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A SCHWARZ LEMMA FOR NON-ANALYTIC FUNCTIONS DEFINED IN THE UNIT DISK

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Abstract. We derive a Schwarz lemma for a class of non-analytic function defined on the unit disk.
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1. INTRODUCTION

Schwarz lemma is a fundamental tool with important consequences in Complex analysis, especially in the Geometric function theory. It has been generalized (Schwarz-Pick, Schwarz-Ahlfors-Pick theorems) in the case of analytic functions defined in the unit disk, or in the case of other domains (see [3] for a generalization in the case of the upper half-plane, for example).

The purpose of this article is to obtain a generalization of the Schwarz lemma for a class of non-analytic functions defined in the unit disk, defined as follows. In [1] and [2], the authors considered the class of non-analytic functions f defined on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ having a series expansion of the form:

$$f(z,\bar{z}) = \sum_{n=1}^{\infty} f_n(z,\bar{z}), \quad z \in U, \tag{1}$$

where $f_n = f_n(z,\bar{z})$ are complex functions defined for $z = x + \mathrm{i}y \in \bar{U}$, real positive homogeneous of degree n and satisfying a certain inequality on the boundary ∂U, and they showed that the maximum modulus principle holds for the functions of this class.

In the present paper we show (Theorem 3.1) that Schwarz lemma holds for functions of the form $f(z,\bar{z})$, where $f(z,\bar{z})$ is a function in the class described above. In particular, for $f_n(z,\bar{z}) = a_n z^n$ $(n = 2, 3, \ldots)$ we obtain as a corollary a new proof of the Schwarz lemma for analytic functions (with an additional hypothesis on the coeffcients of the Taylor series), and for $f_n(z,\bar{z}) = a_n z^{k_n} \bar{z}^{n-k_n}$, $k_n \in \{0, 1, \ldots, n\}$ $(n = 2, 3 \ldots)$ we obtain a version of Schwarz lemma (Corollary 3.4) for a particular class of non-analytic functions.

Some examples and counterexamples are also given, and it is shown that the hypothesis of the main theorem is sharp (cannot be relaxed).

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2. PRELIMINARIES

In order to prove the main result, we begin with some preliminaries needed in the proof.

We denote by $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk in $\mathbb{C}$, and for a complex number $z = x + \mathrm{i}y \in \mathbb{C}$ we denote by $\bar{z} = x - \mathrm{i}y$, $|z| = \sqrt{x^2 + y^2}$ the complex conjugate, respectively the modulus of z.

We need the following result from [1]:

THEOREM 2.1. *Let $f(z, \bar{z})$ defined for $z \in U$ have a series expansion of the form:*

$$f(z, \bar{z}) = \sum_{n=1}^{\infty} f_n(z, \bar{z}), \quad z \in U, \tag{2}$$

where $f_n(z, \bar{z})$ are functions of $z \in \bar{U}$ satisfying

$$f_n(rz, r\bar{z}) = r^n f_n(z, \bar{z}), \tag{3}$$

for all $z \in \bar{U}$ and all real numbers $r > 0$ for which $rz \in \bar{U}$, $n = 1, 2, \ldots$.

If for some $\theta \in [0, 2\pi)$ we have:

$$\sum_{n=2}^{\infty} n \left| f_n(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) \right| \leq \left| f_1(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) \right| \neq 0, \tag{4}$$

then $f(z, \bar{z})$ is an increasing function of $|z|$ on $\arg z = \theta$, that is

$$|f(z_1, \overline{z_1}| < |f(z_2, \overline{z_2}|, \tag{5}$$

for any $z_1 = r_1 \mathrm{e}^{\mathrm{i}\theta}$, $z_2 = r_2 \mathrm{e}^{\mathrm{i}\theta} \in U$ with $0 < r_1 < r_2 < 1$.

In particular, if the condition (4) holds for all $\theta \in [0, 2\pi)$, then $|f|$ is radially increasing in the whole disk U, and it cannot therefore attain its maximum at an interior point of U.

3. MAIN RESULTS

We are now ready to prove the main result, as follows:

THEOREM 3.1. *Let $f(z, \bar{z})$ defined for $z \in U$ have a series expansion of the form:*

$$f(z, \bar{z}) = \sum_{n=2}^{\infty} f_n(z, \bar{z}), \quad z \in U, \tag{6}$$

where $f_n(z, \bar{z})$ are functions of $z \in \bar{U}$ satisfying

$$f_n(rz, r\bar{z}) = r^n f_n(z, \bar{z}), \tag{7}$$

for all $z \in \bar{U}$ and all real numbers $r > 0$ for which $rz \in \bar{U}$, $n = 2, 3, \ldots$.

If $f:\{(z,\bar{z}):z\in U\}\to U$ *and for some* $\theta\in[0,2\pi)$ *we have:*

$$\sum_{n=3}^{\infty}(n-1)\left|f_n(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})\right|\leq\left|f_2(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})\right|, \tag{8}$$

then

$$|f(z,\bar{z})|\leq|z|, \tag{9}$$

for all $z\in U$ *with* $\arg z=\theta$.

In particular, if the inequality (8) *holds for all* $\theta\in[0,2\pi)$, *then* $|f(z,\bar{z})|\leq|z|$ *for all* $z\in U$.

Proof. Let $\theta\in[0,2\pi)$ for which (8) holds be arbitrarily fixed.

If $f_2(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})=0$, from (8) it follows that $f_n(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})=0$ for all $n=2,3,\ldots$, and therefore $f(z,\bar{z})=0$ for all $z\in U$ with $\arg z=\theta$, and therefore (9) holds in this case.

Without loss of generality we can therefore assume $f_2(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})\neq 0$.

Consider the function $F:\{(z,\bar{z}):z\in U\}\to\mathbb{C}$ defined by

$$F(z,\bar{z})=\begin{cases}\frac{1}{z}f(z,\bar{z}), & z\in\bar{U}-\{0\}\\ 0, & z=0\end{cases}.$$

We will show that with this definition, the function $F(z,\bar{z})$ satisfies the hypotheses of Theorem 2.1. From the definition of $F(z,\bar{z})$ and using the series representation (6) of $f(z,\bar{z})$ it follows that we have:

$$F(z,\bar{z})=\sum_{n=1}^{\infty}F_n(z,\bar{z}),$$

where $F_n(z,\bar{z})$ are functions defined for $z\in\bar{U}$ by

$$F_n(z,\bar{z})=\begin{cases}\frac{f_{n+1}(z,\bar{z})}{z}, & z\in\bar{U}-\{0\}\\ 0, & z=0\end{cases},$$

for all $n=1,2,\ldots$.

From the hypothesis (7) on the homogeneity of the functions $f_n(z,\bar{z})$ it follows that

$$\begin{aligned}F_n(rz,r\bar{z}) &= \frac{f_{n+1}(rz,r\bar{z})}{rz}=\frac{r^{n+1}f_{n+1}(z,\bar{z})}{rz}\\ &= r^n\frac{f_{n+1}(z,\bar{z})}{z}=r^nF_n(z,\bar{z}),\end{aligned}$$

for all $z\in\bar{U}-\{0\}$ and $r>0$ for which $rz\in\bar{U}$.

Since the above equality is also true for $z=0$ (and any $r>0$), it follows that we have:

$$F_n(rz,r\bar{z})=r^nF_n(z,\bar{z}),$$

for all $z\in\bar{U}$ and $r>0$ for which $rz\in\bar{U}$, for all $n=1,2,\ldots$.

The condition (4) of Theorem 2.1 reduces to

$$\sum_{n=2}^{\infty} n \left| \frac{f_{n+1}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})}{\mathrm{e}^{\mathrm{i}\theta}} \right| \leq \left| \frac{f_2(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})}{\mathrm{e}^{\mathrm{i}\theta}} \right| \neq 0,$$

which is equivalent to the hypothesis (8).

By Theorem 2.1 it follows that $\left|F(r\mathrm{e}^{\mathrm{i}\theta}, r\mathrm{e}^{-\mathrm{i}\theta})\right|$ is an increasing function of $r \in (0,1)$.

Then for $\rho \in (0,1)$ arbitrarily fixed, we have:

$$\left|F(r\mathrm{e}^{\mathrm{i}\theta}, r\mathrm{e}^{-\mathrm{i}\theta})\right| \leq \left|F(\rho\mathrm{e}^{\mathrm{i}\theta}, \rho\mathrm{e}^{-\mathrm{i}\theta})\right| = \left| \frac{f(\rho\mathrm{e}^{\mathrm{i}\theta}, \rho\mathrm{e}^{-\mathrm{i}\theta})}{f_1(\rho\mathrm{e}^{\mathrm{i}\theta}, \rho\mathrm{e}^{-\mathrm{i}\theta})} \right|,$$

for all $r \in (0, \rho]$.

By hypothesis we have $\left|f(\rho\mathrm{e}^{\mathrm{i}\theta}, \rho\mathrm{e}^{-\mathrm{i}\theta})\right| < 1$, and therefore we obtain:

$$\left| \frac{f(r\mathrm{e}^{\mathrm{i}\theta}, r\mathrm{e}^{-\mathrm{i}\theta})}{r\mathrm{e}^{\mathrm{i}\theta}} \right| = \left|F(r\mathrm{e}^{\mathrm{i}\theta}, r\mathrm{e}^{-\mathrm{i}\theta}\right| < \frac{1}{|\rho\mathrm{e}^{\mathrm{i}\theta}|} = \frac{1}{\rho},$$

for all $r \in (0, \rho]$.

Passing to the limit with $\rho \nearrow 1$ we obtain:

$$\left| \frac{f(r\mathrm{e}^{\mathrm{i}\theta}, r\mathrm{e}^{-\mathrm{i}\theta})}{r\mathrm{e}^{\mathrm{i}\theta}} \right| \leq \lim_{\rho \nearrow 1} \frac{1}{\rho} = 1,$$

for all $r \in (0,1)$, or equivalently:

$$\left|f(r\mathrm{e}^{\mathrm{i}\theta}, r\mathrm{e}^{-\mathrm{i}\theta})\right| \leq \left|r\mathrm{e}^{\mathrm{i}\theta}\right| = r$$

for all $r \in (0,1)$, which gives the first part of the claim.

If the inequality (8) holds for all $\theta \in [0, 2\pi)$, by the previous part of the proof it follows that the inequality (9) holds for all $z \in U$, concluding the proof. □

As a first consequence of the above theorem, we obtain a new proof of Schwarz lemma for analytic functions (with an additional hypothesis on the coefficients of the Taylor series), as follows:

COROLLARY 3.2 (Schwarz lemma for analytic functions). *If $f : U \to U$ is analytic in the unit disk U and has a Taylor series expansion*

$$f(z) = \sum_{n=2}^{\infty} a_n z^n, \quad z \in U,$$

where the coefficients a_n satisfy the inequality

$$\sum_{n=3}^{\infty} (n-1)\,|a_n| \leq |a_2| \tag{10}$$

then $|f(z)| \leq |z|$ for all $z \in U$.

Proof. It follows from Theorem 3.1 by considering the functions $f_n(z,\bar{z}) = a_n z^n$, $n \geq 2$. $\square$

REMARK 3.3. It can be shown that the additional hypothesis (10) in the above corollary is equivalent to the fact that the function $z \in U \longmapsto \frac{f(z)}{z}$ is starlike (with respect to the origin) in the unit disk.

As it is known, the Schwarz lemma holds true without this additional hypothesis; we presented it here as a consequence of our main Theorem 3.1 above, which applies to both analytic and non-analytic functions (of the form indicated there).

More generally, we can obtain a Schwarz lemma for a new class of non-analytic functions, as follows:

COROLLARY 3.4. *If $f = f(z,\bar{z}) : \{(z,\bar{z}) : z \in U\} \to U$ has a series expansion of the form:*

$$f(z,\bar{z}) = \sum_{n=2}^{\infty} a_n z^{k_n} \bar{z}^{n-k_n}, \quad z \in U,$$

where $k_n \in \{0,1,\ldots,n\}$, and the coefficients $a_n \in \mathbb{C}$ satisfy the inequality

$$\sum_{n=3}^{\infty} (n-1)\,|a_n| \leq |a_2|\,, \tag{11}$$

then $|f(z,\bar{z})| \leq |z|$ for all $z \in U$.

Proof. Follows from Theorem 3.1 by considering the functions $f_n(z,\bar{z}) = a_n z^{k_n} \bar{z}^{n-k_n}$ with $k_n \in \{0,1,\ldots,n\}$, $n \geq 2$. $\square$

EXAMPLE 3.5. Consider the family of functions

$$f_a(z,\bar{z}) = \frac{z\bar{z} - a z^2 \bar{z}^2}{1-a} = \frac{|z|^2 - a\,|z|^4}{1-a}$$

defined for $z \in U$, where $a \in \left(0, \frac{1}{2}\right)$ is a real parameter.

Note that the function $f_a(z,\bar{z})$ satisfies the hypotheses (6) and (3) of Theorem 3.1 ($f_2(z,\bar{z}) = \frac{z\bar{z}}{1-a}$, $f_4(z,\bar{z}) = -\frac{a z^2 \bar{z}^2}{1-a}$ and $f_n(z,\bar{z}) \equiv 0$ for $n \in \mathbb{N} - \{2,4\}$), for all values $a \in \left(0, \frac{1}{2}\right)$ of the parameter a.

Also, it is easy to see that $|f_a(z,\bar{z})| < 1$ for all $z \in U$ and $a \in \left(0, \frac{1}{2}\right)$, and therefore f_a maps $\{(z,\bar{z}) : z \in U\}$ into U.

The condition (8) of Theorem 3.1 reduces to $3\,|a| < 1$, and it is therefore satisfied for all $a \in \left(0, \frac{1}{3}\right)$.

From the theorem it follows that $|f_a(z,\bar{z})| \leq |z|$ for all $z \in U$, as it can also be checked by direct computation.

Note that for $a \in \left(\frac{1}{3}, \frac{1}{2}\right)$, the function $\left|\frac{f_a(z,\bar{z})}{z}\right| = |z| - a\,|z|^3$ defined for $z \in U$ has a maximum on $|z| = \frac{1}{\sqrt{3a}} < 1$, the maximum value being equal to

$\frac{2}{3\sqrt{3a}(1-a)} > 1$, for all $a \in \left(\frac{1}{3}, \frac{1}{2}\right)$, and therefore the inequality

$$|f_a(z, \bar{z})| \leq |z|, \quad z \in U$$

does not hold for $a \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

The previous example shows that the condition (8) of Theorem 3.1 is sharp (cannot be relaxed), in the sense that if we replace the constant 1 in the condition (8)

$$\frac{\sum_{n=3}^{\infty}(n-1)\left|f_n(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})\right|}{|f_2(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})|} \leq 1,$$

by a larger constant, then there exist functions for which the conclusion of the theorem fails (as showed above, Theorem 3.1 fails for the functions $f_a(z, \bar{z})$ with $a \in \left(\frac{1}{3}, \frac{1}{2}\right)$, functions for which

$$\frac{\sum_{n=3}^{\infty}(n-1)\left|f_n(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})\right|}{|f_2(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})|} = |3a| \in \left(1, \frac{3}{2}\right),$$

can be chosen as close to 1 as needed).

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