MATHEMATICA, Tome 53(76), N° 1, 2011, pp. 29–34

CORRESPONDENCES FOR COVERING POINTS

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Abstract. Harris and Knörr proved that there is a defect group preserving correspondence between the covering blocks of two Brauer correspondent blocks. A module theoretical version of this result exists and it is due to Alperin [1]. Here we prove that these two results still hold in a more general setting, that is the case of points on some G-algebras over a discrete valuation ring.

MSC 2010. 20C20.

Key words. Pointed group, defect pointed group, divisor, restriction and induction of divisors, Green correspondence.

1. INTRODUCTION AND NOTATION

Throughout this paper we follow the notations from [5] and [4]. Thus G will always denote a finite group, for a G-algebra A over a discrete valuation ring $\mathcal{O}$ and for any subgroup H of G the set $\mathcal{P}(A^H)$ denotes the set of point of the subalgebra of H-fixed elements of A, that is the conjugacy classes of a primitive idempotent of A^H by invertible elements of A^H. If $\alpha \in \mathcal{P}(A^H)$ is such a conjugacy class, the pair (H, α) is denoted by H_α and it represents a pointed group. There is an order relation between pointed groups; if H and K are subgroups of G such that $K \subseteq H$ and if α and β are point of K on A and of H on A respectively, then $K_\alpha \leq H_\beta$ if and only if for any $j \in \beta$ there exists $i \in \alpha$ appearing in a decomposition of j in A^K and that is $ji = ij = i$. We also assume that the field $k = \mathcal{O}/J(\mathcal{O})$ is of characteristic $p > 0$. The maps $\mathrm{Tr}_K^H : A^K \to A^H$ and $r_K^H : A^H \to A^K$ denote the relative trace map and respectively the standard inclusion.

2. GREEN CORRESPONDENCE FOR COVERING POINTS

Let N be a normal subgroup of G and let $\alpha \in \mathcal{P}(A^G)$ and $\beta \in \mathcal{P}(A^N)$ such that $N_\beta \leq G_\alpha$. Suppose P_γ is a defect pointed group of G_α. This means P_γ is minimal with the following property: for any $i \in \alpha$ there exists $e \in \gamma$ and $a, b \in A^P$ such that $i = \mathrm{Tr}_P^G(aeb)$.

Then by Mackey decomposition we have:

$$r_N^G(i) = r_N^G \,\mathrm{Tr}_P^G(aeb) = \sum_{g \in [N \backslash G/P]} \mathrm{Tr}_{N \cap {}^gP}^N \, r_{N \cap {}^gP}^{{}^gP} \left({}^g(aeb)\right).$$

This research has been supported by the Romanian PN-II-IDEI-PCE-2007-1 project ID_532, contract no. 29/01.10.2007.

Let $j \in \beta$ be an primitive appearing in a decomposition of $r_N^G(i)$ in A^N. Then multiplying on both sides the above equality with j we get

$$j = \sum_{g \in [N \backslash G/P]} \mathrm{Tr}_{N \cap {}^gP}^N \left[j r_{N \cap {}^gP}^{{}^gP} ({}^g(aeb)j) \right],$$

and since j is a primitive idempotent, there exists an element $g \in [N \setminus G/P]$ such that $\beta \in A_{N \cap {}^gP}^N$.

Definition 2.1. If $N \cap {}^gP$ is the minimal subgroup of N with the property $\beta \in A_{N \cap {}^gP}^N$, then we say that α *covers* β. In this case there exists a point $\gamma' \in \mathcal{P}(A^Q)$, $Q = N \cap {}^gP$, such that $Q_{\gamma'}$ is a defect pointed group of N_β.

In what follows we give a generalization of Alperin [1, Theorem 1]. Fix a p-subgroup Q of the normal subgroup N of G, and let $K = N_N(Q)$, $L = N_G(Q)$. Let $\beta \in \mathcal{P}(A^N)$ such that N_β has defect pointed group $Q_{\gamma'}$ for some point $\gamma' \in \mathcal{P}(A^Q)$. Since $N_N(Q_{\gamma'}) \subseteq N_N(Q) = K$, there is a unique point $\delta \in \mathcal{P}(A^K)$ corresponding to β under the Green Correspondence, moreover K_δ has $Q_{\gamma'}$ as defect pointed group (see [5, Chapter 3, Theorem 20.1]).

Theorem 2.2. *There is a one-to-one correspondence between points of A^G covering β and points of A^L covering δ. Moreover, if $Q_{\gamma'}$ is a defect pointed group of K_δ, hence a defect pointed group of N_β, and P_γ is a defect pointed group of L_ϵ, hence of G_α then ${}^g(Q_{\gamma'}) \leq P_\gamma$ for some $g \in L$.*

Proof. Let $\epsilon \in \mathcal{P}(A^L)$ be a point covering δ. It follows by definition that $K_\delta \leq L_\epsilon$ and if P_γ is a defect pointed group of L_ϵ then $(P \cap K)_{\gamma'} = Q_{\gamma'}$ is a defect pointed group of K_δ, for some point $\gamma' \in \mathcal{P}(A^Q)$. Since $N_N(Q_{\gamma'}) \subseteq K$ and $Q_{\gamma'} \leq K_\delta \leq N_\beta$, it follows by the Burry-Carlson-Puig theorem ([5, Chapter 3, Theorem 20.4]) that $Q_{\gamma'}$ is also a defect pointed group of N_β.

Because $Q = P \cap K \leq P \cap N < N$ it is clear that $Q = P \cap N$, since Q is a maximal p-subgroup of N, thus

$$N_G(P_\gamma) \subseteq N_G(P) \subseteq N_G(P \cap N) = N_G(Q) = L.$$

In these conditions let $\alpha \in \mathcal{P}(A^G)$ be the Green correspondent point of ϵ. Note that P_γ is a defect pointed group of G_α.

We have the relations $P_\gamma \leq L_\epsilon \leq G_\alpha$ and $K_\delta \leq L_\epsilon$, hence $K_\delta \leq G_\alpha$, and also $K_\delta \leq N_\beta$. By the same Burry-Carlson-Puig theorem we deduce that $Q_{\gamma'}$ is a defect pointed group of G_α and also of N_β, which implies the fact that the point α is the Green correspondent of β such that $Q_{\gamma'} \leq N_\beta \leq G_\alpha$, since $N_N(Q_{\gamma'}) \subseteq N$. This defines a one-to-one map from the points of A^L covering δ to the points of A^G covering β.

We prove that this map is surjective. Let $\alpha_1 \in \mathcal{P}(A^G)$ covering β such that P_γ is a defect pointed group of G_{α_1}. Denote by $\epsilon_1 \in \mathcal{P}(A^L)$ the Green correspondent of α_1. Thus L_{ϵ_1} has defect pointed group P_γ, and consequently,

$m_{\alpha_1} = m_{\epsilon_1} \cap A^G$([5, Chapter 3, Theorem 20.1]), where m_{α_1} and m_{ϵ_1} denote the maximal ideals of A^G and of A^L corresponding to α_1 and ϵ_1 respectively (see [5, Chapter 1, Theorem 1.15]). Since α_1 covers β, by applying [5, Chapter 2, Lemma 13.3] we get $m_\beta \cap A^G \subseteq m_{\alpha_1}$. Also, because δ is the Green correspondence of β, we have: $m_\beta = m_\delta \cap A^N$, hence

$$m_\delta \cap A^N \cap A^G = m_\delta \cap A^G \subseteq m_{\alpha_1} = m_{\epsilon_1} \cap A^G.$$

We interpret the inclusion $m_\delta \cap A^G \subseteq m_{\epsilon_1} \cap A^G$ in terms of inclusion maps, that is,

$$\left(r_K^G\right)^{-1}(m_\delta) \subseteq \left(r_L^G\right)^{-1}(m_{\epsilon_1}).$$

Since $r_K^G = r_K^L r_L^G$, we obtain:

$$\left(r_L^G\right)^{-1}(r_K^L)^{-1}(m_\delta) \subseteq \left(r_L^G\right)^{-1}(m_{\epsilon_1}).$$

Ignoring the first inclusion we obtain $\left(r_K^L\right)^{-1}(m_\delta) \subseteq m_{\epsilon_1}$. The last inclusion is equivalent to $m_\delta \cap A^L \subseteq m_{\epsilon_1}$ which is equivalent to $K_\delta \leq L_{\epsilon_1}$.

Because α_1 covers β we have $P \cap N = Q$, hence $Q = P \cap K$ since $Q \leq K$ and $N \cap K = K$. This proves that ϵ_1 covers δ.

The last statement follows from the fact that $Q_{\gamma'}$ is local such that $Q_{\gamma'} \leq L_\epsilon$, because $Q_{\gamma'} \leq K_\delta \leq L_\epsilon$, and from the fact that L_ϵ is projective relative to P_γ(for details see [5, Chapter 2 Paragraph 14]). Using all the above, the result follows from [5, Chapter 3 Lemma 18.2]. □

Remark 2.3. Let M be a kG module. By applying the above theorem to the G-algebra $A := \mathrm{End}_k(M)$ one obtains Alperin's result on modules. Indeed, by [5, Chapter 2, Example 13.4], we see that in this case, to any point it corresponds an indecomposable direct summand of M. Moreover, the definition of the covering points from this paragraph applied to A yields the definition from [1].

3. A HARRIS-KNÖRR CORRESPONDENCE FOR POINTED GROUPS

In this section we give a generalization to the case of certain graded algebras of the main result in [2]. We assume that A is inductively complete, that is, any idempotent of A has orthogonal trace, see [4, Chapter 5].

Let us recall that any conjugacy class of idempotents of A induces a map $D : \mathcal{P}(A) \to \mathbb{N}$ which is called a divisor of A. If D and D' are two divisors such that $D(\alpha) \leq D'(\alpha)$ for any $\alpha \in \mathcal{P}(A)$, then we say that D' contains D and we write $D \subset D'$. Any idempotent i of A induces a A-divisor, namely $\mu_A^i = \sum_{\alpha \in \mathcal{P}(A)} m_\alpha^i \alpha$. Here m_α^i denotes the multiplicity of α in a decomposition of i. Note that the containment of divisors is transitive. In what follows we will identify any point α with the divisor induced by α. For more detailed explanation see [4, Chapter 3,5].

Thus, if α is a point of H on A and β is a point of K on A, where K and H are two subgroups of G such that $K \subseteq H$, then $K_\beta \leq H_\alpha$ is equivalent to $\beta \subset \mathrm{res}^H_K(\alpha)$, we used here the restriction map $\mathrm{res}^H_K(\mu^i_{A^H}) = \mu^i_{A^K}$ for any $i \in \alpha$. Denote by $\mathrm{ind}^H_K(\beta) = \mu_{A^H}^{\mathrm{Tr}^H_K(j)}$ the induction of β. One can easily verify that this definition does not depend on the choice of $j \in \beta$, and that restriction and induction are linear maps, and for any two divisor of K on A such that $D \subset D'$ we have $\mathrm{ind}^H_K(D) \subset \mathrm{ind}^H_K(D')$ (see [4, Proposition 5.6]).

In the case of a inductively complete G-algebras, a more precise version of the Green correspondence holds.

Theorem 3.1 (The Green Correspondence). *Let A be an inductively complete G-algebra and let P_γ be a local pointed group on A, also let H be a subgroup of G containing $N_G(P_\gamma)$. Then, if α is a point of G on A with defect pointed group P_γ there exists a unique point β of H on A with the same defect pointed group such that $\beta \subset \mathrm{res}^G_H(\alpha)$ or equivalently $\alpha \subset \mathrm{ind}^G_H(\beta)$.*

Proof. By [4, Theorem 5.12], if $i \in \alpha \subset A^G$ and $j \in \beta \subset A^H$ then, we have $\mu^i_{A^G} \subset \mathrm{ind}^G_H(\mu^j_{A^H})$ if and only if there are $a, b \in A^H$ such that $i = \mathrm{Tr}^G_H(ajb)$. Since we are dealing with points we can write $\mu^i_{A^G} \subset \mathrm{ind}^G_H(\mu^j_{A^H})$ as $\alpha \subset \mathrm{ind}^G_H(\beta)$. Now equality $i = \mathrm{Tr}^G_H(ajb)$ for some $a, b \in A^H$ means that G_α is projective relative to H_β(see [5, Lemma 14.1]) and that is one of the properties which the pointed groups associated to the points in the Green correspondence satisfy. Hence applying [5, Theorem 20.1] the proof is complete. □

Let N be a normal subgroup in G and consider the G/N-graded algebra $A = \bigoplus A_\sigma$, where σ runs over the set of the cosets of N in G. Observe that G acts on A such that if $g \in G$ and $a \in A_\sigma$, then ${}^g a \in A_{{}^g a}$. This shows that A is a G-graded G-algebra and A_1 are in fact G-algebras. For any subgroup H of G consider the inclusion $I : A_1^H \to A^H$. If we fix a point α of H on A_1 then we get a divisor of H on A, namely $\mathrm{res}_I(\alpha) = \mu_A^{I(i)}$ for some $i \in \alpha$. Clearly, the inclusion I depends on the choice of the subgroup H, but we do not make any further notation since it will be clear from the context which inclusion is used.

Definition 3.2. For two subgroups K and H of G such that K is normal in H, consider a point β of K on A_1. We say that the point α of H on A *covers* β if $\alpha \subset \mathrm{res}_I(\mathrm{ind}^H_K(\beta))$ and if P is a defect of α we have $P \cap N$ is a defect of β.

Let us emphasize that in the case of the group algebra the last condition in this definition is superfluous as seen in [3, Proposition 4.2]. Still one can show that for a more general algebra there are examples of points satisfying the terms in the definition.

With the notations as in Theorem 2.2, choose a G-stable point β of N on A_1, that is, ${}^g\beta = \beta$ for all $g \in G$.

Remark 3.3. The Green correspondent $\delta \in \mathcal{P}(A_1^K)$ of β is also G-stable. Indeed, if $Q_{\gamma'}$ is a defect pointed group for both β and δ, since $Q_{\gamma'} \leq K_\delta \leq N_\beta$

we get ${}^g(Q_{\gamma'}) \leq K_{{}^g\delta} \leq N_\beta$ for any $g \in$G. The last inclusion takes place because the inclusion of pointed groups is compatible with the action of G as it follows from [5, Exercise 13.5]. Now ${}^g(Q_{\gamma'})$ is a defect pointed group for β hence for ${}^g\delta$ and this implies that ${}^g\delta$ is the Green correspondent of β, then ${}^g\delta$ and δ must coincide.

We fix the G-stable point β and its Green correspondent δ.

Theorem 3.4. *There is a one-to-one correspondence between points of G on A covering β and points of L on A covering δ. This correspondence is induced by the Green correspondence for points and it preserves the defect groups.*

Proof. Since the definition above is similar to the first definition of the paper, all we have to prove is: a point of L covers δ if and only if its Green correspondent covers β.

Thus, let ϵ be a point of L on A covering δ and let α be the point of G on A corresponding to ϵ. By Definition 3.2 and Theorem 3.1 above, we obtain the following inclusions of divisors:

$$\begin{aligned}\alpha \subset \mathrm{ind}_L^G(\epsilon) &\subset \mathrm{ind}_L^G\left(\mathrm{res}_I\left(\mathrm{ind}_K^L(\delta)\right)\right)\\ &\subset \mathrm{ind}_L^G\left(\mathrm{res}_I\left(\mathrm{ind}_K^L\left(\mathrm{res}_K^N(\beta)\right)\right)\right)\\ &= \mathrm{res}_I\left(\mathrm{ind}_K^G\left(\mathrm{res}_K^N(\beta)\right)\right)\\ &= n \cdot \mathrm{res}_I\left(\mathrm{ind}_N^G(\beta)\right),\end{aligned}$$

where n is the index of K in N. Hence $\alpha \subset \mathrm{res}_I(\mathrm{ind}_N^G(\beta))$ and this proves that α covers β.

Conversely, let α be a point of G on A covering β and let ϵ be the corresponding point of α. We prove that ϵ covers δ. Again by Definition 3.2 and Theorem 3.1, we obtain:

$$\begin{aligned}\epsilon \subset \mathrm{res}_L^G(\alpha) &\subset \mathrm{res}_L^G\left(\mathrm{res}_I\left(\mathrm{ind}_N^G(\beta)\right)\right)\\ &= \mathrm{res}_I\left(\mathrm{res}_L^G\left(\mathrm{ind}_N^G(\beta)\right)\right)\\ &\subset \mathrm{res}_I\left(\mathrm{res}_L^G\left(\mathrm{ind}_K^G(\delta)\right)\right)\\ &= \mathrm{res}_I\left(\sum_{x\in[L/G\backslash K]} \mathrm{ind}_{L\cap {}^xK}^L\left(\mathrm{res}_{L\cap {}^xK}^{{}^xK}({}^x\delta)\right)\right)\\ &= \mathrm{res}_I\left(\mathrm{ind}_K^L\left(\mathrm{res}_K^K\left(\sum_{x\in[L/G\backslash K]} {}^x\delta\right)\right)\right)\\ &= m \cdot \mathrm{res}_I\left(\mathrm{ind}_K^L(\delta)\right),\end{aligned}$$

where we used Mackey decomposition formula and the fact the δ is G-stable. As a conclusion we get $\epsilon \subset \mathrm{res}_I\left(\mathrm{ind}_K^L(\delta)\right)$. □

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Received July 27, 2009
Accepted September 10, 2009

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