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A FAMILY OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH A CONVOLUTION OPERATOR

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Abstract. In this paper we introduce and investigate a new class of harmonic univalent functions defined by convolution. Among other results, we obtain coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combinations for the above family of harmonic functions.

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1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain D, function f can be written as $f = h + \overline{g}$, where h and g are analytic in D. Then h is called the analytic part of f and g is called the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D.

As usual, let S_H denote the family of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g}$ belonging to S_H we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1. \tag{1.1}$$

The class S_H reduces to the class S of analytic and univalent functions in the open unit disk U, if the co-analytic part is $g \equiv 0$.

In 1984, Clunie and Sheil-Small [3] investigated the class S_H and studied some sufficient bounds. Since then there have been several papers published related to S_H and its subclasses. In fact, by introducing new subclasses Ahuja [1], Jahangiri [7], Sheil Small [12], Silverman [13], Silverman and Silvia [14] presented a systematic and unified study of harmonic univalent functions.

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Furthermore we refer to Duren [4], Ponnusamy [9] and references therein for basic results on the subjects.

In [2], Ahuja, Aghalary and Joshi introduced and studied the class $G_H(k, \beta, t)$ consisting of functions which are k-uniformly starlike harmonic univalent functions satisfying the condition:

$$\mathrm{Re}\left(\frac{zf'(z)}{z'f_t(z)}\right) \geq k\left|\frac{zf'(z)}{z'f_t(z)} - 1\right| + \beta,$$

where $z' = \frac{\partial}{\partial\theta}(z = r\mathrm{e}^{\mathrm{i}\theta}),\ f'(z) = \frac{\partial}{\partial\theta}f(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{i}(h'(z) - \overline{zg'(z)}),\ f_t(z) = (1 - t)z + t(h(z) + \overline{g(z)}), 0 \leq t \leq 1, 0 \leq \beta < 1$ and $k \geq 0$.

The convolution or Hadamard product of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ is defined by

$$(f * \phi)(z) = z + \sum_{n=2}^{\infty} a_n \lambda_n z^n. \tag{1.2}$$

For a detailed study see [11].

Recently Rosy et al. [10] defined the subclass $G_H(\gamma) \subset S_H$ consisting of harmonic univalent functions $f(z)$ satisfying the condition:

$$\mathrm{Re}\left\{(1 + \mathrm{e}^{\mathrm{i}\alpha})\frac{zf'(z)}{z'f(z)} - \mathrm{e}^{\mathrm{i}\alpha}\right\} \geq \gamma,\ 0 \leq \gamma < 1,\ \alpha \in R.$$

They proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty}\left[\frac{(2n - 1 - \gamma)}{(1 - \gamma)}|a_n| + \frac{(2n + 1 + \gamma)}{(1 - \gamma)}|b_n|\right] \leq 2,\ 0 \leq \gamma < 1, \tag{1.3}$$

then f is in $G_H(\gamma)$.

This condition is proved to be also necessary by Rosy et al. if h and g are of the form:

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n,\ g(z) = \sum_{n=1}^{\infty}|b_n|z^n. \tag{1.4}$$

Motivated by the work of Rosy et al. [10], now we introduce a class k-$S_H(\phi, \psi; \gamma; t)$ of functions $f = h + \overline{g}$ of the form (1.1) that satisfy the following condition:

$$\mathrm{Re}\left\{\frac{(1 + k\mathrm{e}^{\mathrm{i}\alpha})[\phi(z) * h(z) - \overline{\psi(z) * g(z)}]}{f_t(z)} - k\mathrm{e}^{\mathrm{i}\alpha}\right\} \geq \gamma, \tag{1.5}$$

where $\phi(z) = z + \sum_{n=2}^{\infty}\lambda_n z^n,\ \psi(z) = z + \sum_{n=2}^{\infty}\mu_n z^n$ with $\lambda_n,\ \mu_n \geq 0,\ 0 \leq k < \infty,\ \alpha \in R,\ 0 \leq \gamma < 1$ and $f_t(z) = (1 - t)z + t(h(z) + \overline{g(z)})$ with $0 \leq t \leq 1$.

We further let k-$TS_H(\phi, \psi; \gamma; t)$ denote the subclass of k-$S_H(\phi, \psi; \gamma; t)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form (1.4).

The family $k\text{-}TS_H(\phi,\psi;\gamma;t)$ is of special interest because it contains various classes of well known harmonic univalent functions as well as many new ones.

It is worthy to mention that by specializing the various parameters, we obtain the following interesting classes studied in earlier research.

(1) $0\text{-}S_H(\phi,\psi;\gamma;1) = S_H(\phi,\psi;\gamma)$ and $0 - T_H(\phi,\psi;\gamma;1) \equiv TS_H(\phi,\psi;\gamma)$, comprehensive family of harmonic univalent functions studied by Frasin [5].

(2) $0\text{-}S_H\left(\dfrac{z}{(1-z)^2},\dfrac{z}{(1-z)^2};\gamma;1\right) \equiv T_H(\gamma)$, starlike harmonic functions studied by Jahangiri [7].

(3) $1\text{-}S_H\left(\dfrac{z}{(1-z)^2},\dfrac{z}{(1-z)^2};\gamma;1\right) \equiv G_H(\gamma)$, Goodman-Ronning type harmonic univalent functions studied by Rosy et al. [10].

(4) $k\text{-}S_H\left(\dfrac{z}{(1-z)^2},\dfrac{z}{(1-z)^2};\gamma;t\right) \equiv G(k,\gamma,t)$ studied by Ahuja, Aghalary and Joshi [2].

Thus, in this paper we make a systematic and unified study by introducing the above mentioned new and interesting classes. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for functions in $k\text{-}TS_H(\phi,\psi;\gamma;t)$ are obtained.

2. MAIN RESULTS

In our first theorem we prove a sufficient coefficient bound for harmonic functions in $k\text{-}S_H(\phi,\psi;\gamma;t)$.

THEOREM 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{n=2}^{\infty}\frac{\lambda_n(k+1)-t(k+\gamma)}{1-\gamma}|a_n| + \sum_{n=1}^{\infty}\frac{\mu_n(k+1)+t(k+\gamma)}{1-\gamma}|b_n| \le 1, \tag{2.1}$$

where $\lambda_n \ge 0$, $\mu_n \ge 0$, $n(1-\gamma) \le \lambda_n(k+1) - t(k+\gamma)$ and $n(1-\gamma) \le \mu_n(k+1)+t(k+\gamma)$ with $0 \le t \le 1$ and $0 \le \gamma < 1$, then f is sense-preserving harmonic univalent in U and $f \in k\text{-}S_H(\phi,\psi;\gamma;t)$.

Proof. Note that f is sense-preserving. For this, we have:

$$\begin{aligned}
|h'(z)| &\ge 1 - \sum_{n=2}^{\infty} n|a_n|\ |z|^{n-1} \ge 1 - \sum_{n=2}^{\infty} n|a_n| \\
&\ge 1 - \sum_{n=2}^{\infty}\frac{\lambda_n(k+1)-t(k+\gamma)}{1-\gamma}|a_n| \ge \sum_{n=1}^{\infty}\frac{\mu_n(k+1)+t(k+\gamma)}{1-\gamma}|b_n| \\
&> \sum_{n=1}^{\infty} n|b_n|\cdot|z|^{n-1} \ge |g'(z)|.
\end{aligned}$$

Also, f is locally univalent. For this, we have:

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\frac{\sum\limits_{n=1}^{\infty} n|b_n|}{1-\sum\limits_{n=2}^{\infty} n|a_n|}$$

$$\geq 1-\frac{\sum\limits_{n=1}^{\infty} \frac{\mu_n(k+1)+t(k+\gamma)}{1-\gamma}|b_n|}{1-\sum\limits_{n=2}^{\infty} \frac{\lambda_n(k+1)-t(k+\gamma)}{1-\gamma}|a_n|} \geq 0.$$

Using the fact that $\operatorname{Re} w \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it sufficies to show that:

$$\begin{aligned}
&|(1-\gamma)f_t(z)+(1+k\mathrm{e}^{\mathrm{i}\alpha})[\phi * h-\overline{g*\psi}]-k\mathrm{e}^{\mathrm{i}\alpha}f_t(z)| \\
&-|(1+\gamma)f_t(z)-(1+k\mathrm{e}^{\mathrm{i}\alpha})[\phi * h-\overline{g*\psi}]+k\mathrm{e}^{\mathrm{i}\alpha}f_t(z)| \\
&\geq (2-\gamma)|z|-\sum_{n=2}^{\infty}\{(1-\gamma)t+(1+k)\lambda_n-kt\}|a_n|\ |z|^n \\
&-\sum_{n=1}^{\infty}\{-(1-\gamma)t+(1+k)\mu_n+kt\}|b_n|\ |z|^n-\gamma|z| \\
&-\sum_{n=2}^{\infty}\{(1+k)\lambda_n-kt-(1+\gamma)t\}|a_n|\ |z|^n \\
&-\sum_{n=1}^{\infty}\{(1+\gamma)t+(k+1)\mu_n+kt\}|b_n|\ |z|^n \\
&> 2(1-\gamma)\left[1-\sum_{n=2}^{\infty}\frac{\{(k+1)\lambda_n-(k+\gamma)t\}}{1-\gamma}|a_n|\right. \\
&\left.-\sum_{n=1}^{\infty}\frac{\{(k+1)\mu_n+(k+\gamma)t\}}{1-\gamma}|b_n|\right].
\end{aligned}$$

The last expression is non-negative by (2.1), so the proof is complete.

The harmonic functions

$$\begin{aligned}
f(z) = z &+ \sum_{n=2}^{\infty}\frac{(1-\gamma)}{(k+1)\lambda_n-(k+\gamma)t}x_n z^n \\
&+ \sum_{n=1}^{\infty}\frac{(1-\gamma)}{(k+1)\mu_n+(k+\gamma)t}\overline{y}_n\overline{z}^n,
\end{aligned}$$

where $\sum\limits_{n=2}^{\infty}|x_n|+\sum\limits_{n=1}^{\infty}|y_n|=1$, show that the coefficient bound given by (2.1) is sharp. $\square$

Our next theorem shows that the above sufficient condition is also necessary for functions in $k\text{-}TS_H(\phi,\psi;\gamma;t)$.

THEOREM 2.2. *Let $f=h+\overline{g}$ be given by (1.4). Then $f\in k\text{-}TS_H(\phi,\psi;\gamma;t)$ if and only if*

$$\sum_{n=2}^{\infty}\big\{\lambda_n(k+1)-(k+\gamma)t\big\}|a_n|+\sum_{n=1}^{\infty}\big\{(k+1)\mu_n+(k+\gamma)t\big\}|b_n|\le 1-\gamma. \tag{2.2}$$

Proof. Since $k\text{-}T\ S_H(\phi,\psi;\gamma;t)\subset k\text{-}S_H(\phi,\psi;\gamma;t)$, we only need to prove the "only if" part of Theorem 2.2. To this end, for functions f of the form (1.4), we notice that the condition (1.5) is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\gamma)z-\sum\limits_{n=2}^{\infty}[(1+k\mathrm{e}^{\mathrm{i}\alpha})\lambda_n-(k\mathrm{e}^{\mathrm{i}\alpha}+\gamma)t]|a_n|z^n}{z-\sum\limits_{n=2}^{\infty}|a_n|tz^n+\sum\limits_{n=1}^{\infty}|b_n|t\overline{z}^n}\right\}$$
$$-\operatorname{Re}\left\{\frac{\sum\limits_{n=1}^{\infty}\{(1+k\mathrm{e}^{\mathrm{i}\alpha})\mu_n+(k\mathrm{e}^{\mathrm{i}\alpha}+\gamma)t\}|b_n|\overline{z}^n}{z-\sum\limits_{n=2}^{\infty}|a_n|tz^n+\sum\limits_{n=1}^{\infty}|b_n|t\overline{z}^n}\right\}\ge 0.$$

The above condition must hold for all values of z, $|z|=r<1$. Upon choosing the values of z on the positive real axis, where $0\le z=r<1$, we must have:

$$\operatorname{Re}\left\{\frac{(1-\gamma)r-\sum\limits_{n=2}^{\infty}(\lambda_n-\gamma t)|a_n|r^n-\sum\limits_{n=1}^{\infty}(\mu_n+\gamma t)|b_n|r^n}{r-\sum\limits_{n=2}^{\infty}|a_n|tr^n+\sum\limits_{n=1}^{\infty}|b_n|tr^n}\right\}$$
$$-\operatorname{Re}\left\{\frac{\mathrm{e}^{\mathrm{i}\alpha}\left\{\sum\limits_{n=2}^{\infty}k(\lambda_n-t)|a_n|r^n+\sum\limits_{n=1}^{\infty}k(\mu_n+t)|b_n|r^n\right\}}{r-\sum\limits_{n=2}^{\infty}|a_n|tr^n+\sum\limits_{n=1}^{\infty}|b_n|tr^n}\right\}\ge 0.$$

Since $\mathrm{Re}(-e^{i\alpha}) \geq -|e^{i\alpha}| = -1$, the above inequality reduces to the following:

$$\tag{2.3} \frac{(1-\gamma) - \sum_{n=2}^{\infty}\{(k+1)\lambda_n - (k+\gamma)t\}|a_n|r^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n|tr^{n-1} + \sum_{n=1}^{\infty}|b_n|tr^{n-1}} - \frac{\sum_{n=1}^{\infty}\{(k+1)\mu_n + (k+\gamma)t\}|b_n|r^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n|tr^{n-1} + \sum_{n=1}^{\infty}|b_n|tr^{n-1}} \geq 0.$$

If the condition (2.2) does not hold, then the numerator in (2.3) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient (2.3) is negative. This contradicts the condition for $f \in k\text{-}S_H(\phi,\psi;\gamma;t)$ and hence the result is proved. □

Now we determine the extreme points of the class $k\text{-}TS_H(\phi,\psi;\gamma;t)$.

THEOREM 2.3. *Let f be given by (1.4). Then $f \in k\text{-}TS_H(\phi,\psi;\gamma;t)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty}\{x_n h_n(z) + y_n g_n(z)\},$$

where $h_1(z) = z$, $h_n(z) = z - \frac{(1-\gamma)}{\{(k+1)\lambda_n - (k+\gamma)t\}}z^n$ $(n = 2,3,\ldots)$, $g_n(z) = z + \frac{(1-\gamma)}{\{(k+1)\mu_n + (k+\gamma)t\}}\overline{z}^n$ $(n = 1,2,3,\ldots)$ with $\sum_{n=1}^{\infty}(x_n + y_n) = 1$, $x_n \geq 0$, $y_n \geq 0$. Therefore the extreme points of the class $k\text{-}TS_H(\phi,\psi;\gamma;t)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions f of the form (1.4), we have:

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\{x_n h_n(z) + y_n g_n(z)\} \\ &= \sum_{n=1}^{\infty}(x_n + y_n)z - \sum_{n=2}^{\infty}\frac{(1-\gamma)}{(k+1)\lambda_n - (k+\gamma)t}x_n z^n \\ &\quad + \sum_{n=1}^{\infty}\frac{(1-\gamma)}{(k+1)\mu_n + (k+\gamma)t}y_n \overline{z}^n. \end{aligned}$$

Then $f \in k\text{-}TS_H(\phi,\psi;\gamma;t)$, because

$$\sum_{n=2}^{\infty}\frac{(k+1)\lambda_n-(k+\gamma)t}{(1-\gamma)}|a_n|+\sum_{n=1}^{\infty}\frac{(k+1)\mu_n+(k+\gamma)t}{(1-\gamma)}|b_n|$$
$$=\sum_{n=2}^{\infty}x_n+\sum_{n=1}^{\infty}y_n=1-x_1\le 1.$$

Conversely, suppose that $f \in k\text{-}TS_H(\phi,\psi;\gamma;t)$. Then:

$$|a_n|\le\frac{(1-\gamma)}{(k+1)\lambda_n-(k+\gamma)t},\quad |b_n|\le\frac{(1-\gamma)}{(k+1)\mu_n+(k+\gamma)t}.$$

Set

$$x_n=\frac{(k+1)\lambda_n-(k+\gamma)t}{(1-\gamma)}\ (n\ge 2),\quad y_n=\frac{(k+1)\mu_n+(k+\gamma)t}{(1-\gamma)}\ (n\ge 1).$$

Then note that by Theorem 2.2, $0\le x_n\le 1$ $(n\ge 2)$ and $0\le y_n\le 1$, $(n\ge 1)$. Define $x_1=1-\sum_{n=2}^{\infty}x_n-\sum_{n=1}^{\infty}y_n\ge 0$, by Theorem 2.2. Consequently, we see that $f(z)$ can be expressed as $f(z)=\sum_{n=1}^{\infty}\{x_nh_n(z)+y_ng_n(z)\}$, as required. □

THEOREM 2.4. *Let $f \in k\text{-}TS_H(\phi,\psi;\gamma;t)$. Then for $|z|=r<1$, we have:*

$$|f(z)|\le(1+|b_1|)r+\left(\frac{1-\gamma}{(k+1)\lambda_2-(k+\gamma)t}-\frac{(k+1)\mu_1+(k+\gamma)t}{(k+1)\lambda_2-(k+\gamma)t}|b_1|\right)r^2$$

and

$$|f(z)\ge|(1-|b_1|)r-\left(\frac{1-\gamma}{(k+1)\lambda_2-(k+\gamma)t}-\frac{(k+1)\mu_1+(k+\gamma)t}{(k+1)\lambda_2-(k+\gamma)t}|b_1|\right)r^2.$$

Proof. We only prove the right hand side inequality. The proof for the left hand side inequality is similar and will be omitted. Let $f \in k\text{-}TS_H(\phi,\psi;\gamma;t)$. Taking the absolute value of f, we obtain:

$$|f(z)|\le(1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$
$$\le(1+|b_1|)r+\frac{(1-\gamma)}{\lambda_2(k+1)-(k+\gamma)t}\sum_{n=2}^{\infty}\left\{\frac{\lambda_n(k+1)-(k+\gamma)t}{(1-\gamma)}|a_n|\right.$$
$$\left.+\frac{\mu_n(k+1)+(k+\gamma)t}{(1-\gamma)}|b_n|\right\}r^2$$
$$\le(1+|b_1|)r+\left[\frac{(1-\gamma)}{\lambda_2(k+1)-(k+\gamma)t}-\frac{\mu_1(k+1)+(k+\gamma)t}{(1-\gamma)}|b_1|\right]r^2,$$

as required. □

The following covering result follows from the left hand side inequality of Theorem 2.4.

Corollary 2.5. Let $f \in k\text{-}TS_H(\phi, \psi; \gamma; t)$ and $\lambda_2 - \gamma \leq \lambda_n - \gamma \leq \mu_n + \gamma$, for $n \geq 2$. Then

$$\left\{ w : |w| < \frac{(k+1)\lambda_2 - (k+\gamma)t - (1-\gamma)}{(k+1)\lambda_2 - (k+\gamma)t} - \frac{(k+1)(\lambda_2 - \mu_1) - 2(k+\gamma)t}{(k+1)\lambda_2 - (k+\gamma)t}|b_1| \right\}$$

is included in $f(U)$.

We next show that the class $k\text{-}TS_H(\phi, \psi; \gamma; t)$ is invariant under convolution and convex combinations of its functions.

For this purpose, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n,$$

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n,$$

we define the convolution of $f(z)$ and $F(z)$ as:

$$(2.4) \qquad (f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n| \cdot |A_n| z^n + \sum_{n=1}^{\infty} |b_n| \cdot |B_n| \overline{z}^n.$$

Theorem 2.6. *For $0 \leq \beta \leq \gamma < 1$, let $f \in k\text{-}TS_H(\phi, \psi; \gamma; t)$ and $F \in k\text{-}TS_H(\phi, \psi; \beta; t)$. Then $f * F \in k\text{-}TS_H(\phi, \psi; \gamma; t) \subset k\text{-}TS_H(\phi, \psi; \beta; t)$.*

Proof. Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \in k\text{-}TS_H(\phi, \psi; \gamma; t)$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n \in k\text{-}TS_H(\phi, \psi; \beta; t)$. Then the convolution of $f * F$ is given by (2.4). We wish to show that the coefficients of $f * F$ satisfy the required condition given in Theorem 2.2. For $F \in k\text{-}TS_H(\phi, \psi; \beta; t)$, we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now, for the convolution function $f * F$, we obtain:

$$\sum_{n=2}^{\infty} \frac{\lambda_n(k+1) - (k+\gamma)t}{(1-\gamma)} |a_n| \cdot |A_n| + \sum_{n=1}^{\infty} \frac{\mu_n(k+1) + (k+\gamma)t}{(1-\gamma)} |b_n| \cdot |B_n|$$

$$\leq \sum_{n=2}^{\infty} \frac{\lambda_n(k+1) - (k+\gamma)t}{(1-\gamma)} |a_n| + \sum_{n=1}^{\infty} \frac{\mu_n(k+1) + (k+\gamma)t}{(1-\gamma)} |b_n| \leq 1,$$

since $f \in k\text{-}TS_H(\phi, \psi; \gamma; t)$. Therefore we have $f * F \in k\text{-}TS_H(\phi, \psi; \gamma; t) \subset k\text{-}TS_H(\phi, \psi; \beta; t)$. □

Theorem 2.7. *The family $k\text{-}TS_H(\phi, \psi; \gamma; t)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i(z) \in k\text{-}TS_H(\phi, \psi; \gamma; t)$, where $f_i(z) = z - \sum_{n=2}^{\infty} |a_{in}| z^n + \sum_{n=1}^{\infty} |b_{in}| \overline{z}^n$. Then by Theorem 2.2 we have:

$$\sum_{n=2}^{\infty} \frac{\lambda_n(k+1) - (k+\gamma)t}{(1-\gamma)} |a_{in}| + \sum_{n=1}^{\infty} \frac{\mu_n(k+1) + (k+\gamma)t}{(1-\gamma)} |b_{in}| \leq 1. \tag{2.5}$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{in}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{in}| \right) \overline{z}^n.$$

Then by (2.5) we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\lambda_n(k+1) - (k+\gamma)t}{(1-\gamma)} \left(\sum_{i=1}^{\infty} t_i |a_{in}| \right) \\
&+ \sum_{n=1}^{\infty} \frac{\mu_n(k+1) + (k+\gamma)t}{(1-\gamma)} \left(\sum_{i=1}^{\infty} t_i |b_{in}| \right) \\
&= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\lambda_n(k+1) - (k+\gamma)t}{(1-\gamma)} |a_{in}| + \sum_{n=1}^{\infty} \frac{\mu_n(k+1) + (k+\gamma)t}{(1-\gamma)} |b_{in}| \right) \\
&\leq \sum_{i=1}^{\infty} t_i = 1,
\end{aligned}$$

and therefore we have $\sum_{i=1}^{\infty} t_i f_i(z) \in k\text{-}TS_H(\phi, \psi; \gamma; t)$ by Theorem 2.2. $\square$

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