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MODULES WITH DEDEKIND FINITE ENDOMORPHISM RINGS

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Abstract. This article is a survey of modules whose endomorphism rings are Dedekind finite, Hopfian or co-Hopfian. We summarise the properties of such modules and present unified proofs of known results and generalisations to new structure theorems.

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1. INTRODUCTION

A unital ring R is called *Dedekind* (or *directly*, or *von Neumann*) *finite* if $ba = 1$ whenever $ab = 1$. Equivalently, R is Dedekind finite if whenever a is left or right invertible, then a is invertible. It follows immediately that commutative rings and unit regular rings (but not regular rings in general) are Dedekind finite.

Clearly $ab = 1$ implies that ba is a non-zero idempotent, so R is Dedekind finite if and only if R is not isomorphic to any proper left or right ideal direct summand. This criterion provides an easy proof that several classes of rings with other finiteness conditions are Dedekind finite; for example Artinian and Noetherian rings, rings with finite Goldie dimension, Abelian rings (i.e., rings in which all idempotents are central), reduced rings (i.e., rings having no nilpotent elements), and I-finite rings (i.e., rings having no infinite set of orthogonal idempotents).

Let R be any unital ring and M a unital R-module. We say that M is a *DF module* if its ring of R-endomorphisms, $\mathrm{End}(M)$, is Dedekind finite. Consequently, M is a DF module if and only if M is not isomorphic to any proper direct summand of itself. An immediate but important consequence of this is that indecomposable modules are DF. Note that if $f, g \in \mathrm{End}(M)$ satisfy $fg = 1$ then f is necessarily epic and g monic. Hence to show that M is DF, it suffices to show that in this situation, f is monic or g epic.

By definition, a DF module is one in which every left or right invertible endomorphism is invertible. One can strengthen this characterization, and thereby describe a more restrictive class of modules, by requiring that every monic endomorphism, or every epic endomorphism be invertible. This leads to the classes of *co-Hopfian* and *Hopfian* modules, respectively.

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It is the purpose of this paper to survey what is known about closure properties of the classes of Dedekind finite rings and DF modules, with special attention to abelian groups. We relate their properties to those of Hopfian and co-Hopfian modules and provide unified results for all three classes of modules.

Henceforth, R denotes a unital ring and M a unital right R-module. When R is considered as right (left) R-module, we denote it as R_R $({}_RR)$. Clearly, R is Dedekind finite as a ring if and only if R_R is Dedekind finite as R-module.

We denote the center of R by $Z(R)$, its unit group by $U(R)$ and its Jacobson radical by $\mathrm{Rad}(R)$. As usual, the phrase *left zero divisor* means a non-zero element $a \in R$ such that for some non-zero $b \in R$, $ab = 0$. The element b is then a *right zero divisor*.

The word *group* will be used to denote an abelian group G; $T(G)$ is its torsion subgroup and G_p its p-primary component for each prime p. For other terminology concerning abelian groups we refer to Fuchs [10], and for rings and modules to Lam [15].

The main results of this paper are the following. Section 2 examines the closure properties of the class of Dedekind finite rings. In Section 3 we consider DF modules, summarising known results on and establishing new ones. In Section 4 we study Hopfian and co-Hopfian modules, and characterize them, as well as DF modules, as limits of certain ascending and descending series of submodules.

2. DEDEKIND FINITE RINGS

Throughout this section, R denotes a unital ring and $\mathcal{DF}$ the class of Dedekind finite rings.

Various closure properties of $\mathcal{DF}$ are well known or easy to deduce. For example, $\mathcal{DF}$ is closed under direct products, finite direct sums and (unital) subrings, but not under homomorphic images. Since its defining axioms are the equations defining a unital ring together with the Horn sentence

$$\forall a,\ b \in R,\ ab = 1 \Rightarrow ba = 1,$$

$\mathcal{DF}$ is a quasi-variety [5, Section VI.4]. An obvious property is that $R \in \mathcal{DF}$ if and only if its opposite ring $R^{\mathrm{op}} \in \mathcal{DF}$.

By [14, Proposition (4.8)], $R/\mathrm{Rad}(R) \in \mathcal{DF}$ if and only if $R \in \mathcal{DF}$. Consequently, local and semilocal rings are in $\mathcal{DF}$. Since the semilocal property is preserved by matrix rings [9, p. 6], the matrix rings over a semilocal ring are in $\mathcal{DF}$. Conversely, R is stably finite (see [15, Section 1B]) if all matrix rings over R are Dedekind finite. In general, however, $R \in \mathcal{DF}$ does not imply that matrix rings over R are Dedekind finite. For example Lam [15, Exercise 1.18, with solution] quotes an example of Shepherdson of the 2×2 matrix ring over a Dedekind finite ring which is not Dedekind finite.

There is an elementary sufficient condition for R to be in $\mathcal{DF}$.

Proposition 2.1. *If R has no right or left zero divisors, then $R \in \mathcal{DF}$.*

Proof. Let $ab = 1$ in R. Then $a \neq 0$ and $a(ab - ba) = 0$, so $ba = ab = 1$. The proof for left zero divisors is similar. $\square$

The converse of course is false, as witnessed by any finite ring with zero divisors. We determine now the polynomial and power series rings in $\mathcal{DF}$. Let X be a set of indeterminates of arbitrary cardinality. Let $R[X]$ and $R[\tilde{X}]$ denote the rings of polynomials in commuting elements of X and polynomials in non-commuting elements of X respectively. That is, $R[\tilde{X}]$ is the free word algebra over X with coefficients from R. Let $R[[X]]$ be the power series ring in X.

Proposition 2.2. *The following are equivalent:*
(1) $R \in \mathcal{DF}$.
(2) $R[X] \in \mathcal{DF}$.
(3) $R[\tilde{X}] \in \mathcal{DF}$.
(4) $R[[X]] \in \mathcal{DF}$.

Proof. Since R is a subring, each of (2), (3) and (4) implies (1), so it remains to show that (1) implies each of (2), (3) and (4). In each case, suppose $f(X)g(X) = 1$, and let f_0 and g_0 be the corresponding terms of degree 0. Then $f_0 g_0 = 1$ so $g_0 f_0 = 1$. In cases (2) and (3), this implies that f_0 and g_0 are not zero divisors, so that $f(X)$ and $g(X)$ are invertible. In case (4), $g_0 f_0 = 1$ implies immediately that $f(X)$ and $g(X)$ are invertible. $\square$

Corollary 2.3. *$\mathcal{DF}$ is not closed under homomorphic images.*

Proof. Let R have no zero divisors and let $R[x, y]$ be the polynomial ring over R in non-commuting indeterminates x and y. Let I be the ideal of $R[x, y]$ generated by $xy-1$. Then $x+I$ is right invertible but not invertible in R/I. $\square$

Proposition 2.4. *Let $e^2 = e \in R$. Then $R \in \mathcal{DF}$ implies $eRe \in \mathcal{DF}$.*

Proof. Let $ab = e$ for $a, b \in eRe$ and $\overline{e} = 1-e$. Then $(a+\overline{e})(b+\overline{e}) = ab+\overline{e} = e + \overline{e} = 1$ and so $(b + \overline{e})(a + \overline{e}) = 1$. Hence $ba = 1 - \overline{e} = e$, as desired. $\square$

On the other hand, it is not true that for some idempotent e, eRe and $(1 - e)R(1 - e) \in \mathcal{DF}$ imply $R \in \mathcal{DF}$. The result Lam [15, Exercise 1.18] quoted above provides a counterexample. Lam's result [14, Proposition (4.8)] on the Jacobson radical noted above readily extends to nilpotent ideals:

Theorem 2.5. *Let I be a nilpotent ideal in a ring R. Then $R \in \mathcal{DF}$ if and only if $R/I \in \mathcal{DF}$.*

Proof. Suppose $R \in \mathcal{DF}$ and let $(a + I)(b + I) = ab + I = 1 + I$ in R/I. Then $ab \in 1+I \subseteq U(R)$, so that a is left invertible and hence invertible. Thus $a + I$ is invertible in R/I so that $R/I \in \mathcal{DF}$.

Conversely, let $R/I \in \mathcal{DF}$ and suppose $ab = 1$. Then $(a+I)(b+I) = 1+I = (b + I)(a + I)$ so $ba \in U(R)$. Hence a is left invertible and $R \in \mathcal{DF}$. $\square$

Corollary 2.6. *Let R be a ring and let n be a positive integer. Let $T(n,R)$ be the ring of upper [lower] triangular matrices over R. Then $R \in \mathcal{DF}$ if and only if $T(n,R) \in \mathcal{DF}$.*

Proof. The ideal I of R consisting of the strictly upper triangular matrices is nilpotent and R/I is isomorphic to the direct sum of n copies of R. By Lemma 2.5, R is Dedekind finite if and only if $T(n,R)$ is. □

Similarly, we have:

Corollary 2.7. *Let A, B be rings and ${}_AM_B$ a bimodule. Then the ring $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is Dedekind finite if and only if A and B are both Dedekind finite.*

3. DEDEKIND FINITE MODULES

Beaumont and Pierce's 1964 results in [4] concerning abelian groups which have isomorphic proper summands, can be expressed in terms of DF groups as follows.

Proposition 3.1. (1) *If G is a DF group, then its torsion subgroup $T(G)$ is a DF group.*

(2) *If $T(G)$ and $G/T(G)$ are DF groups then G is a DF group.*
(3) *$T(G)$ is a DF group if and only if every primary component G_p of $T(G)$ is a DF group,*
(4) *A reduced p-group G is a DF group if and only if for every positive integer n, the nth Ulm invariant is finite.*
(5) *If G is a reduced p-group such that $|G| > 2^{\aleph_0}$, then G is not a DF group.*
(6) *Let $G = R \oplus D$ where R is reduced and D divisible. Then G is a DF group if and only if both R and D are DF groups.*
(7) *If G is divisible, then G is a DF group if and only if G has finite torsion-free rank and finite p-rank for every prime p.*
(8) *If G is reduced and for all primes p, G/pG is finite, then G is a DF group. The converse is false for both torsion and torsion-free groups.*

Other results from [4] are easily modified to apply to DF modules over arbitrary rings.

Proposition 3.2. *Let M be an R-module.*

(1) *There is a monomorphism $f \in \mathrm{End}(M)$ with $\mathrm{im}\, f$ a proper direct summand if and only if there is an epimorphism $g \in \mathrm{End}(M)$ with $\ker g$ a proper direct summand.*
(2) *If M is a DF module, then so is any direct summand of M.*
(3) *If M is the direct sum of infinitely many copies of the same non-zero module, then M is not a DF module.*

(4) *Let $f \in \mathrm{End}(M)$ be a monomorphism of M onto a direct summand, and let N be a fully invariant submodule which is a DF module. Then $f(N) = N$.*

(5) *Let N be a fully invariant submodule of M. If N and M/N are DF modules, then M is a DF module.*

(6) *If M is the direct sum of fully invariant submodules N_i, then M is a DF module if and only if every N_i is.*

(7) *If M is the direct product of (infinitely many) fully invariant submodules N_i, then M is a DF module if and only if every N_i is.*

Proof. (1) Suppose such an f exists and let $M = K \oplus \operatorname{im} f$. Let $h : \operatorname{im} f \to M$ be any isomorphism and define g to be zero on K and h on $\operatorname{im} f$. Conversely, let $M = \ker g \oplus N$. Note that $M \cong M/\ker g = N$ and let $f : M \to N$ be any isomorphism, regarded as an endomorphism of M.

(2) Let $M = N \oplus K$. If L is a proper direct summand of N isomorphic to N, then $L \oplus K$ is a direct summand of M isomorphic to M, a contradiction.

(3) Suppose $M = \oplus_{i \in I} N_i$ for some well-ordered index set I, such that for all $i \in I$, $N_i \cong N$ for some module N. Then for all $i \in I$, there exists an isomorphism $t_i : N_i \to N_{i+1}$. Then the t_i induce the *right shift map* $f : M \to M$ by $x \mapsto t_i(x)$ for all $x \in N_i$. Similarly, the t_i induce the *left shift map* $g : M \to M$ by $x \mapsto t_i^{-1}x$ for all $x \in N_{i+1}$ and g maps N_0 to zero. Clearly f and g are endomorphisms of M satisfying $gf = 1$ but $fg \neq 1$, so M is not DF.

(4) Let $M = f(M) \oplus H$ so that $f(N) \subseteq N \cap f(M)$. By (1), there is an endomorphism $g \in \mathrm{End}(M)$ such that $gf = 1$. Let $x \in M$ such that $f(x) \in N$. Then $x = gf(x) \in N$, since N is fully invariant. Hence $f(N) = f(M) \cap N$. Consequently, $N = f(N) \oplus (H \cap N)$. Since N is DF, $f(N) = N$.

(5) Once again, let f be a monomorphism of M into M such that $M = f(M) \oplus H$. By (4), it follows that $N = f(N) \subseteq f(M)$. Hence

$$M/N = f(M)/N \oplus (H + N)/N \text{ and } f(M)/N = f(M)/f(N) \cong M/N$$

Since M/N is DF, $H + N = N = f(N)$. Thus $H = 0$ so M is DF.

(6) This follows immediately from (2) and (4).

(7) Let $M = \prod_{i \in I} N_i$. If M is DF, then so is each N_i since it is a direct summand of M. Conversely, suppose each N_i is DF, and let ϕ be a monomorphism of M onto a direct summand H. Then for all $i \in I$, $\phi|_{N_i}$ is an isomorphism so that $H = M$. □

Other known closure properties of DF modules include:

Example 3.3. (1) Let $G = A \oplus B$ with $\mathrm{Hom}(A, B) = 0$. Then G is DF if and only if A and B are DF. (This is an immediate consequence of Proposition 2.7.)

(2) If M is not DF then M contains an infinite direct sum $\oplus_{n \in \mathbb{N}} N_n$, where each $N_n \cong N$ by [16, Ex 1.8].

(3) Neither submodules nor factor modules of DF modules need be DF. For example, the $\mathbb{Z}$-module $\oplus_{n\in\mathbb{N}}\mathbb{Z}(p^n)$ is DF but has $\oplus_{n\in\mathbb{N}}\mathbb{Z}(p)$ as both submodule and factor module.

(4) The class of DF-modules is not closed with respect to general extensions, finite direct sums or endomorphic images.

We know from [15, Exercise 1.18] that there exists a DF ring R such that the ring of 2×2 matrices over R is not DF. Therefore the R-module $R\oplus R$ is not DF. We shall see in Section 4 that there exists a DF abelian group A such that $A\oplus A$ is not DF. Let $A=\prod_p \mathbb{Z}(p)$. Then A and its torsion subgroup $T(A)$ are DF, but $A/T(A)\cong\mathbb{Q}^\omega$ is not DF.

(5) The property stated in Proposition 3.1(1) cannot be extended to general torsion theories. To see this we consider the ring $R=\begin{bmatrix} K & K^{(I)} \\ 0 & K\end{bmatrix}$, where K is a field and I is an infinite set. This is a DF ring as a consequence of Corollary 2.7. Then $M=(K,K^{(I)})$ is DF as a right R-module. However, if we consider the projective module $P=(0,K)$, the class $\mathcal{T}=\{\oplus_I P \mid I \text{ is a set}\}$ is a torsion class. Then $(0,K^{(I)})$ is not a DF module, but it is the torsion part of M relative to the torsion theory induced by $\mathcal{T}$.

Goodearl [12] discusses in particular injective DF modules. He shows, for example [12, Section 6 B], that an injective submodule of a DF module is DF, and that the following conditions are equivalent for any injective module M: (1) M is DF; (2) M has no proper submodule satisfying $M\cong M^2$; (3) M has no proper summand satisfying $M\cong M^2$. Finally, he shows [12, Theorem 6.14] that if R is a right nonsingular ring, then the direct sum of nonsingular injective DF modules is DF, whereas this is no longer true [12, Example 6.11] if the modules are not injective.

THEOREM 3.4. *Let M be an R-module and N a fully invariant submodule of M.*

(1) *If N is an essential DF module, then M is DF.*
(2) *If N is superfluous and M/N is DF, then M is DF.*

Proof. Let f, g be endomorphisms of M such that $fg=1$.

(1) The restrictions $f'=f|_N$ and $g'=g|_N$ are endomorphisms of N such that $f'g'=1_N$. Since $0=\ker(g')=\ker(g)\cap N$, and N is essential, it follows that $\ker(g)=0$, and M is DF.

(2) Let f' and g' the induced endomorphisms of M/N. From $f'g'=1_{M/N}$ we deduce that f' is an epimorphism, hence $f(M)+N=M$. But N is superfluous, so $f(M)=M$ and hence M is DF. □

For the following corollary, recall that the socle $\mathrm{Soc}(M)$ of a module M is the sum of its minimal submodules, with $\mathrm{Soc}(M)=0$ if M has no minimal submodules; and the radical $\mathrm{Rad}(M)$ is the intersection of its maximal submodules, with $\mathrm{Rad}(M)=M$ if M has no maximal submodules.

Corollary 3.5. *Let M be an R-module.*

(1) *If* $\mathrm{Soc}(M)$ *is essential and DF then M is DF.*
(2) *If* $\mathrm{Rad}(M)$ *is superfluous and* $M/\mathrm{Rad}(M)$ *is DF then M is DF.*

Example 3.6. Abelian group examples show that both conditions of Corollary 3.5 are necessary:

(1) If $M = \oplus_{n\in\mathbb{N}}\mathbb{Z}(p)$ then $M = \mathrm{Soc}(M)$ is essential but M is not DF. If $M = \oplus_{n\in\mathbb{N}}\mathbb{Z}$ then $\mathrm{Soc}(M) = 0$ is DF but M is not DF.

(2) If $M = \oplus_{n\in\mathbb{N}}\mathbb{Q}$ then $M/\mathrm{Rad}(M) = 0$ is DF but M is not DF. If $M = \oplus_{n\in\mathbb{N}}\mathbb{Z}$ then $\mathrm{Rad}(M) = 0$ is superfluous but M is not DF.

The converse of the results of Corollary 3.5 are not valid in general.

Example 3.7. (1) $\mathbb{Z}$ is a DF $\mathbb{Z}$-module whose socle, 0 is not essential and $\mathbb{Z}_p$, the group of p-adic integers, is a DF $\mathbb{Z}$-module whose radical $p\mathbb{Z}$ is not superfluous.

(2) The group $G = \oplus_{n>0}\mathbb{Z}(p^n)$ is DF and its socle is essential, but not DF. Another example with these properties can be obtained using the module M from Example 3.3(5).

(3) The group $G = \oplus_{n>0}\mathbb{Z}(p^n)$ is DF, its radical pG is superfluous, but G/pG is not DF. Another example with these properties can be obtained using [16, Exercise 6.1.B].

(4) G. Bergman gave an example of module of finite uniform dimension, which has a factor module which is not DF (see [16, Exercise 6.1.B]).

Proposition 3.8. [16, Exercise 6.31] *Let M be a module. If its injective envelope $E(M)$ is DF, then M is DF. The converse is valid if M is quasi-injective.*

Proof. Suppose that M is not DF. Then $M \cong M \oplus N$ with $N \neq 0$. Then $E(M) \cong E(M) \oplus E(N)$, hence $E(M)$ is not DF, a contradiction.

For the converse, we note that every quasi-injective module is fully invariant in its injective envelope, hence we can apply the previous results. □

In general, the converse is not valid.

Example 3.9. If p is a prime, the $\mathbb{Z}$-module $G = \oplus_{n>0}\mathbb{Z}(p^n)$ is DF, but its injective envelope $E(G) = \oplus_{n>0}\mathbb{Z}(p^\infty)$ is not a DF $\mathbb{Z}$-module.

There are however dual sufficient conditions for an arbitrary submodule or factor module of a DF module to be DF:

Theorem 3.10. *Let N be a submodule of a DF module M.*

(1) *If* $\mathrm{Hom}(M/N, M) = 0$ *and every* $f \in \mathrm{End}(N)$ *extends to* $\mathrm{End}(M)$*, then N is DF.*
(2) *If* $\mathrm{Hom}(M, N) = 0$ *and every* $f \in \mathrm{End}(M/N)$ *lifts to* $\mathrm{End}(M)$*, then M/N is DF.*

Proof. (1) Let $f \in \mathrm{End}(N)$ such that $fg = 1_N$. Suppose that g and h are endomorphisms of M such that $g|_N = h|_N$. Then $N \subseteq \ker(g-h)$, so the homomorphism $M/\ker(g-h) \to M$ induced by $g-h$ is 0. Then every endomorphism f of N extends uniquely to an endomorphism f' of M. It is not hard to see that the correspondence $f \mapsto f'$ is a ring monomorphism, and it follows that $\mathrm{End}(M)$ is DF.

(2) Note that if $h \in \mathrm{End}(M/N)$ lifts to $\overline{h}$ and $\overline{h}'$ in $\mathrm{End}(M)$, then $(\overline{h} - \overline{h}')(M) \subseteq N$. Hence the condition $\mathrm{Hom}(M,N) = 0$ implies that h lifts uniquely. As in (1) we observe that $h \mapsto \overline{h}$ defines a ring monomorphism $\mathrm{End}(M/N) \to \mathrm{End}(M)$, so M/N is DF. □

We consider now the important class of pure-injective or algebraically compact DF modules. To start with, there is a well known decomposition theorem [13, Lemma 1.2.24].

LEMMA 3.11. *Let M be a pure-injective module. Then $M = K \oplus N$ where K is zero or a pure-injective hull of a direct sum of indecomposable pure-injective modules, and N is zero or a superdecomposable pure-injective module (i.e., N has no indecomposable direct summands).*

It is also well known that indecomposable pure-injectives have local endomorphism rings, so that the Krull-Schmidt-Azumaya Theorem applies to direct sums of indecomposable pure-injective modules. Furthermore, any module is pure essential in its pure-injective hull, so we have immediately:

PROPOSITION 3.12. *Let M be a pure-injective module with no superdecomposable direct summand. Then M is a pure-injective hull of a direct sum $\oplus_{i\in I} M_i$ of indecomposable pure-injectives M_i and M is DF if and only if each isomorphism class contains only finitely many M_i.*

As far as superdecomposable DF modules are concerned, little is known except for the case in which $\mathrm{End}(M)$ is commutative. It is readily checked that M is superdecomposable if and only if $\mathrm{End}(M)$ has no primitive idempotents, so we need modules whose endomorphism ring is commutative and has no primitive idempotents.

PROPOSITION 3.13. *There are countable superdecomposable DF groups.*

Proof. Corner [6] showed that there is a commutative ring R whose additive group is free and of countable rank which has no primitive idempotents and therefore a countable torsion-free group G whose endomorphism ring is isomorphic with R. Thus G is a countable superdecomposable DF group. □

Moreover, over certain rings there are arbitrarily large DF superdecomposable modules. In [11], Fuchs and Göbel construct integral domains R for which there exist superdecomposable cotorsion-free modules M of arbitrary rank. For example, let μ denote an infinite cardinal and T_1 the monoid whose elements are the finite subsets of μ with the commutative multiplication defined via $\sigma \cdot \tau = \sigma \cup \tau$. The $\mathbf{Z}$-algebra of T_1 is superdecomposable and realizable as the endomorphism ring of a cotorsion-free module.

Beaumont and Pierce [4, Theorem 3.8] proved a structure theorem for abelian groups G that are not DF. Their proof uses no properties of abelian groups that are not true for arbitrary modules, so carries over verbatim to modules that are not DF:

Proposition 3.14. *Let M be an R-module which is not DF. Then there exists a monomorphism $\phi \in \mathrm{End}_R(M)$ such that $M = \phi(M) \oplus H$, with $H \neq 0$. Let $K = \cap_{n \in \mathbb{N}} \phi^n(M)$ and let $H_n = \phi^n(H) \cong H$. Let $S = \oplus_{n \in \mathbb{N}} H_n$ and $P = \prod_{n \in \mathbb{N}} H_n$. Then $\phi|_K$ is an automorphism and there is a module T such that $S \subseteq T \subseteq P$ and M is an extension of K by T.*

4. HOPFIAN AND CO-HOPFIAN MODULES

In considering Dedekind finiteness of modules, it is convenient to consider two stronger concepts. A right (left) R-module M is called right (left) *Hopfian* if every surjective R-endomorphism is invertible; it is *co-Hopfian* if every injective R-endomorphism is invertible.

In the case that $M = R_R$, since every $f \in \mathrm{End}_R(R_R)$ is realised by a left multiplication, these definitions imply that R_R is right Hopfian if and only if for all $a \in R$, $aR = R$ implies that $Ra = R$ and R_R is right co-Hopfian if and only if every $a \in R$ is either a left zero divisor or invertible. Dual properties of course occur when right is replaced everywhere by left.

When R_R (${}_RR$) is right (left) Hopfian, the ring R is called right (left) Hopfian, and similarly for co-Hopfian. The following results are expressed for right Hopfian and co-Hopfian rings. Of course corresponding results hold for the case when right is replaced by left. Let $\mathcal{H}$ denote the class of Hopfian, and co$-\mathcal{H}$ the class of co-Hopfian rings. The following relations between these classes of rings and the class $\mathcal{DF}$ are well known (see [17, Section 1]).

Proposition 4.1. co$-\mathcal{H} \subsetneq \mathcal{DF}$ *and* $\mathcal{H} = \mathcal{DF}$.

Proof. Let $R \in$ co$-\mathcal{H}$ and suppose $a,\ b \in R$ satisfy $ab = 1$. Then considered as an endomorphism of R, b is injective and hence invertible. It follows that $ba = 1$ so $R \in \mathcal{DF}$. To verify that the inclusion is proper, note that $\mathbb{Z}$ is Dedekind finite but not co-Hopfian. More generally, a similar example works for R any integral domain or Ore ring which is not a division ring.

A similar proof shows that $\mathcal{H} \subseteq \mathcal{DF}$. For the reverse inclusion, let $R \in \mathcal{DF}$ and suppose that $aR = R$ for some $a \in R$. Then $ab = 1$ for some $b \in R$ so that $ba = 1$. Hence for every $c \in R,\ c = cba \in Ra$. $\square$

The equality in the above proposition implies that the Hopficity property of a ring is left-right symmetric, [17, Theorem 1.3]. However, the co-Hopficity property is not left-right symmetric, as it is proved in [17, Example 1.6].

We return to the study of Hopfian and co-Hopfian modules.

Proposition 4.2. *If an R-module M is Hopfian or co-Hopfian, then it is DF.*

Proof. Suppose M is not DF, so M has a proper isomorphic summand N.

If M is Hopfian, then the canonical projection of M on N, composed with an isomorphism of N onto M is an epimorphism in $\mathrm{End}(M)$ containing a non-trivial kernel, a contradiction.

If M is co-Hopfian, then any isomorphism of M onto N is a monomorphism in $\mathrm{End}(M)$ which is not surjective, again a contradiction. $\square$

To see that neither converse holds, note that the abelian group $\mathbb{Z}$ is DF but not co-Hopfian, while $\mathbb{Z}(p^\infty)$ is DF but not Hopfian.

Closure properties for the classes of Hopfian and co-Hopfian modules are discussed in [17]. For instance, these classes are closed with respect direct summands, but they are not closed with respect direct sums or finite powers. Such examples can be constructed even for abelian groups. Corner [7] provided examples of non-Hopfian torsion-free abelian groups G_1, G_2 and G_3, necessarily of infinite rank since finite rank groups are Hopfian, such that $\mathrm{Aut}(G_1) = \{1, -1\}$, $G_2 = A \oplus B$ with A and B both Hopfian and $G_3 \cong C \oplus C$ with C Hopfian.

Let M be a right R-module, and let $\mathrm{Mon}(M)$ denote the set of monomorphisms in $\mathrm{End}(M)$, $\mathrm{Epi}(M)$ the set of epimorphisms in $\mathrm{End}(M)$, $\mathrm{Left}(M)$ the set of left units in $\mathrm{End}(M)$, i.e., $\{f \in \mathrm{End}(M) : \exists\, g \in \mathrm{End}(M)$ such that $fg = 1\}$ and $\mathrm{Right}(M)$ the set of right units in $\mathrm{End}(M)$.

Associated with these sets of endomorphisms of M, there are sets of invariant submodules, namely

$$\mathcal{P}(\mathrm{Mon}(M)) = \{N \subseteq M : f(N) \subseteq N \text{ for all } f \in \mathrm{Mon}(M)\}$$
$$\mathcal{P}(\mathrm{Epi}(M)) = \{N \subseteq M : f(N) \subseteq N \text{ for all } f \in \mathrm{Epi}(M)\}$$
$$\mathcal{P}(\mathrm{Left}(M)) = \{N \subseteq M : f(N) \subseteq N \text{ for all } f \in \mathrm{Left}(M)\}.$$
$$\mathcal{P}(\mathrm{Right}(M)) = \{N \subseteq M : f(N) \subseteq N \text{ for all } f \in \mathrm{Right}(M)\}.$$

We also use the following notation: if $N \subseteq M$ is invariant under $f \in \mathrm{End}(M)$ then $f_N \in \mathrm{End}(N)$ denotes the restriction of f to N; and $f^N \in \mathrm{End}(M/N)$ denotes the induced endomorphism.

Lemma 4.3. *Let N be a submodule of a module M and $f \in \mathrm{End}(M)$.*

(1) *If $N \in \mathcal{P}(\mathrm{Mon}(M))$ and $f \in \mathrm{Mon}(M)$, then $f_N \in \mathrm{Mon}(N)$.*
(2) *If $N \in \mathcal{P}(\mathrm{Epi}(M))$ and $f \in \mathrm{Epi}(M)$, then $f^N \in \mathrm{Epi}(M/N)$.*
(3) *If $N \in \mathcal{P}(\mathrm{Left}(M)) \cap \mathcal{P}(\mathrm{Right}(M))$ with $fg = 1$ in $\mathrm{End}(M)$, then $f_N \in \mathrm{Left}(N)$, $g_N \in \mathrm{Right}(N)$ and $f_N g_N = 1_N$ in $\mathrm{End}(N)$.*

Proof. (1) and (2). Both cases follow immediately from the definitions of f_N, f^N, $\mathrm{Mon}(N)$ and $\mathrm{Epi}(M/N)$.

(3) $fg = 1$ implies that $f \in \mathrm{Left}(M)$ and $g \in \mathrm{Right}(M)$. Hence f_N and g_N are well-defined. Let $n \in N$ and suppose that $g(m) = n \in N$ for some $m \in M$. Then $m = fg(m) = f(n) \in N$. Hence $n = fg(n) = f_N g_N(n)$, i.e., $f_N g_N = 1_N$. The rest follows from the definitions. $\square$

Baumslag [3, Theorem 1] proved that an abelian group is Hopfian if and only if the intersection of the fully invariant subgroups with Hopfian quotient is zero. The following proposition improves this result and provides a corresponding characterization of co-Hopfian, Hopfian and DF modules.

PROPOSITION 4.4. (1) *A module M is co-Hopfian if and only if M is the union of a family of co-Hopfian submodules $H \in \mathcal{P}(\mathrm{Mon}(M))$.*

(2) *A module M is Hopfian if and only if there is a family $\mathcal{H}$ of submodules $H \in \mathcal{P}(\mathrm{Epi}(M))$ with Hopfian quotients M/H such that $\cap_{H\in\mathcal{H}} H = 0$.*

(3) *A module M is DF if and only if there is a family $\mathcal{H}$ of DF submodules $H \in \mathcal{P}(\mathrm{Left}(M)) \cap \mathcal{P}(\mathrm{Right}(M))$ such that $\cup_{H\in\mathcal{H}} H = H$.*

(4) *A module M is DF if and only if there is a family $\mathcal{H}$ of submodules $H \in \mathcal{P}(\mathrm{Left}(M)) \cap \mathcal{P}(\mathrm{Right}(M))$ with DF quotients M/H such that $\bigcap_{H\in\mathcal{H}} H = 0$.*

Proof. In all four cases, the direct implication is obvious, so we just need to prove the converses.

(1) Let $\mathcal{H}$ be a family of co-Hopfian submodules in $\mathcal{P}(\mathrm{Mon}(M))$ such that $M = \cup_{H\in\mathcal{H}} H$. Let $f \in \mathrm{Mon}(M)$. If $y \in M$ then there exists $H \in \mathcal{H}$ such that $y \in H$. By Lemma 4.3 (1), the restriction $f_H \in \mathrm{Mon}(H)$ so is an isomorphism. Thus $y \in f(M)$ so f is an isomorphism.

(2) Let $\mathcal{H}$ be a family of submodules of M satisfying the condition of statement (2). If $f \in \mathrm{Epi}(M)$, then by Lemma 4.3 (2), for all $H \in \mathcal{H}$, the induced map $f^H \in \mathrm{Epi}(M/H)$ so is an isomorphism. Hence $\ker(f) \subseteq H$ for all $H \in \mathcal{H}$, so that $\ker(f) = 0$ and f is an isomorphism.

(3) Let $\mathcal{H}$ be a family of submodules of M satisfying the condition of statement (3). If f and g are endomorphisms of M such that $fg = 1$, then by Lemma 4.3 (3), for all $H \in \mathcal{H}$ the restrictions f_H and g_H are right, respectively left invertible as endomorphisms of H. It follows that these restrictions are automorphisms for H. Then $\ker(f) \cap H = 0$ for all $H \in \mathcal{H}$, hence $\ker(f) = 0$.

(4) Let $\mathcal{H}$ be a family of submodules of M satisfying the condition of statement (4). If $f : M \to M$ is left invertible, then for all $H \in \mathcal{H}$, the induced map $f^H \in \mathrm{End}(M/H)$ is an isomorphism. Then $\ker(f) \subseteq H$ for all $H \in \mathcal{H}$, so that $\ker(f) = 0$. $\square$

Baer [1] (using different terminology) also characterised Hopfian and co-Hopfian groups as limits of ascending and descending chains of subgroups.

Recall that a continuous descending [ascending] filtration of a module M is a well ordered descending [ascending] chain N_ν, $\nu \le \kappa$, of submodules such that $N_0 = M$ [$N_0 = 0$] and $N_\kappa = 0$ [$N_\kappa = M$] and if $\mu \le \kappa$ is a limit ordinal, then $N_\mu = \bigcap_{\nu<\mu} N_\nu$ [$N_\mu = \bigcup_{\nu<\mu} N_\nu$]. We now prove a unified version of Baer's characterization of co-Hopfian, Hopfian and DF modules in terms of continuous descending or ascending filtrations.

THEOREM 4.5. *Let M be a module.*

(1) *The following are equivalent:*

(a) *M is co-Hopfian;*

(b) *M has a continuous ascending filtration of modules N_ν, $\nu \leq \kappa$, in $\mathcal{P}(\mathrm{Mon}(M))$ such that for all $\nu < \kappa$, $N_{\nu+1}/N_\nu$ is co-Hopfian;*

(2) *The following are equivalent:*

(a) *M is Hopfian;*

(b) *M has a continuous descending filtration of modules N_ν, $\nu \leq \kappa$, in $\mathcal{P}(\mathrm{Epi}(M))$ such that for all $\nu < \kappa$, $N_\nu/N_{\nu+1}$ is Hopfian.*

(3) *The following are equivalent:*

(a) *M is DF;*

(b) *M has a continuous ascending filtration of modules N_ν, $\nu \leq \kappa$ in $\mathcal{P}(\mathrm{Left}) \cap \mathcal{P}(\mathrm{Right})$ such that for all $\nu < \kappa$, N_ν and $N_{\nu+1}/N_\nu$ are DF;*

(c) *M has a continuous descending filtration of N_ν, $\nu \leq \kappa$ of modules in $\mathcal{P}(\mathrm{Left}) \cap \mathcal{P}(\mathrm{Right})$ such that for all $\nu < \kappa$, $N_\nu/N_{\nu+1}$ is DF.*

Proof. (1) For (a)$\Rightarrow$(b) it is enough to take $\kappa = 1$.

(b)$\Rightarrow$(a) Let $f : M \to M$ be a monomorphism, and we denote by $f_\nu : N_\nu \to N_\nu$ the restrictions on f to N_ν. We will prove by induction that all f_ν are isomorphisms. It is obvious that f_0 is an isomorphism, and suppose that f_ρ are isomorphisms for all $\rho < \nu$.

If ν is a limit ordinal then for all $n \in N_\nu$ there is $\rho < \nu$ such that $n \in N_\rho$, hence we can find an element $m \in N_\rho \subseteq N_\nu$ such that $f(m) = n$. Then f_ν is an isomorphism.

If $\nu = \rho + 1$ then the induced map $\overline{f_\nu} : N_\nu/N_\rho \to N_\nu/N_\rho$ is an isomorphism. Moreover, $f_\nu(N_\rho) = N_\rho$ by the induction hypothesis, hence f_ν is an isomorphism. The proof is complete.

(2) (a)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(a) Let $f : M \to M$ be an epimorphism. We denote by $f^\nu : M/N_\nu \to M/N_\nu$ the maps induced by f. We observe that $f(N_0) = N_0$ and the map $f^0 : M/N_0 \to M/N_0$ induced by f is an isomorphism. Suppose that $f(N_\rho) = N_\rho$ and that the maps $f^\rho : M/N_\rho \to M/N_\rho$ are isomorphisms for all $\rho < \nu$.

If ν is a limit ordinal, let $n \in N_\nu$. There exists $m \in M$ such that $f(m) = n$. Let $\rho < \nu$. Since the map $f^\rho : M/N_\rho \to M/N_\rho$ induced by f is an isomorphism and $n \in N_\rho$, it follows that $m \in N_\rho$. Then $m \in N_\nu$, hence $f(N_\nu) = N_\nu$. Moreover, if $m \in M$ is such that $f(m) \in N_\nu$ then $f(m) \in N_\rho$ for all $\rho < \nu$, and by induction hypothesis $m \in N_\rho$ for all $\rho < \nu$, hence f^ν is an isomorphism.

If $\nu = \rho + 1$, then the map $f' : N_\rho/N_\nu \to N_\rho/N_\nu$ induced by f is an epimorphism, hence it is an isomorphism. Then for every $m \in N_\rho \setminus N_\nu$, $f(m) \notin N_\nu$. Therefore, if $n \in N_\nu$ and $m \in N_\rho$ are such that $f(m) = n$ then $m \in N_\nu$. Then $f(N_\nu) = N_\nu$. Suppose that f^ν is not an isomorphism. Let $0 \neq m \in \ker f^\nu$. Then there is $\rho < \nu$ such that $m \in M_\rho \setminus M_{\rho+1}$. The map $f' : N_\rho/N_{\rho+1} \to N_\rho/N_{\rho+1}$ induced by f is an epimorphism by the induction hypothesis. Using the hypothesis it follows that f' is an isomorphism, and this contradicts $f'(m + M_{\rho+1}) = 0$. Then f^ν is an isomorphism.

Then f^κ is an isomorphism, hence M is Hopfian.

To prove (3) we observe that (a) implies (b) and (c) by taking $\kappa = 1$.

To show that (b) implies (a), we consider $f, g \in \mathrm{End}(M)$ such that $fg = 1$. Suppose that f is not a monomorphism. Then there is $0 \neq m \in M$ such that $f(m) = 0$. Since $m \notin N_0$ there is a largest $\rho < \kappa$ such that $m \notin N_\rho$, hence $m \in N_{\rho+1}$. Since N_ρ and $N_{\rho+1}$ are invariant with respect f and g, the maps f and g induce two endomorphisms f' and g' of $N_{\rho+1}/N_\rho$. It is not hard to see that $f'g' = 1$, and it follows that $M_{\rho+1}/M_\rho$ is not DF, a contradiction.

To show that (c) implies (a), let f, $g \in \mathrm{End}(M)$ with $fg = 1$. It suffices to show that f is injective. If not, then there exists $0 \neq m \in M$ such that $f(m) = 0$. Note that there exists $\nu < \kappa$ such that $m \in N_\nu \setminus N_{\nu+1}$. We can restrict the induced maps $\overline{f}, \overline{g} : M/N_{\nu+1} \to M/N_{\nu+1}$ to $f^*, g^* : N_\nu/N_{\nu+1} \to N_\nu/N_{\nu+1}$. Since $N_\nu/N_{\nu+1}$ is DF, $f^*g^* = 1$ implies that f^* is monic, hence $f^*(m + N_{\nu+1}) \neq 0$, a contradiction. □

Corollary 4.6. *Let $N \subseteq M$.*

(1) *If $N \in \mathcal{P}(\mathrm{Mon}(M))$ and N and M/N are both co-Hopfian, then M is co-Hopfian.*
(2) *If $N \in \mathcal{P}(\mathrm{Epi}(M))$ and N and M/N are both Hopfian, then M is Hopfian.*
(3) *If $N \in \mathcal{P}(\mathrm{Left}) \cap \mathcal{P}(\mathrm{Right})$ and N and M/N are both DF, then M is DF.*

Since fully invariant submodules are contained in $\mathcal{P}(\mathrm{Mon}(M))$, $\mathcal{P}(\mathrm{Epi}(M))$ and $\mathcal{P}(\mathrm{Left}) \cap \mathcal{P}(\mathrm{Right})$, we have a generalization of Proposition 3.2 (5):

Corollary 4.7. *Let N be fully invariant in M. If N and M/N are each co-Hopfian, Hopfian or DF, then so is M.*

Remark 4.8. Hopfian modules cannot be characterised using ascending filtrations in a similar manner to Theorem 4.5 (3). To see this, it is enough to consider the group $H = \mathbb{Z}(p^\infty)$, which is not Hopfian but has an ascending chain of fully invariant subgroups $H_n = H[p^n]$, $0 \leq n \leq \omega$ such that each H_{n+1}/H_n is Hopfian.

We need instead, using again ideas from [1], to introduce a more restrictive property of the submodules in the filtration, the class

$$\mathcal{SP}(\mathrm{Epi}(M)) = \{N \subseteq M : f(N) = N \text{ for all } f \in \mathrm{Epi}(M)\}.$$

Theorem 4.9. *Let M be a module. The following are equivalent:*

(1) *M is Hopfian;*
(2) *M has a continuous ascending filtration of modules N_ν, $\nu \leq \kappa$, in $\mathcal{SP}(\mathrm{Epi}(M))$ such that for all $\nu < \kappa$, N_ν and $N_{\nu+1}/N_\nu$ are Hopfian.*

Proof. Suppose that M has a continuous ascending filtration as in (2), and let $f : M \to M$ be an epimorphism. If f is not monic, then there is $0 \neq$

$m \in \ker f$. Since $m \notin N_0$, there exists a smallest ordinal ρ such that $m \in N_{\rho+1} \setminus N_\rho$, hence the epimorphism $\overline{f} : N_{\rho+1}/N_\rho \to N_{\rho+1}/N_\rho$, induced by f, is not a monomorphism. Since $N_{\rho+1}/N_\rho$ is Hopfian, it follows that $\overline{f}$ is not an epimorphism, a contradiction. $\square$

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