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ON THE SEMILOCAL CONVERGENCE OF STEFFENSEN'S METHOD

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Abstract. We provide a semilocal convergence analysis of a general Steffensen's method in a Banach space setting. In some interesting special cases, we expand the applicability of this method. A numerical example involving the solution of a nonlinear two boundary problem is also provided in this study.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet–differentiable operator defined on a nonempty convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in $\mathcal{X}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We shall use Steffensen's method (SM):

$$\begin{aligned} & x_{n+1} = x_n - A_n^{-1}\, F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \\ & A_n := A(x_n) = [x_n, g(x_n); F], \end{aligned} \tag{2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $g : \mathcal{X} \longrightarrow \mathcal{X}$ is a continuous operator, and $[x, y; F] \in \mathcal{L}(\mathcal{X})$ denotes a divided difference of order one of F at points x, y, satisfying

$$[x, y; F]\,(y - x) = F(y) - F(x), \quad x, y \in \mathcal{D}, \quad x \neq y, \tag{3}$$

$$[x, x; F] = F'(x), \quad (x \in \mathcal{D}) \tag{4}$$

(see [4], [6]).

If $g(x) = x$, $(x \in \mathcal{D})$, (SM) reduces to Newton's method (NM) given by

$$y_{n+1} = y_n - F'(y_n)^{-1}\, F(y_n) \quad (n \geq 0), \quad (y_0 = x_0 \in \mathcal{D}). \tag{5}$$

The popular choice for g is given by $g(x) = x + \lambda\, F(x)$ $(x \in \mathcal{D}\ )$, $\lambda \in \mathbb{R}$. Other choices for g can be found in [4], [6]. (SM) is a usefull alternative to (NM) in case where operator $F'(x)^{-1}$, $(x \in \mathcal{D})$ is not available, or difficult or impossible to compute.

A semilocal convergence analysis for (SM) under different choices for g and using different hypothese have been given by several authors [1]–[14]. A survey of such results can be also found in [4], [6], and the references there.

In this study, we present a semilocal convergence analysis for (SM) by introducing recurrent functions and using conditions simpler than before. We show that the order of convergence is quadratic. In the special case of (NM), we show that our convergence analysis is finer than before. Finally, we provide a numerical example to show that (SM) can be faster than (NM). Moreover, we provide a theoretical justification to showing why this can happen. The paper is organized as follows. In section 2, we provide the semilocal convergence analysis of (SM) and the numerical example in presented in section 3.

2. SEMILOCAL CONVERGENCE ANALYSIS OF (SM)

DEFINITION 2.1. Let $\alpha \geq 0$, $\beta > 0$, $\eta > 0$, $M_0 \geq 0$, $M_1 > 0$, and

$$M = \begin{cases} > M_0 & \text{if} \quad M_0 = 0 \\ \geq M_0 & \text{if} \quad M_0 \neq 0 \end{cases} \tag{6}$$

be given constants. Define:

$$L_0 = \gamma\, \eta + M_0, \quad \gamma = \frac{\alpha}{2}\,(2\,\alpha\, M_1 + M), \tag{7}$$

$$L = 2\,\alpha\, M_1 + M, \tag{8}$$

$$\eta_0 = \begin{cases} \dfrac{2}{M_0 + \sqrt{M_0^2 + 4\,\gamma}} & \text{if} \quad \alpha \neq 0 \\ \dfrac{1}{M_0} & \text{if} \quad \alpha = 0, \end{cases} \tag{9}$$

$$\eta_1 = \begin{cases} \dfrac{2\,(\alpha\, M_1 + M - M_0)}{\gamma} & \text{if} \quad \alpha \neq 0 \\ \text{Any positive number} & \text{if} \quad \alpha = 0, \end{cases} \tag{10}$$

η_2 to be the minimal positive zero of polynomial:

$$h_1(t) = 2\,\gamma^2\,t^4 + 4\,\gamma\,M_0\,t^3 + 2\,(M_0^2 - 2\,\gamma)\,t^2 - (4\,M_0 + L)\,t + 2 \tag{11}$$

if it exists. Otherwise, let η_2 be any positive number,

$$\eta_3 = \begin{cases} \dfrac{8}{L_0 + 4\,M_0 + \sqrt{(L + 4\,M_0)^2 + 64\,\gamma}} & \text{if} \quad \alpha \neq 0 \\ \dfrac{4}{L + 4\,M_0} & \text{if} \quad \alpha = 0, \end{cases} \tag{12}$$

$$\eta_4 = \min\{\eta_i,\ i = 0, \ldots, 3\}, \tag{13}$$

$$\eta_5 = \min\left\{\eta_2,\ \frac{1}{L + 4\,M_0}\right\}. \tag{14}$$

It is simple algebra to show:
(i) if $\eta < \eta_0$, then $h_2(t) = \gamma\,t^2 + M_0\,t < 1$ for all $t \in [0, \eta_0]$;
(ii) if $\eta \leq \eta_1$, then $L_0 \leq L$;
(iii) if $\eta \leq \eta_2$, then $h_1(t) \geq 0$ for all $t \in [0, \eta_2]$, and

$$q_A = \overline{L}\,\eta \leq \frac{1}{2}, \tag{15}$$

where

$$\overline{L} = \frac{1}{8}\,(L + 4\,L_0 + \sqrt{L^2 + 8\,L_0\,L}); \tag{16}$$

(iv) if $\eta < \eta_3$, then $h_3(t) = 4\,\gamma\,t^2 + (L + 4\,M_0)\,t - 4 < 0$ for all $t \in [0, \eta_3]$.
Items (i)–(iv) can be written in a condensed form provided that the following conditions hold:

$$\eta \leq \eta^\star \begin{cases} \begin{cases} \leq \eta_4 & \text{if} \quad \alpha \neq 0,\ \eta_4 \neq \eta_3 \\ < \eta_4 & \text{if} \quad \alpha \neq 0,\ \eta_4 = \eta_3 \end{cases} \\ \begin{cases} \leq \eta_5 & \text{if} \quad \alpha = 0,\ M_0 \neq 0,\ \eta_5 = \eta_2 \\ < \dfrac{1}{L + 4\,M_0} & \text{if} \quad \alpha = 0,\ M_0 \neq 0,\ \eta_5 = \dfrac{1}{L + 4\,M_0} \end{cases} \\ \leq \dfrac{2}{L} \qquad \text{if} \quad \alpha = M_0 = 0. \end{cases}$$

We need the following result on majorizing sequences for (SM).

LEMMA 2.2 ([5]). *Assume there exist constants* $L_0 \geq 0$, $L \geq 0$, *and* $\eta \geq 0$, *with* $L_0 \leq L$, *such that:*

$$q_A = \overline{L}\,\eta \begin{cases} \leq \dfrac{1}{2} & \text{if} \quad L_0 \neq 0 \\ < \dfrac{1}{2} & \text{if} \quad L_0 = 0\ , \end{cases} \tag{17}$$

where $\overline{L}$ *is given by* (16)*. Then the sequence* $\{t_k\}$ $(k \geq 0)$ *given by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L\ (t_k - t_{k-1})^2}{2\ (1 - L_0\ t_k)} \qquad (k \geq 1), \tag{18}$$

is nondecreasing, bounded from above by $t^{\star\star}$*, and converges to its unique least upper bound* $t^\star \in [0, t^{\star\star}]$*, where*

$$t^{\star\star} = \frac{2\ \eta}{2 - \delta}, \tag{19}$$

$$\delta = \frac{4\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} < 2 \quad \text{for } L_0 \neq 0. \tag{20}$$

Moreover, the following estimates hold:

$$L_0\ t^\star \leq 1, \tag{21}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}\ (t_k - t_{k-1}) \leq \ldots \leq \left(\frac{\delta}{2}\right)^k \eta \quad (k \geq 1), \tag{22}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\ q_A)^{2^k - 1}\ \eta \quad (k \geq 0), \tag{23}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\ q_A)^{2^k - 1}\ \eta}{1 - (2\ q_A)^{2^k}} \quad (2\ q_0 < 1), \quad (k \geq 0). \tag{24}$$

We shall show the following semilocal convergence theorem for (SM).

Theorem 2.3. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{X}$ *be a Fréchet–differentiable operator,* $g : \mathcal{D} \longrightarrow \mathcal{X}$ *be a continuous operator,* $[x, y; F]$ *be a divided difference of order one of* F *on* $\mathcal{D}$*, satisfying* (3)*, and let* $A(x) \in \mathcal{L}(\mathcal{X})$ *given in* (2)*. Assume that there exist* $x_0 \in \mathcal{D}$*, and constants* $\alpha \geq 0$*,* $\beta > 0$*,* $\mu > 0$*,* $M_0 \geq 0$*,* M *satisfying* (6)*,* $M_1 > 0$*, and* $M_2 > 0$*, such that for all* $x, y \in \mathcal{D}$*:*

$$\Gamma = F'(x_0)^{-1} \in \mathcal{L}(\mathcal{X}), \tag{25}$$

$$\| \Gamma\ F(x_0) \| \leq \eta, \tag{26}$$

$$\| \Gamma\ (F'(x) - F'(y)) \| \leq M\ \ \| x - y \|, \tag{27}$$

$$\| \Gamma\ (F'(x) - F'(x_0)) \| \leq M_0\ \ \| x - x_0 \|, \tag{28}$$

$$\| \Gamma\ (A(x) - F'(x)) \| \leq M_1\ \ \| x - g(x) \|, \tag{29}$$

$$\| \Gamma\ ([x, y; F] - F'(x_0)) \| \leq M_2\ (\| x - x_0 \| + \| y - x_0 \|), \tag{30}$$

$$\| x - g(x) \| \leq \alpha\ \ \| \Gamma\ F(x) \|, \tag{31}$$

$$\eta \leq \eta^\star, \tag{32}$$

$$\overline{U}(x_0, r) = \{x \in \mathcal{X}\ : \| x - x_0 \| \leq r\} \subseteq \mathcal{D}, \tag{33}$$

where

$$r = \alpha\ \eta + t^\star, \tag{34}$$

and q_A, $\eta^\star$, $t^\star$ *are given in Definition* 2.1, *and Lemma* 2.2. *Then the sequence* $\{x_n\}$ $(n \geq 0)$ *generated by (SM) is well defined, remains in* $\overline{U}(x_0, r)$ *for all* $n \geq 0$, *and converges to a solution* $x^\star$ *of equation* $F(x) = 0$ *in* $\overline{U}(x_0, r)$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\| x_n - x^\star \| \leq t^\star - t_n, \tag{35}$$

where the sequence $\{t_n\}$ $(n \geq 0)$ *is given in Lemma* 2.2. *Furthermore, if*

$$t^\star \leq \frac{1}{2\ M_2} \tag{36}$$

then the vector $x^\star$ *is the only solution of equation* $F(x) = 0$ *in* $U(x_0, R)$, *where*

$$R \in \left[t^\star,\ \frac{1}{M_2} - t^\star\right]. \tag{37}$$

Proof. We shall show using induction on $n \geq 0$:

$$\| x_{n+1} - x_n \| \leq t_{n+1} - t_n, \tag{38}$$

$$\overline{U}(x_{n+1}, t^\star - t_{n+1}) \subseteq \overline{U}(x_n, t^\star - t_n), \tag{39}$$

$$\| \Gamma\ F(x_n) \| \leq t_{n+1} - t_n, \tag{40}$$

$$\| g(x_n) - x_0 \| \leq r. \tag{41}$$

For every $z \in \overline{U}(x_1, t^\star - t_1)$,

$$\| z - x_0 \| \leq \| z - x_1 \| + \| x_1 - x_0 \| \leq t^\star - t_1 + t_1 - t_0 = t^\star - t_0,$$

implies $z \in \overline{U}(x_0, t^\star - t_0)$. We also have $\| x_1 - x_0 \| = \| \Gamma\ F(x_0) \| \leq \eta = t_1 - t_0$ by (18), (26). In view of (31), and (40), we get

$$\| g(x_0) - x_0 \| \leq \alpha\ \ \| \Gamma\ F(x_0) \| \leq \alpha\ \eta \leq \alpha\ \eta + t^\star = r.$$

Hence, estimates (38)–(41) hold for $n = 0$. Let us assume these estimates hold for all integers $k \leq n$. Then we have

$$\| x_{n+1} - x_0 \| \leq \sum_{i=1}^{n+1} \| x_i - x_{i-1} \| \leq \sum_{i=1}^{n+1} (t_i - t_{i-1}) = t_{n+1} - t_0 = t_{n+1} \leq t^\star,$$

$$\| x_n + \theta\,(x_{n+1} - x_n) - x_0 \| \leq t_n + \theta\,(t_{n+1} - t_n) \leq t^\star,$$

for all $\theta \in (0, 1)$, and

$$\begin{aligned} \| g(x_n) - x_0 \| &\leq \| g(x_n) - x_n \| + \| x_n - x_0 \| \leq \alpha\ \ \| \Gamma\, F(x_n) \| + t^\star \\ &\leq \alpha\ (t_{n+1} - t_n) + t^\star \leq \alpha\ \eta + t^\star = r. \end{aligned}$$

For simplicity, denote $x_{\theta,n} = x_n + \theta\,(x_{n+1} - x_n)$ and $\Lambda_n = x_{n+1} - x_n$. In view of (SM), we obtain the identity:

$$F(x_{n+1}) = (F'(x_n) - A_n)\,\Lambda_n + \int_0^1 (F'(x_{\theta,n}) - F'(x_n))\,\Lambda_n\,\mathrm{d}t. \tag{42}$$

Then using the induction hypotheses, (27)–(31), (42), we have in turn:

$$\begin{aligned}
&\| \Gamma F(x_{n+1}) \| \\
&= \| \Gamma \left((F'(x_n) - A_n) \Lambda_n + \int_0^1 (F'(x_{\theta,n}) - F'(x_n)) \Lambda_n \, \mathrm{d}t \right) \| \\
&\leq \| \Gamma (F'(x_n) - A_n) \Lambda_n \| + \int_0^1 \| \Gamma (F'(x_{\theta,n}) - F'(x_n)) \Lambda_n \| \, \mathrm{d}t \\
&\leq M_1 \| x_n - g(x_n) \| \| \Lambda_n \| + \frac{\overline{M}}{2} \| \Lambda_n \|^2 \qquad (43) \\
&\leq \alpha \, M_1 \| \Gamma F(x_n) \| \| \Lambda_n \| + \frac{\overline{M}}{2} \| \Lambda_n \|^2 \\
&\leq (\alpha M_1 + \frac{\overline{M}}{2}) (t_{n+1} - t_n)^2 = \frac{1}{2} (2 \alpha M_1 + \overline{M}) (t_{n+1} - t_n)^2 \\
&\leq \frac{1}{2} L_1 (t_{n+1} - t_n)^2,
\end{aligned}$$

where

$$\overline{M} = \begin{cases} M_0 & \text{if} \quad n = 0 \\ M & \text{if} \quad n > 0 \end{cases} \quad \text{and} \quad L_1 = \begin{cases} L_0 & \text{if} \quad n = 0 \\ L & \text{if} \quad n > 0. \end{cases} \qquad (44)$$

In view of (9), (21), (28)–(32), and the induction hypotheses, we obtain:

$$\begin{aligned}
&\| \Gamma (A_{n+1} - F'(x_0)) \| \\
&= \| \Gamma \left((A_{n+1} - F'(x_{n+1})) + (F'(x_{n+1}) - F'(x_0)) \right) \| \\
&\leq \| \Gamma (A_{n+1} - F'(x_{n+1})) \| + \| \Gamma (F'(x_{n+1}) - F'(x_0)) \| \\
&\leq M_1 \| g(x_{n+1}) - x_{n+1} \| + M_0 \| x_{n+1} - x_0 \| \\
&\leq \alpha M_1 \| \Gamma F(x_{n+1}) \| + M_0 \| x_{n+1} - x_0 \| \qquad (45) \\
&\leq \frac{\alpha M_1 L_1}{2} (t_{n+1} - t_n)^2 + M_0 t_{n+1} \\
&\leq \left(\frac{\alpha M_1 L_1}{2} (t_{n+1} - t_n) + M_0 \right) t_{n+1} \\
&\leq \left(\frac{\alpha M_1 L_1}{2} \eta + M_0 \right) t_{n+1} \leq L_0 t_{n+1} < 1.
\end{aligned}$$

It follows from (45), and the Banach lemma on invertible operators [4], [6] that A_{n+1}^{-1} exists, such that

$$\| A_{n+1}^{-1} F'(x_0) \| \leq (1 - L_0 \, t_{n+1})^{-1}. \qquad (46)$$

Using (2), we obtain the approximation $x_{n+2} - x_{n+1} = -A_{n+1}^{-1} F(x_{n+1}) = -(A_{n+1}^{-1} F'(x_0)) (\Gamma F(x_{n+1}))$. Using (43), and (46), we get:

$$\begin{aligned}
\| x_{n+2} - x_{n+1} \| &\leq \| A_{n+1}^{-1} F'(x_0) \| \| \Gamma F(x_{n+1}) \| \\
&\leq \frac{L \, (t_{n+1} - t_n)^2}{2 \, (1 - L_0 \, t_{n+1})} = t_{n+2} - t_{n+1},
\end{aligned} \qquad (47)$$

which completes the induction for (38).

Thus, for every $z \in \overline{U}(x_{n+2}, t^\star - t_{n+2})$, we have:

$$\begin{aligned} \| z - x_{n+1} \| &\leq \| z - x_{n+2} \| + \| x_{n+2} - x_{n+1} \| \\ &\leq t^\star - t_{n+2} + t_{n+2} - t_{n+1} = t^\star - t_{n+1}, \end{aligned}$$

which implies $z \in \overline{U}(x_{n+1}, t^\star - t_{n+1})$. That is (39) holds for $n+1$ replacing n.

Furthemore, it follows from (21), and (47):

$$\| \Gamma \ F(x_{n+1}) \| \leq (1 - L\, t_{n+1})\,(t_{n+2} - t_{n+1}) \leq t_{n+2} - t_{n+1}, \tag{48}$$

which completes the induction for (40).

We also have:

$$\begin{aligned} \| g(x_{n+1}) - x_0 \| &\leq \| g(x_{n+1}) - x_{n+1} \| + \| x_{n+1} - x_0 \| \\ &\leq \alpha \ (t_{n+2} - t_{n+1}) + r \leq \alpha \ (t_1 - t_0) + t^\star = \alpha \ \eta + t^\star = r. \end{aligned}$$

Hence, the induction for (38)–(41) is completed.

It follows from Lemma 2.2 that the sequence $\{t_n\}$ is Cauchy. In view of (38) and (39), the sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy too in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, r)$ (since $\overline{U}(x_0, r)$ is a closed set). By letting $n \longrightarrow \infty$ in (43), we obtain $F(x^\star) = 0$. Estimate (35) follows from (38) by using standard majorization techniques [4], [6].

To show the uniqueness part, let $y^\star \in U(x_0, R)$ be a solution of equation $F(x) = 0$. Using (30), (36), and (37), we get:

$$\begin{aligned} \| \Gamma \ ([x^\star, y^\star; F] - F'(x_0)) \| &\leq M_2 \ (\| x^\star - x_0 \| + \| y^\star - x_0 \|) \\ &< M_2 \ (t^\star + R) \leq 1. \end{aligned} \tag{49}$$

It follows by (49), and the Banach lemma on invertible operators that the linear operator $[x^\star, y^\star; F]$ is invertible. Then using the estimate $0 = F(x^\star) - F(y^\star) = [x^\star, y^\star; F]\,(x^\star - y^\star)$, we deduce $x^\star = y^\star$. That completes the proof of Theorem 2.3. $\square$

REMARK 2.4. (a) The number $t^\star$ can be replaced by $t^{\star\star}$ (given in closed form in (19)) in (33), (34), (36), and (37).

(b) Conditions (27), (28), and (30) certainly hold if replaced by the stronger

$$\| \Gamma\,([x, y; F] - F'(z)) \| \leq M_3 \ (\| x - z \| + \| y - z \|), \tag{50}$$

for all $x, y, z \in \mathcal{D}$, and some $M_3 > 0$. In this case, we can set $M = 2\,M_3$, and $M_0 = 2\,M_2$ or $M_0 = 2\,M_3$. Note also that $M_2 \leq M_3$.

(c) If $g(x) = x$ $(x \in \mathcal{D})$, then (SM) reduces to (NM). Then we can set $\alpha = M_1 = 0$, $L_0 = M_0$, $M_2 = \dfrac{M_0}{2}$, and $L = M$. Then it is simple algebra to show that the sufficient convergence condition (32) reduces to (17). The corresponding to (17) Newton–Kantorovich hypothesis [4], [6] is given by

$$q_K = L \ \eta \leq \frac{1}{2}. \tag{51}$$

Note that in general $M_0 \leq M$ (i.e., $L_0 \leq L$) holds, and $\dfrac{M}{M_0}$ can be arbitrarily large [3]–[6]. Note also that $q_K \leq \dfrac{1}{2}$ implies that $q_A \leq \dfrac{1}{2}$, but not necessarily vice versa unless $M_0 = M$. In [3]–[6], we have provided numerical examples, where (17) holds but (51) is violated, and $L_0 < L$. The Newton–Kantorovich [4], [6] majorizing sequence corresponding to (18) is

$$s_0 = 0, \quad s_1 = \eta, \quad s_{n+2} = s_{n+1} + \frac{M\ (s_{n+1} - s_n)^2}{2\ (1 - M\ s_{n+1})} \qquad (n \geq 0). \tag{52}$$

A simple inductive argument shows that under (51) if $L_0 < L$, then $t_n < s_n$ $(n > 1)$, $t_{n+1} - t_n < s_{n+1} - s_n$ $(n \geq 1)$, $t^\star - t_n < s^\star - s_n$ $(n \geq 1)$ and $t^\star < s^\star$, where $s^\star = \lim_{n \to \infty} s_n$.

Note also that estimates (23), and (24) are also finer, since $q_A < q_K$. Hence, under the same computational cost, the applicability of (NM) is expanded under our approach. Note that in the general case of (SM) (i.e., $g(x) \neq x$, $x \in \mathcal{D}$), we do not have a related Kantorovich–type theorem to compare our results. The existing sufficient convergence conditions for (SM) involve stronger hypotheses than ours [1], [7]–[14].

(d) In view of the proof of Theorem 2.3, it follows (see (43), and (44)) that the scalar sequence $\{r_n\}$ given by

$$r_0 = 0, \quad r_1 = \eta, \quad r_2 = r_1 + \frac{L_0\ (r_1 - r_1)^2}{2\ (1 - L_0\ r_1)}, \quad r_{n+2} = r_{n+1} + \frac{L\ (r_{n+1} - r_n)^2}{2\ (1 - L_0\ r_{n+1})}$$

$(n \geq 1)$ is also majorizing sequence for $\{x_n\}$. Moreover, for $L_0 < L$, we have $r_n < t_n$ $(n > 1)$, $r_{n+1} - r_n < t_{n+1} - t_n$ $(n \geq 1)$, $r^\star - r_n < t^\star - t_n$ $(n \geq 1)$, and $r^\star \leq t^\star$, where $r^\star = \lim_{n \to \infty} r_n$.

(e) If $\mathcal{X} = \mathbb{R}^k$, then the divided difference $[x, y; F]$ can be given by $J_n := [x_n, g(x_n); F] = (F(x_n + D_n\, e^1) - F(x_n), \ldots, F(x_n + D_n\, e^n) - F(x_n))\ D_n^{-1}$ $(n \geq 0)$, where $D_n = \operatorname{diag}(f_1(x_n), f_2(x_n), \ldots, f_k(x_n))$, $F(x) = (f_1(x), f_2(x), \ldots, f_k(x))$, $f_i : \mathbb{R}^k \longrightarrow \mathbb{R}$ $(i = 1, \ldots, k)$ and $x = (x_1, x_2, \ldots, x_k)$. (SM) in the form $x_{n+1} = x_n - J_n^{-1}\ F(x_n)$ $(n \geq 0)$ avoids the evaluation of $F'(x_n)^{-1}$ which may be too expensive or impossible to invert, but has the same convergence order with (NM).

At the end of the study, we provide a numerical example, where we show that (SM) can be faster than (NM).

3. EXAMPLE

Example 3.1 ([8]). We consider the non-linear second order boundary value problem:

$$\frac{d^2 w(t)}{d\, t^2} = e^{w(t)}, \quad w(0) = w(1) = 0. \tag{53}$$

As in [2], we approximate the second derivative by

$$(54)\qquad w'' \approx \frac{w_{i-1} - 2\,w_i + w_{i+1}}{h^2}, \quad i = 1, \ldots, k-1, \quad q_i = i\,h, \quad h = \frac{1}{k}.$$

In view of (54), equation (53) becomes $F(w) = A\ w - h^2\ p(w)$, where $F : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}^{k-1}$,

$$A = \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & 1 & -2 \end{pmatrix},$$

$w = (w_1, w_2, \ldots, w_{k-1})^T$ and $p(x) = (e^{w_1}, e^{w_2}, \ldots, e^{w_{k-1}})^T$. Let us solve equation $F(w) = 0$, using (SM), and (NM) (5) for comparison reasons. Let us choose $k = 12$, and initial iterate $w_i(0) = q_i\,(q_i - 1)$, $i = 1, \ldots, 11$. Then we get the following table, which justifies the claim made at the end of Section 2:

Comparison Table between (SM) and (NM) on $\parallel F(w^{(k)}) \parallel$

n	(SM) : (2)	(NM): (5)
1	1.4448e-009	1.9887e-009
2	1.9664e-017	4.5194e-017

REMARK 3.2. (a) Under the hypotheses of Theorem 2.3, we have $M_0 \leq L_0$ and $M \leq L$ (by (7), and (8)). It then follows from (15), (17), and (32) that (NM) defined by (5) converges. Moreover, the predicted estimates by (23), and (24) show that (NM) is faster than (SM). (b) We can justify the results of the Comparison Table on $\parallel F(x_n) \parallel$ (i.e., on $\parallel F'(x_0)^{-1}\ F(x_n) \parallel$). In view of (43), (48), we have

$$(55)\quad \parallel \Gamma\, F(x_n) \parallel \leq \left(M\ \parallel x_n - g(x_n) \parallel + \frac{\overline{M}}{2}\ \parallel x_{n+1} - x_n \parallel \right)\ \parallel x_{n+1} - x_n \parallel,$$

$$(56)\qquad \parallel \Gamma\, F(x_{n+1}) \parallel \leq (1 - L_0\ t_{n+1})\ (t_{n+2} - t_{n+1}),$$

respectively. Moreover, we have $\parallel F'(y_0)^{-1}\, F(y_{n+1}) \parallel \leq \dfrac{M}{2} \parallel y_{n+1} - y_n \parallel^2$ and $\parallel F'(y_0)^{-1}\, F(y_{n+1}) \parallel \leq (1 - M\ s_{n+1})\ (s_{n+2} - s_{n+1})$, for (5). It then follows that if e.g. $M < L_0$, $t_n \approx s_n$ $(n \geq 0)$, at least theoretically predicted upper bounds on $\parallel \Gamma\, F(x_{n+1}) \parallel$ can be smaller than $\parallel \Gamma\, F(y_{n+1}) \parallel$. This observation explains at least theoretically the results of the Comparison Table.

CONCLUSION

We presented a convergence analysis of Steffensen's method to solve a nonlinear equations in Banach spaces under some Lipschitz–type conditions and using the divided difference operator of order one.

A numerical example of non–linear second order boundary value problem, further validating the theoretical results, a comparison between Steffensen's and Newton's method, and some remarks are also presented in this study.

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