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ON STRONGLY STARLIKE AND CONVEX FUNCTIONS OF ORDER α AND TYPE β

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Abstract. In this note we investigate the inclusion relationship between the class of strongly starlike functions of order α and type β, $\alpha \in (0,1]$ and $\beta \in [0,1)$, which satisfy

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

and the class of strongly convex functions of order α and type β which satisfy

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

in the unit disk, where f is an analytic function defined on the unit disk and satisfies $f(0) = f'(0) - 1 = 1$. Some applications of our main result are also presented which contains various classical results for the typical subclasses of starlike and convex functions.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Let α be a real number with $\alpha \in (0,1]$. A function $f \in \mathcal{A}$ is called *strongly starlike of order* α if it satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha$$

for all $z \in \mathbb{D}$, and *strongly convex of order* α if

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha$$

for all $z \in \mathbb{D}$. Let us denote by $\mathcal{S}^*(\alpha)$ the class of functions strongly starlike of order α, and by $\mathcal{K}(\alpha)$ the class of functions strongly convex of order α. The class $\mathcal{S}^*(\alpha)$ was introduced first by Stankiewicz [13] and by Brannan and Kirwan [2], independently. It is clear from the definitions that $\mathcal{S}^*(\alpha_1) \subset \mathcal{S}^*(\alpha_2)$ and $\mathcal{K}(\alpha_1) \subset \mathcal{K}(\alpha_2)$ for $0 < \alpha_1 < \alpha_2 \leq 1$. The case when $\alpha = 1$, i.e., $\mathcal{S}^*(1)$ and $\mathcal{K}(1)$ correspond to well known classes of starlike and convex functions respectively, and therefore all the functions which belong to $\mathcal{S}^*(\alpha)$ or $\mathcal{K}(\alpha)$ are univalent in $\mathbb{D}$. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the classes of starlike and convex functions. For the general reference of classes of starlike and convex functions, see, for instance [3].

Mocanu [9] obtained the following result (see also [11]). Here, set

$$\rho(\alpha) = \mathrm{Tan}^{-1} \frac{\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \sin\left[\frac{\pi}{2}(1-\alpha)\right]}{1+\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \cos\left[\frac{\pi}{2}(1-\alpha)\right]} \tag{1}$$

and $\gamma(\alpha) = \alpha + \frac{2}{\pi}\rho(\alpha)$.

Theorem A. *$\mathcal{K}(\gamma(\alpha)) \subset \mathcal{S}^*(\alpha)$ for each $\alpha \in (0,1]$.*

We remark that the function $\gamma(\alpha)$ is continuous and strictly increases from 0 to 1 when α moves from 0 to 1. Further investigations for the above theorem can be found in [5].

Now we shall introduce the class of functions $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{K}(\alpha,\beta)$, $\alpha \in (0,1]$ and $\beta \in [0,1)$, whose members satisfy the conditions:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha \quad \text{and} \quad \left|\arg\left\{1 + \frac{zf''(z)}{f'(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha$$

for all $z \in \mathbb{D}$, respectively. We call a function $f \in \mathcal{S}^*(\alpha,\beta)$ *strongly starlike of order α and type β*. In the same way, a function $f \in \mathcal{K}(\alpha,\beta)$ is *strongly convex of order α and type β*. It is obvious that $\mathcal{S}^*(\alpha,0) = \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha,0) = \mathcal{K}(\alpha)$. Also the following relations are true from the definitions:

$$\begin{array}{rl} \text{i)} & \mathcal{S}^*(\alpha_1,\beta) \subset \mathcal{S}^*(\alpha_2,\beta), \\ \text{ii)} & \mathcal{K}(\alpha_1,\beta) \subset \mathcal{K}(\alpha_2,\beta), \\ \text{iii)} & \mathcal{S}^*(\alpha,\beta_1) \supset \mathcal{S}^*(\alpha,\beta_2), \\ \text{iv)} & \mathcal{K}(\alpha,\beta_1) \supset \mathcal{K}(\alpha,\beta_2), \end{array} \tag{2}$$

for $0 < \alpha_1 < \alpha_2 \le 1$ and $0 \le \beta_1 < \beta_2 < 1$. That is why all functions belong to $\mathcal{S}^*(\alpha,\beta)$ or $\mathcal{K}(\alpha,\beta)$ are univalent on $\mathbb{D}$.

A sufficient condition for which $f \in \mathcal{A}$ lies in $\mathcal{S}^*(\alpha,\beta)$ was proved by the second author et al. [12]. The authors also proposed in [12] the open problem about a inclusion relationship between $\mathcal{K}(\alpha,\beta)$ and $\mathcal{S}^*(\alpha,\beta)$. However, it seems that no results concerning this question have been known.

Our main result is the following:

Theorem 1. *$\mathcal{K}(\gamma(\alpha),\beta) \subset \mathcal{S}^*(\alpha,\beta)$ for each $\alpha \in (0,1]$ and $\beta \in [0,1)$.*

The above theorem includes Theorem A as the case when $\beta = 0$.

We should notice the reader that this estimation is not sharp for each $\alpha \in (0,1]$ and $\beta \in [0,1)$ (see also [5]). We will discuss about this problem in section 2 with the proof of Theorem 1. Our main theorem yields several applications which will be shown in the last section.

2. PROOF OF THEOREM 1

Our proof relies on the following lemma which was obtained by the second author [10, 11].

LEMMA B. *Let $p(z)$ be analytic and satisfies $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{D}$. Let us assume that there exists a point $z_0 \in \mathbb{D}$ such that $|\arg p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \pi\alpha/2$ where $\alpha > 0$. Then we have:*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{i}\alpha k,$$

where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$ when $\arg p(z_0) = \pi\alpha/2$ and $k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right)$ when $\arg p(z_0) = -\pi\alpha/2$, where $p(z_0)^{1/\alpha} = \pm\mathrm{i}a$ and $a > 0$.

The next result will be used later:

LEMMA 2. $\mathrm{Tan}^{-1}\alpha \geq \rho(\alpha)$ *for all $\alpha \in (0, 1]$, where ρ is defined by* (1).

Proof. Put $\phi(\alpha) = (1/(1-\alpha))\left((1-\alpha)/(1+\alpha)\right)^{\frac{1+\alpha}{2}}$. It is enough to prove that

$$\alpha \geq \frac{\alpha\phi(\alpha)\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\phi(\alpha)\cos\left[\pi(1-\alpha)/2\right]}$$

for all $\alpha \in (0, 1]$. Since $\phi(\alpha) < 1$, because of $\phi(0) = 1$ and $\phi'(\alpha) < 0$, we obtain $\alpha > \alpha\phi(\alpha)$ and therefore

$$\frac{\alpha\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\cos\left[\pi(1-\alpha)/2\right]} > \frac{\alpha\phi(\alpha)\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\phi(\alpha)\cos\left[\pi(1-\alpha)/2\right]}.$$

It remains to show that

$$\alpha \geq \frac{\alpha\sin\left[\pi(1-\alpha)/2\right]}{1 + \alpha\cos\left[\pi(1-\alpha)/2\right]}$$

for all $\alpha \in (0, 1]$ and this is clear. □

Proof of Theorem 1. Let us suppose that f satisfies the assumption of the theorem and let

$$p(z) = \frac{1}{1-\beta}\left(\frac{zf'(z)}{f(z)} - \beta\right).$$

Then $p(0) = 1$, and calculations show that

$$1 + \frac{zf''(z)}{f'(z)} - \beta = (1-\beta)p(z)\left\{1 + \frac{\frac{zp'(z)}{p(z)}}{(1-\beta)p(z) + \beta}\right\}. \tag{3}$$

We note that $p(z) \neq 0$ holds for all $z \in \mathbb{D}$ since $1 + zf''(z)/f'(z) - \beta \neq \infty$ on $\mathbb{D}$ from our assumption.

Now we derive a contradiction by using Lemma B. If there exists a point z_0 such that $|\arg p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \pi\alpha/2$, where $\alpha \in (0, 1]$, then by Lemma B, p must satisfy $z_0 p'(z_0)/p(z_0) = \mathrm{i}\alpha k$ where

$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right)$ when $\arg p(z_0) = \pi\alpha/2$ and $k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right)$ when $\arg p(z_0) = -\pi\alpha/2$, where $p(z_0)^{1/\alpha} = \pm \mathrm{i}a$ and $a > 0$.

First we suppose that $\arg p(z_0) = \pi\alpha/2$. Then from (3) we have:

$$\begin{aligned}\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\beta\right\} &= \arg\left[(1-\beta)p(z_0)\left\{1+\frac{\frac{z_0 p'(z_0)}{p(z_0)}}{(1-\beta)p(z_0)+\beta}\right\}\right] \\ &= \frac{\pi}{2}\alpha + \arg\left\{1+\frac{\mathrm{i}\alpha k}{(1-\beta)p(z_0)+\beta}\right\}.\end{aligned}$$

We shall estimate the second term of the second line of above. Geometric observations show that the point $1+[\mathrm{i}\alpha k/\{(1-\beta)p(z_0)+\beta\}]$ lies on the subarc C of the circle which passes through 1, $1+\mathrm{i}\alpha k$ and $1+[\mathrm{i}\alpha k/p(z_0)]$, where C connects $1+\mathrm{i}\alpha k$ and $1+[\mathrm{i}\alpha k/p(z_0)]$ and does not pass through 1. Further, we can find out that the value $\{\arg z : z \in C\}$ attains its minimum at the end points of C. Therefore we have:

$$\arg\left\{1+\frac{\mathrm{i}\alpha k}{(1-\beta)p(z_0)+\beta}\right\} \geq \min\left\{\arg\{1+i\alpha k\},\, \arg\left\{1+\frac{i\alpha k}{p(z_0)}\right\}\right\}. \tag{4}$$

Here, the first value in the above minimum can be evaluated by $\arg\{1+\mathrm{i}\alpha k\} \geq \mathrm{Tan}^{-1}\alpha$ since $k \geq 1$. For the second value, we note that $a^{1-\alpha}+a^{-1-\alpha}$ takes its minimum value at $a = \sqrt{(1+\alpha)/(1-\alpha)}$. Therefore

$$\begin{aligned}\arg\left\{1+\frac{\mathrm{i}\alpha k}{p(z_0)}\right\} &= \arg\left\{1+\mathrm{e}^{\frac{\pi}{2}(1-\alpha)\mathrm{i}}\cdot\frac{\alpha}{2}[a^{1-\alpha}+a^{-1-\alpha}]\right\} \\ &\geq \arg\left\{1+\mathrm{e}^{\frac{\pi}{2}(1-\alpha)\mathrm{i}}\cdot\frac{\alpha}{2}\left[\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{2}}+\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{-1-\alpha}{2}}\right]\right\} \\ &= \rho(\alpha).\end{aligned}$$

By Lemma 2 we conclude that

$$\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\beta\right\} \geq \frac{\pi}{2}\alpha + \min\left\{\mathrm{Tan}^{-1}\alpha,\, \rho(\alpha)\right\} = \frac{\pi}{2}\gamma(\alpha)$$

and this contradicts our assumption.

In the same fashion, if $\arg p(z_0) = -\pi\alpha/2$ then a similar argument shows that

$$\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\beta\right\} \leq -\frac{\pi}{2}\alpha + \max\left\{\mathrm{Tan}^{-1}(-\alpha),\, -\rho(\alpha)\right\} = -\frac{\pi}{2}\gamma(\alpha).$$

This also contradicts our assumption and our proof is completed. □

We remark that we expect this theorem to be room for improvement in our method, because the inequality (4) is a rough estimation except the case when $\beta = 0$, whereas it seems to be not easy to give a precise estimation for the left hand side of (4).

3. APPLICATIONS

We would like to give a further discussion to the relationship between $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{K}(\alpha,\beta)$ by using Theorem 1.

3.1. It is well known that a convex function is a starlike function, that is, $\mathcal{K} \subset \mathcal{S}^*$. Furthermore, Mocanu [8] showed that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$ for all $\alpha \in (0,1]$. Now we give the next result which includes these properties as special cases.

Corollary 3. *$\mathcal{K}(\alpha,\beta) \subset \mathcal{S}^*(\alpha,\beta)$ for each $\alpha \in (0,1]$ and $\beta \in [0,1)$.*

Proof. Since $\alpha \le \gamma(\alpha)$ for all $\alpha \in (0,1]$, $\mathcal{K}(\alpha,\beta) \subset \mathcal{K}(\gamma(\alpha),\beta) \subset \mathcal{S}^*(\alpha,\beta)$ by ii) in (2) and Theorem 1 which is our desired inclusion. $\square$

Corollary 3 yields the following property.

Corollary 4. *If $zf'(z) \in \mathcal{S}^*(\alpha,\beta)$, then $f \in \mathcal{S}^*(\alpha,\beta)$.*

Proof. It is obvious that $g \in \mathcal{K}(\alpha,\beta)$ if and only if $zg'(z) \in \mathcal{S}^*(\alpha,\beta)$. Thus if $zg'(z) \in \mathcal{S}^*(\alpha,\beta)$ then $g \in \mathcal{K}(\alpha,\beta) \subset \mathcal{S}^*(\alpha,\beta)$ from Corollary 3. Hence our assertion follows if we put $f(z) = zg'(z)$. $\square$

This corollary is equivalent to the following: *$\mathcal{S}^*(\alpha,\beta)$ is preserved by the Alexander transformation, where the Alexander transformation* [1] *is the integral transformation defined by $f(z) \mapsto \int_0^z \frac{f(u)}{u}du$ for $f \in \mathcal{A}$.*

3.2. If $\alpha = 1$, then the class $\mathcal{S}^*(1,\beta)$ and $\mathcal{K}(1,\beta)$ is called *starlike of order β* and *convex of order β*, respectively. It is easy to see that $f \in \mathcal{S}^*(1,\beta)$ satisfies $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$ and $f \in \mathcal{K}(1,\beta)$ satisfies $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta$. Marx [7] and Strohhäcker [14] showed that $\mathcal{K}(1,0) \subset \mathcal{S}^*(1,1/2)$. Jack [4] proposed the more general problem: What is the largest number β_0 which satisfies $\mathcal{K}(1,\beta) \subset \mathcal{S}^*(1,\beta_0)$? Later MacGregor [6] and Wilken and Feng [15] answered the problem to give the exact value of β_0.

Theorem C. *$\mathcal{K}(1,\beta) \subset \mathcal{S}^*(1,\delta(\beta))$ for all $\beta \in [0,1)$, where*

$$\delta(\beta) = \begin{cases} \dfrac{1-2\beta}{2^{2-2\beta}(1-2^{2\beta-1})} & \text{if} \quad \beta \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{2\log 2} & \text{if} \quad \beta = \dfrac{1}{2}. \end{cases}$$

This estimation is sharp for each $\beta \in [0,1)$.

Setting $\beta = 0$, we have the result of Marx and Strohhäcker. We can obtain a similar estimation to above that "$\mathcal{K}(\gamma(\alpha),\delta(\beta)) \subset \mathcal{S}^*(\alpha,\beta)$ for all $\alpha \in (0,1]$ and $\beta \in [0,1)$" by Theorem 1 since $\beta < \delta(\beta)$ for all $\beta \in [0,1)$. However, the following problem is still open.

Open Problem. *$\mathcal{K}(\gamma(\alpha),\beta) \subset \mathcal{S}^*(\alpha,\delta(\beta))$ for each $\alpha \in (0,1]$ and $\beta \in [0,1)$.*

This problem implies Theorem 1, because $\mathcal{S}^*(\alpha, \delta(\beta)) \subset \mathcal{S}^*(\alpha, \beta)$ for all $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, and Theorem C as the case when $\alpha = 1$.

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