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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper we study the uniqueness of meromorphic functions concerning differential polynomials, proving the following theorem: Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, k be two positive integers with $n \geq 12k+20$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 IM (ignoring multiplicities), then either $[f^n(z)(f(z)-1)]^{(k)}[g^n(z)(g(z)-1)]^{(k)} \equiv 1$ or $f(z) \equiv g(z)$. This generalizes and improves some results given by M.L. Yang, S.S. Bhoosnurmath and R.S. Dyavanal.

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1. INTRODUCTION AND RESULTS

Let f be a nonconstant meromorphic function defined in the whole complex plane. We use the standard notations in Nevanlinna theory of meromorphic functions such as the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$ and the proximity function $m(r, f)$ and so on. For any nonconstant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set of r of finite linear measure. We refer the reader to Hayman [2], Yang [4], Yi and Yang [5] and for more details.

Let f and g be two nonconstant meromorphic functions. Let a be a finite complex number. We say that f and g share the value a CM (counting multiplicities) if f and g have the same a-points with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $N_{11}\left(r, \frac{1}{f-1}\right)$ the counting function for common simple 1-points of f and g where multiplicity is not counted. $\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right)$ is the counting function for 1-points of both $f^{(k)}$ and $g^{(k)}$ about which $f^{(k)}$ has larger multiplicity than $g^{(k)}$, with multiplicity being not counted. For any constant a, we define

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

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Let f be a nonconstant meromorphic function, a a finite complex number and k a positive integer. We denote by $N_{k)}\left(r,\frac{1}{f-a}\right)$ (or $\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)$) the counting function for zeros of $f-a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $N_{(k}\left(r,\frac{1}{f-a}\right)$(or $\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$) the counting function for zeros of $f-a$ with multiplicity at least k (ignoring multiplicities). Set

$$N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\ldots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

We further define $\delta_k(a,f)=1-\overline{\lim\limits_{r\to\infty}}\frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)}$.

Fang [3] proved the following result.

THEOREM 1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n,k be two positive integers with $n\geq 2k+8$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 CM, then $f(z)\equiv g(z)$.*

Recently, S.S Bhoosnurmath[1] and R.S. Dyavanal extended Theorem 1 and proved the following theorem.

THEOREM 2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions satisfying $\Theta(\infty,f)>\frac{3}{n+1}$, and let n,k be two positive integers with $n\geq 3k+13$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 CM, then $f(z)\equiv g(z)$.*

It is natural to ask the following question: what can be said if CM shared value is replaced by an IM shared value in Theorem 1 and 2? In this paper, we answer the question by proving the following theorem.

THEOREM 3. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions satisfying $\Theta(\infty,f)>\frac{2}{n}$, let n , k be two positive integers with $n\geq 12k+20$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 IM, then either $[f^n(z)(f(z)-1)]^{(k)}[g^n(z)(g(z)-1)]^{(k)}\equiv 1$ or $f(z)\equiv g(z)$.*

2. SOME LEMMAS

For the proof of our result we need the following lemmas.

LEMMA 1. (See [2]) *Let f be nonconstant meromorphic function, and let $a_0,a_1,\ldots,a_n$ be finite complex numbers such that $a_n\neq 0$. Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\ldots+a_0)=nT(r,f)+S(r,f).$$

LEMMA 2. (See [2]) *Let f be a nonconstant meromorphic function, k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned}T(r,f)&\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\\&\leq\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f).\end{aligned}$$

Here $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ *is the counting function which only counts those points such that* $f^{(k+1)} = 0$ *but* $f(f^{(k)} - c) \neq 0$.

LEMMA 3. (See [5]) *Let* f *be a transcendental meromorphic function, and let* $a_1(z)$, $a_2(z)$ *be two meromorphic functions such that* $T(r, a_i) = S(r, f), i = 1, 2$. *Then* $T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r, f)$.

LEMMA 4. (See [6]) *Let* f *be a nonconstant meromorphic function, and* k, p *be two positve integers. Then* $N_p\left(r\frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r\frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f)$. *Clearly* $\overline{N}\left(r\frac{1}{f^{(k)}}\right) = N_1\left(r\frac{1}{f^{(k)}}\right)$.

LEMMA 5. *Let* $f(z)$ *and* $g(z)$ *be two meromorphic functions, and let* k *be a positive integer. If* $f^{(k)}$ *and* $g^{(k)}$ *share the value 1 IM and*

$$\begin{aligned}\Delta = (2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + (k+2)\Theta(0, f) \\ + (2k+3)\Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 7k + 13,\end{aligned} \tag{1}$$

then either $f^{(k)}g^{(k)} \equiv 1$ *or* $f \equiv g$.

Proof. Let

$$h(z) = \frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z) - 1} - \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} + 2\frac{g^{(k+1)}(z)}{g^{(k)}(z) - 1}. \tag{2}$$

If z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2), we see that z_0 is a zero of $h(z)$. Thus, we have:

$$\begin{aligned}N_{11}\left(r, \frac{1}{f^{(k)} - 1}\right) = N_{11}\left(r, \frac{1}{g^{(k)} - 1}\right) \leq \overline{N}\left(r, \frac{1}{h}\right) \\ \leq T(r, h) + O(1) \leq N(r, h) + S(r, f) + S(r, g).\end{aligned} \tag{3}$$

By our assumptions, $h(z)$ has poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g, and those 1-points of $f^{(k)}$ and $g^{(k)}$ whose multiplicities are distinct from the multiplicities of corresponding 1-points of $g^{(k)}$ and $f^{(k)}$ respectively. Thus, we deduce from (2) that

$$\begin{aligned}N(r, h) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) \\ + N_0\left(r, \frac{1}{g^{(k+1)}}\right) + \overline{N}_L\left(r, \frac{1}{f^{(k)} - 1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)} - 1}\right).\end{aligned} \tag{4}$$

Here $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ has the same meaning as in Lemma 2. By Lemma 2, we have:

$$T(r, f) \leq \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f), \tag{5}$$

$$(6)\quad T(r,g) \le \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-c}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g).$$

Since $f^{(k)}$ and $g^{(k)}$ share the value 1 IM, we have:

$$(7)\quad \begin{aligned} &\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) \\ &\le N_{11}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right) \\ &\le N_{11}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) + T(r,f^{(k)}) + O(1) \\ &\le N_{11}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) + m(r,f) \\ &+ m\left(r,\frac{f^{(k)}}{f}\right) + N(r,f) + k\overline{N}(r,f) + S(r,f) \\ &\le N_{11}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) + T(r,f) + k\overline{N}(r,f) + S(r,f). \end{aligned}$$

Note that by Lemma 4 we have:

$$(8)\quad \begin{aligned} \overline{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r\frac{1}{f^{(k)}}\right) &\le N_{1+k}\left(r\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \\ &\le (k+1)\overline{N}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f), \end{aligned}$$

$$\begin{aligned} \overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right) &\le N\left(r,\frac{1}{f^{(k)}-1}\right) - \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) \le N\left(r,\frac{f^{(k)}}{f^{(k+1)}}\right) \\ &\le N\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) + S(r,f) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \end{aligned}$$

So, we have:

$$(9)\quad \overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right) \le (k+1)\overline{N}(r,f) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + S(r,f).$$

Similarly

$$(10)\quad \overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right) \le (k+1)\overline{N}(r,g) + (k+1)\overline{N}\left(r,\frac{1}{g}\right) + S(r,g).$$

From (3)–(10) we obtain:

$$\begin{aligned} T(r,g) &\le (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r,\frac{1}{f}\right) \\ &+ (2k+3)\overline{N}\left(r,\frac{1}{g}\right) + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \le T(r,g)$ for $r \in I$. Hence

$$\begin{aligned} T(r,g) \le \{[(7k+14) - (2k+3)\Theta(\infty,f) - (2k+4)\Theta(\infty,g) - (k+2)\Theta(0,f) \\ - (2k+3)\Theta(0,g) - \delta_{k+1}(0,f) - \delta_{k+1}(0,g)] + \varepsilon\}T(r,g) + S(r,g), \end{aligned} \tag{11}$$

for $r \in I$ and $0 < \varepsilon < \Delta - (7k+13)$. Thus, we obtain from (1) and (11) that $T(r,g) \le S(r,g)$for $r \in I$, a contradiction. Hence, we get $h(z) \equiv 0$; that is:

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} = \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}.$$

By solving this equation, we obtain:

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}, \tag{12}$$

where a, b are two constants. Next, we consider three cases.

Case 1: $b \neq 0$ and $a = b$.

Subcase 1: $b = -1$. Then we deduce from (12) that $f^{(k)}(z)g^{(k)}(z) \equiv 1$.

Subcase 2. $b \neq -1$. Then we get from (12) that $\frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)}-1}$, and so

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \le \overline{N}\left(r, \frac{1}{f^{(k)}}\right) \tag{13}$$

From (13) and (8), we get:

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \le (k+1)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

By Lemma 2 we have:

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{b+1}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) \\ &\le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + k\overline{N}(r,f) + (k+1)\overline{N}\left(r, \frac{1}{f}\right) + S(r,f) + S(r,g) \\ &\le (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r, \frac{1}{f}\right) + (2k+3)\overline{N}\left(r, \frac{1}{g}\right) \\ &+ N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g). \end{aligned}$$

That is $T(r,g) \le (7k+14-\Delta)T(r,g) + S(r,g)$ for $r \in I$. Thus, by (1), we obtain that $T(r,g) \le S(r,g)$ for $r \in I$, a contradiction.

Case 2: $b \neq 0$ and $a \neq b$.

Subcase 1. $b = -1$. Then we obtain from (12) that $f^{(k)} = \frac{a}{-g^{(k)}+a+1}$, so

$$\overline{N}\left(r, \frac{a}{-g^{(k)}+a+1}\right) = \overline{N}\left(r, f^{(k)}\right) = \overline{N}(r,f).$$

By Lemma 2 we have:

$$
\begin{aligned}
T(r,g) &\le \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-(a+1)}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g)\\
&\le \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) + S(r,g)\\
&\le (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r,\frac{1}{f}\right) + (2k+3)\overline{N}\left(r,\frac{1}{g}\right)\\
&+ N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g).
\end{aligned}
$$

Using an argument as in Case 1, we get a contradiction.

Subcase 2. $b \neq -1$. Then we get from (12) that $f^{(k)} - (1+\frac{1}{b}) = \frac{-a}{b^2(g^{(k)}+\frac{a-b}{b})}$. Therefore

$$\overline{N}\left(r,\frac{1}{g^{(k)}+\frac{a-b}{b}}\right) = \overline{N}\left(r, f^{(k)} - (1+\frac{1}{b})\right) = \overline{N}(r,f).$$

By Lemma 2, we have:

$$
\begin{aligned}
T(r,g) &\le \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}+\frac{a-b}{b}}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g)\\
&\le \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) + S(r,g)\\
&\le (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r,\frac{1}{f}\right) + (2k+3)\overline{N}\left(r,\frac{1}{g}\right)\\
&+ N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g).
\end{aligned}
$$

Using an argument as in Case 1, we get a contradiction.

Case 3: $b = 0$. From (12), we obtain:

$$f = \frac{1}{a}g + P(z), \tag{14}$$

where $P(z)$ is a polynomial. If $P(z) \neq 0$, then by Lemma 3 we have:

$$
\begin{aligned}
T(r,f) &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-P}\right) + S(r,f)\\
&\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f).
\end{aligned} \tag{15}
$$

From (14), we obtain $T(r,f) = T(r,g) + S(r,f)$. Hence, substituting this into (15), we get:

$$T(r,f) \le \{3 - [\Theta(\infty,f) + \Theta(0,f) + \Theta(0,g)] + \varepsilon\}\, T(r,f) + S(r,f),$$

where

$$0<\varepsilon<1-\delta_{k+1}(0,f)+1-\delta_{k+1}(0,g)+(2k+2)[1-\Theta(\infty,f)] \\ +(2k+4)[1-\Theta(\infty,g)]+[1-\Theta(0,f)]+2[1-\Theta(0,g)].$$

Therefore $T(r,f)\le[7k+14-\Delta]T(r,f)+S(r,f)$. Then $[\Delta-(7k+13)]T(r,f)<S(r,f)$. Hence, by (1), we deduce that $T(r,f)\le S(r,f)$ for $r\in I$, a contradiction. Therefore, we deduce that $P(z)\equiv 0$, that is $f=\frac{1}{a}g$. If $a\neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 IM, we deduce that $g^{(k)}\neq 1$. That is $\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)=0$. Next, we can deduce a contradiction as in Case 1. Thus, we get that $a=1$, that is $f\equiv g$. Thus the proof of Lemma 5 is completed. □

3. PROOF OF THEOREM 3

Let $F(z)=f^n(f-1)$ and $G(z)=f^n(f-1)$. We have:

$$\Delta=(2k+3)\Theta(\infty,F)+(2k+4)\Theta(\infty,G)+(k+2)\Theta(0,F) \\ +(2k+3)\Theta(0,G)+\delta_{k+1}(0,F)+\delta_{k+1}(0,G).$$

Consider

$$\Theta(0,F)=1-\varlimsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f^n(f-1)}\right)}{(n+1)T(r,F)}=1-\varlimsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)}{(n+1)T(r,f)} \\ \ge 1-\varlimsup_{r\to\infty}\frac{2T(r,f)}{(n+1)T(r,f)}\ge\frac{n-1}{n+1}.$$

Similarly we have:

$$\Theta(0,G)\ge\frac{n-1}{n+1},\quad \Theta(\infty,F)\ge\frac{n}{n+1},\quad \Theta(\infty,G)\ge\frac{n}{n+1}.$$

Next, it follows that

$$\delta_{k+1}(0,F)=1-\varlimsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{F})}{T(r,F)}=1-\varlimsup_{r\to\infty}\frac{(k+1)N(r,\frac{1}{f^n(f-1)})}{(n+1)T(r,f)} \\ \ge 1-\varlimsup_{r\to\infty}\frac{(k+2)T(r,f)}{(n+1)T(r,f)}\ge 1-\frac{k+2}{n+1}=\frac{n-(k+1)}{n+1}.$$

Similarly $\delta_{k+1}(0,G)\ge\frac{n-(k+1)}{n+1}$. From the above equalities we get:

$$\Delta=(2k+3)\frac{n}{n+1}+(2k+4)\frac{n}{n+1}+(k+2)\frac{n-1}{n+1} \\ +(2k+3)\frac{n-1}{n+1}+\frac{n-(k+1)}{n+1}+\frac{n-(k+1)}{n+1}.$$

Since $n>12k+20$, we get $\Delta>7k+13$. Considering $F^{(k)}(z)=[f^n(z)]^{(k)}$ and $G^{(k)}(z)=[g^n(z)]^{(k)}$ share the value 1 IM, then by Lemma 5 we deduce that either $F^{(k)}(z)G^{(k)}(z)\equiv 1$ or $F\equiv G$.

Next we consider two cases.

Case 1. $F^{(k)}(z)G^{(k)}(z) \equiv 1$, so $[f^n(z)(f(z)-1)]^{(k)}[g^n(z)(g(z)-1)]^{(k)} \equiv 1$.

Case 2. $F \equiv G$, so $f^n(f-1) \equiv g^n(g-1)$.

Suppose $f \neq g$. Then we consider two cases:

(i) Let $H = \frac{f}{g}$ be a constant. Then it follows that $H \neq 1, H^n \neq 1, H^{n+1} \neq 1$ and $g = \frac{1-H^n}{1-H^{n+1}}$ is a constant, which leads to a contradiction.

(ii) Let $H = \frac{f}{g}$ be not a constant. Since $f \neq g$, we have $H \neq 1$ and hence we deduce that $g = \frac{1-H^n}{1-H^{n+1}}$ and $f = \frac{1-H^n}{1-H^{n+1}}H = \frac{(1+H+H^2+\cdots+H^{n-1})H}{1+H+H^2+\cdots+H^n}$, where H is a non-constant meromorphic function. It follows that $T(r,f) = T(r,gH) = nT(r,H) + S(r,f)$.

On the other hand, by the second fundamental theorem, we deduce that $\overline{N}(r,f) = \sum\limits_{j=1}^{n} \overline{N}\left(r, \frac{1}{H-\alpha_j}\right) \geq (n-2)T(r,H) + S(r,f)$, where $\alpha_j \neq 1$ $(j = 1, 2, \ldots, n)$ are distinct roots of the algebraic equation $H^{n+1} = 1$. We have:

$$\begin{aligned}\Theta(\infty,f) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)} &\leq 1 - \overline{\lim_{r\to\infty}}\frac{(n-2)T(r,H)+S(r,f)}{T(r,f)} \\ &\leq 1 - \overline{\lim_{r\to\infty}}\frac{(n-2)T(r,H)+S(r,f)}{nT(r,H)+S(r,f)} \leq 1 - \frac{n-2}{n} = \frac{2}{n},\end{aligned}$$

which contradicts the assumption $\Theta(\infty,f) > \frac{2}{n}$.

Thus $f \equiv g$. This completes the proof.

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