraff-Suffridge and Roper-Suffridge operators may be found in [11], [16], [5], respectively.
In this paper we shall prove that the Roper-Suffridge operator and the Pfaltzgraff-Suffridge operators preserve the notion of generalized almost starlikeness.

2. MAIN RESULTS

We begin this section with the main result of this paper.

THEOREM 2.1. *Assume f is a generalized almost starlike mapping. Then $F = \Phi_n(f)$ is also a generalized almost starlike mapping.*

Proof. Let $F : B^{n+1} \times [0, \infty) \to \mathbb{C}^{n+1}$, be given by

$$F(z,t) = (f(z',t), z_{n+1}\mathrm{e}^{\frac{t}{n+1}}[J_{f_t}(z')]^{\frac{1}{n+1}}) \tag{2.1}$$

for $z = (z', z_{n+1}) \in B^{n+1}$ and $t \geq 0$. We choose the branch of the power function such that $[J_{f_t}(z')]^{\frac{1}{n+1}}|_{z'=0} = \mathrm{e}^{\frac{nt}{n+1}}$. Using the Schwarz-type lemma for the Jacobian determinant of a holomorphic mapping from B^n into B^n and the automorphisms of B^n, the authors have proved in Theorem 2.1. (see [7]) that $F(z,t)$ is a Loewner chain. The fact that f is generalized almost starlike is equivalent to the statement that $f(z',t) = \mathrm{e}^{t-a(t)} f(\mathrm{e}^{a(t)}z')$, $z \in B^n, t \geq 0, a : [0,\infty) \to \mathbb{C}, \eta \leq \mathrm{Re}\ a(t) \leq 0, \eta < 0, a$ of class C^∞, is a Loewner chain. With this choice of $f(z',t)$, we deduce that $F(z,t)$ given by (2.1) is a Loewner chain.

On the other hand a simple computation yields that

$$F(z,t) = \mathrm{e}^{t-a(t)}(f(\mathrm{e}^{a(t)}z'), z_{n+1} \cdot \mathrm{e}^{a(t)}[J_f(\mathrm{e}^{a(t)}z')]^{\frac{1}{n+1}}) = \mathrm{e}^{t-a(t)} F(\mathrm{e}^{a(t)}z),$$

for $z \in B^{n+1}$, and $t \geq 0, a : [0,\infty) \to \mathbb{C}, \eta \leq \mathrm{Re}\, a(t) \leq 0, \eta < 0, a$ of C^∞ class.

Thus $F = \Phi_n(f)$ is a generalized almost starlike mapping on B^{n+1}. This completes the proof. □

We next consider some particular cases of the above result.

If $a'(t) = \tan\alpha$ we obtain the following particular case [7].

Corollary 2.2. *Assume f is a spirallike mapping of α type. Then $F = \Phi_n(f)$ is a spirallike mapping of α type.*

If in the proof of Theorem 2.1, we consider that $a'(t) = \frac{\alpha}{\alpha-1}, \alpha \in [0,1)$, we obtain the following result.

Corollary 2.3. *Assume f is an almost starlike mapping of order α. Then $F = \Phi_n(f)$ is an almost starlike mapping of order α.*

Note that, if $n = 1$ in Corollary 2.3, we deduce that the Roper-Suffridge operator preserves the notion of almost starlikeness of order α (see[16]).

Considering $a'(t) \equiv \lambda, \mathrm{Re}\,\lambda \leq 0$, we obtain in view of Theorem 2.1 the following particular case (see [1]).

Corollary 2.4. *Assume f is an almost starlike mapping of complex order λ. Then $F = \Phi_n(f)$ is also an almost starlike mapping of complex order λ.*

Another consequence of Theorem 2.1 is given in the following result, which provides a positive answer to the question of Pfaltzgraff and Suffridge regarding the preservation of starlikeness under the operator Φ_n(see [7] and [11]). To prove this result, it suffices to consider $a'(t) \equiv 0$ in Theorem 2.1.

Corollary 2.5. *Assume that $f \in S^*(B^n)$.Then $F = \Phi_n(f) \in S^*(B^{n+1})$.*

We close this paper with the following compactness result.

Theorem 2.6. *The set $\Phi_n[S_g^*(B^n)]$ is compact.*

Proof. Since $\Phi_n[S_g^*(B^n)] \subset S_g^*(B^{n+1})$ and $S^*(B^{n+1})$ is compact by [2], we deduce that $\Phi_n[S_g^*(B^n)]$ is a locally uniformly bounded family. It remains to prove that $\Phi_n[S^*(B^n)]$ is closed. To this end, let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence in $\Phi_n[S_g^*(B^n)]$ which converges locally uniformly on B^{n+1} to a mapping F as $k \to \infty$. Also let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $S_g^*(B^n)$ be such that $F_k = \Phi_n(f_k), k \in \mathbb{N}$. Since $\{f_k\}_{k\in\mathbb{N}}$ is a locally uniformly bounded sequence on B^n, there is a subsequence $\{f_{k_p}\}_{p\in\mathbb{N}}$ of $\{f_k\}_{k\in\mathbb{N}}$ which converges locally uniformly on B^n to a mapping f. Since $S_g^*(B^n)$ is a compact set, we deduce that $f \in S_g^*(B^n)$. Also it is easy to see that the subsequence $\{\Phi_n(f_{k_p})\}_{p\in\mathbb{N}}$ converges locally uniformly on B^{n+1} to $\Phi_n(f)$, and thus we must have $F = \Phi_n(f)$. Hence $F \in \Phi_n[S_g^*(B^n)]$. This completes the proof. □

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