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# PRE-SCHWARZIAN NORM ESTIMATES OF FUNCTIONS FOR A SUBCLASS OF STRONGLY STARLIKE FUNCTIONS

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**Abstract.** For normalized analytic functions $f$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we consider

$$\mathcal{S}^*(\alpha,\beta) = \left\{ f : \frac{zf'(z)}{f(z)} \prec \left(\frac{1+(1-2\beta)z}{1-z}\right)^{\alpha},\ z \in \mathbb{D} \right\},$$

where $0 < \alpha \leq 1$ and $0 \leq \beta < 1$. There exists a close connection between Bloch functions and univalent functions. In this paper, we present an optimal, but not sharp, estimate of the Bloch semi-norm of the function $\log f'$ for $f \in \mathcal{S}^*(\alpha,\beta)$.

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**Key words.** Pre-Schwarzian derivative, univalent, starlike and strongly starlike functions, subordination.

## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f$ analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$ and $\mathcal{LU}$ denote the subclass of $\mathcal{A}$ consisting of all locally univalent functions, namely, $\mathcal{LU} = \{f \in \mathcal{A} : f'(z) \neq 0,\ z \in \mathbb{D}\}$. We may regard $\mathcal{LU}$ as a vector space over $\mathbb{C}$, not in the usual sense, but in the sense of Hornich operations (see [5, 7, 15]) and we define the norm of $f \in \mathcal{LU}$ by

$$\|f\| = \sup_{z\in\mathbb{D}} (1-|z|^2) \left| \frac{f''(z)}{f'(z)} \right|.$$

Here we note that the quantity $T_f := f''/f'$ is called the pre-Schwarzian derivative of $f$. This norm has significance in the theory of Teichmüller spaces (see e.g. [1]). The norm $\|f\|$ is nothing but the Bloch semi-norm of the function $\log f'$ (see for example [12]). It is well known that $\|f\| \leq 6$ if $f$ is univalent in $\mathbb{D}$, and conversely if $\|f\| \leq 1$ then $f$ is univalent in $\mathbb{D}$, and these bounds are sharp (see [2]). Furthermore, $\|f\| < \infty$ if and only if $f$ is uniformly locally univalent; that is, there exists a constant $\rho = \rho(f),\ 0 < \rho \leq 1$, such that $f$ is univalent in each disk of hyperbolic radius $\tanh^{-1}\rho$ in $\mathbb{D}$, i.e. in each Appolonius disk

$$\left\{ w : \left| \frac{w-z}{1-\bar{z}w} \right| < \rho \right\}, \quad z \in \mathbb{D}$$

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(see [15, 16]). The set of all $f$ with $\|f\| < \infty$ is a nonseparable Banach space (see [15, Theorem 1]). For more geometric and analytic properties of $f$ relating the norm, see [8]. Many authors have given norm estimates for classical subclasses $\mathcal{S}$ of univalent functions (see [4, 6, 9, 10, 11, 13, 17]).

In addition, let $\mathcal{H}$ denote the class of functions $f$ analytic in the unit disk $\mathbb{D}$ and $\mathcal{H}_a$ be the subclass $\{f \in \mathcal{H} : f(0) = a\}$ of it, for $a \in \mathbb{C}$.

We say that a function $\varphi \in \mathcal{H}$ is subordinate to $\psi \in \mathcal{H}$ and write $\varphi \prec \psi$ or $\varphi(z) \prec \psi(z)$, if there is a function $\omega \in \mathcal{H}_0$ with $|\omega(z)| < 1$ satisfying $\varphi = \psi \circ \omega$. Note that the condition $\varphi \prec \psi$ is equivalent to the conditions $\varphi(\mathbb{D}) \subset \psi(\mathbb{D})$ and $\varphi(0) = \psi(0)$ when $\psi$ is univalent.

In this paper, we consider the subclass $\mathcal{S}^*(\alpha, \beta)$ of $\mathcal{A}$ defined by

$$\mathcal{S}^*(\alpha,\beta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h_{\alpha,\beta}(z) \equiv \left(\frac{1+(1-2\beta)z}{1-z}\right)^{\alpha} \right\},$$

for $0 < \alpha \le 1$ and $0 \le \beta < 1$.

Since functions in $\mathcal{S}^*(\alpha,\beta)$ belong to $\mathcal{S}^*(1,0) \equiv \mathcal{S}^*$, $\mathcal{S}^*(\alpha,\beta) \subsetneq \mathcal{S}$ for $0 < \alpha \le 1$ and $0 \le \beta < 1$.

The class $\mathcal{S}^*(\alpha,\beta)$ has been studied by Wesołowski in [14]. With $0 < \alpha \le 1$ and $0 < \beta < 1$, we have

$$h_{\alpha,\beta}(e^{i\theta}) = (\beta + i(1-\beta)\cot(\theta/2))^{\alpha}$$

from which we easily see that the univalent function $h_{\alpha,\beta}(z)$ maps $\mathbb{D}$ onto a convex domain bounded by the curve given by

$$w = \left(\frac{\beta}{\cos\phi}\right)^{\alpha} e^{i\alpha\phi}, \quad -\pi/2 < \phi < \pi/2,$$

where $\phi$ and $\theta$ satisfy the relation $(1-\beta)\cot(\theta/2) = \beta\tan\phi$. In particular, functions in the class $\mathcal{S}^*(\alpha) \equiv \mathcal{S}^*(\alpha,0)$ are called the strongly starlike functions of order $\alpha$; equivalently, $f \in \mathcal{S}^*(\alpha)$ if and only if $|\arg(zf'(z))/f(z)| < \pi\alpha/2$, for $z \in \mathbb{D}$. Every strongly starlike function $f$ of order $\alpha < 1$ is bounded (see [3]). Further, this class of functions has been studied by many authors, for example Sugawa (see [13]). In the same paper Sugawa has presented the sharp norm estimate for $f \in \mathcal{S}^*(\alpha)$. The aim of this article is to generalize the result of Sugawa [13, Theorem 1.1] in the following form:

**MAIN THEOREM.** *Let $0 < \alpha < 1$ and $0 \le \beta < 1$. If $f \in \mathcal{S}^*(\alpha,\beta)$, then*

$$\|f\| \le L(\alpha,\beta) + 2\alpha, \tag{1}$$

*where*

$$L(\alpha,\beta) = \frac{4(1-\beta)(k-\beta)(k^{\alpha}-1)}{(k-1)(k+1-2\beta)} \tag{2}$$

*and $k$ is the unique solution of the following equation in $x \in (1,\infty)$:*

$$\begin{aligned}(1-\alpha)x^{\alpha+2} + \beta(3\alpha-2)x^{\alpha+1} + [(1-2\beta)(1+\alpha) + 2\beta^2(1-\alpha)]x^{\alpha} \\ -\alpha\beta(1-2\beta)x^{\alpha-1} - x^2 + 2\beta x = (1-\beta)^2 + \beta^2.\end{aligned} \tag{3}$$

For $\alpha = 1$, it is well known that $\|f\| \leq 6 - 4\beta$ and equality holds if and only if $f(z) = \overline{\mu}\Phi(\mu z)$, where $\Phi(z) = z/(1-z)^{2(1-\beta)}$ and $\mu$ is a unimodular constant (see [17]). Moreover, if $\alpha = 1$ as well as $\beta = 0$, it is known that $\|f\| \leq 6$; and equality holds for the Koebe function $k(z) = z/(1-z)^2$. Now we shall prove the main theorem by using the method adopted by Sugawa [13].

## 2. PROOF OF THE MAIN THEOREM

Let $p(z) = P_f(z) = zf'(z)/f(z)$ and $f$ belong to the class $\mathcal{S}^*(\alpha,\beta)$. Then, by the definition, $p(z)$ is subordinate to the univalent function

$$q(z) = \left(\frac{1+(1-2\beta)z}{1-z}\right)^{\alpha}, \quad z \in \mathbb{D},$$

and therefore, there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that

$$p = q \circ \omega = \left(\frac{1+(1-2\beta)\omega}{1-\omega}\right)^{\alpha}. \tag{4}$$

Let $F \in \mathcal{A}$ be the function with $P_F = q$, i.e.

$$F(z) = z \exp\left(\int_0^z \frac{q(t)-1}{t}\,\mathrm{d}t\right).$$

We split the proof into two cases. Assume first that $0 \leq \beta \leq 1/2$. Logarithmic differentiation of (4) yields that

$$1 + \frac{zf''}{f'} - \frac{zf'}{f} = \frac{2\alpha(1-\beta)z\omega'}{(1-\omega)(1+(1-2\beta)\omega)}.$$

We thus have

$$T_f(z) = \frac{2\alpha(1-\beta)\omega'(z)}{(1-\omega(z))(1+(1-2\beta)\omega(z))} + \frac{p(z)-1}{z}. \tag{5}$$

By triangle inequality and Schwarz-Pick lemma, we obtain

$$\begin{aligned}
|T_f(z)| &\leq \frac{2\alpha(1-\beta)|\omega'(z)|}{|1-2\beta\omega(z)-(1-2\beta)\omega^2(z)|} + \frac{|p(z)-1|}{|z|} \\
&\leq \frac{2\alpha(1-\beta)(1-|\omega(z)|^2)}{(1-|z|^2)(|1-2\beta\omega(z)|-(1-2\beta)|\omega(z)|^2)} + \frac{|q(\omega(z))-1|}{|z|} \\
&\leq \frac{2\alpha(1-\beta)(1-|\omega(z)|^2)}{(1-|z|^2)(1-2\beta|\omega(z)|-(1-2\beta)|\omega(z)|^2)} + \frac{|q(\omega(z))-1|}{|z|} \\
&\leq \frac{2\alpha(1-\beta)(1+|\omega(z)|)}{(1-|z|^2)(1+(1-2\beta)|\omega(z)|)} + \frac{|q(\omega(z))-1|}{|z|}.
\end{aligned}$$

Using a similar argument, namely the triangle inequality (as we did in the denominator above), we see that

$$\begin{aligned}
|q(z)-1| &= \left|\int_0^z q'(t)\,\mathrm{d}t\right| \\
&= \left|\int_0^z \left(\frac{1+(1-2\beta)t}{1-t}\right)^{\alpha} \frac{2\alpha(1-\beta)}{(1-t)(1+(1-2\beta)t)}\,\mathrm{d}t\right| \\
&\le \int_0^{|z|} \left(\frac{1+(1-2\beta)t}{1-t}\right)^{\alpha} \frac{2\alpha(1-\beta)}{(1-t)(1+(1-2\beta)t)}\,\mathrm{d}t \\
&= q(|z|)-1.
\end{aligned}$$

So, using this inequality and the fact $|\omega(z)|\le |z|$, we get

$$\begin{aligned}
|T_f(z)| &\le \frac{2\alpha(1-\beta)(1+|\omega(z)|)}{(1-|z|^2)(1+(1-2\beta)|\omega(z)|)} + \frac{q(|\omega(z)|)-1}{|z|} \\
&\le \frac{2\alpha(1-\beta)(1+|z|)}{(1-|z|^2)(1+(1-2\beta)|z|)} + \frac{q(|z|)-1}{|z|} \\
&= T_F(|z|),
\end{aligned}$$

where the second inequality is strict provided $\omega(z)/z$ is not a unimodular constant. Therefore, we see that $\|f\|\le\|F\|$.

Since

$$(1-t^2)T_F(t) = \frac{2\alpha(1-\beta)(1+t)}{1+(1-2\beta)t} + \frac{1-t^2}{t}(q(t)-1) \to 2\alpha \text{ as } t\to 1^-,$$

the equality $\|f\|=\|F\|$ holds only if $|T_f(z_0)| = T_F(|z_0|)$ for some $z_0\in\mathbb{D}$. Hence we conclude that equality holds if $P_f(z)=q(\mu z)$ for some unimodular constant $\mu$.

We next consider the case $1/2\le\beta<1$. If we use triangle inequality again without multiplying the factors in the denominator, we obtain

$$|q(z)-1|\le q(|z|)-1.$$

Now using the same argument as in the first case, we get

$$\begin{aligned}
(1-|z|^2)|T_f(z)| &\le \frac{2\alpha(1-\beta)(1-|\omega^2(z)|)}{|1-\omega(z)|\,|1+(1-2\beta)\omega(z)|} + \frac{1-|z|^2}{|z|}(q(|\omega(z)|)-1) \\
&\le \frac{2\alpha(1-\beta)(1+|\omega(z)|)}{1+(1-2\beta)|\omega(z)|} + \frac{1-|z|^2}{|z|}(q(|\omega(z)|)-1) \\
&\le \frac{2\alpha(1-\beta)(1+|z|)}{1+(1-2\beta)|z|} + \frac{1-|z|^2}{|z|}(q(|z|)-1) \\
&= (1-|z|^2)T_F(|z|).
\end{aligned}$$

This shows that $\|f\| \le \|F\|$ and the inequality is sharp (as in the argument of the previous case). Thus, it is enough to compute $\|F\|$. Now, we write

$$L(\alpha,\beta) = \sup_{0<t<1} \frac{1-t^2}{t}(q(t)-1) = \sup_{x>1} g(x),$$

where

$$g(x) = \frac{4(1-\beta)(x-\beta)(x^\alpha - 1)}{(x-1)(x+1-2\beta)}$$

with the substitution $x = [1+(1-2\beta)t]/(1-t)$. Logarithmic derivative of $g(x)$ yields

$$\frac{g'(x)}{g(x)} = -\frac{h(x)}{(x-\beta)(x^\alpha-1)(x-1)(x+1-2\beta)},$$

where $h(x)$ is given by

$$\begin{aligned} h(x) &= (1-\alpha)x^{\alpha+2} + \beta(3\alpha-2)x^{\alpha+1} + [(1+\alpha)(1-2\beta)+2\beta^2(1-\alpha)]x^\alpha \\ &- \alpha\beta(1-2\beta)x^{\alpha-1} - x^2 + 2\beta x - (1-\beta)^2 - \beta^2. \end{aligned}$$

Differentiations give easily the following:

$$\begin{aligned} h'(x) &= (1-\alpha)(\alpha+2)x^{\alpha+1} + \beta(3\alpha-2)(\alpha+1)x^\alpha \\ &+ \alpha[(1+\alpha)(1-2\beta)+2\beta^2(1-\alpha)]x^{\alpha-1} \\ &- \alpha\beta(\alpha-1)(1-2\beta)x^{\alpha-2} - 2x + 2\beta \\ h''(x) &= (1-\alpha)(\alpha+2)(\alpha+1)x^\alpha + \alpha\beta(3\alpha-2)(\alpha+1)x^{\alpha-1} \\ &+ \alpha(\alpha-1)[(1+\alpha)(1-2\beta)+2\beta^2(1-\alpha)]x^{\alpha-2} \\ &- \alpha\beta(\alpha-1)(\alpha-2)(1-2\beta)x^{\alpha-3} - 2 \\ h'''(x) &= (1-\alpha)(\alpha+1)(\alpha+2)\alpha x^{\alpha-1} \\ &+ \alpha\beta(3\alpha-2)(\alpha+1)(\alpha-1)x^{\alpha-2} \\ &+ \alpha(\alpha-1)(\alpha-2)[(1+\alpha)(1-2\beta)+2\beta^2(1-\alpha)]x^{\alpha-3} \\ &- \alpha\beta(\alpha-1)(\alpha-2)(\alpha-3)(1-2\beta)x^{\alpha-4} \\ &= \alpha(1-\alpha)x^{\alpha-4}\phi(x), \end{aligned}$$

where $\phi(x) = (\alpha+1)(\alpha+2)x^3 - \beta(3\alpha-2)(\alpha+1)x^2 - (\alpha-2)[(1+\alpha)(1-2\beta)+2\beta^2(1-\alpha)]x + \beta(1-2\beta)(\alpha-2)(\alpha-3)$.

It follows that

$$\phi'(x) = 3(\alpha+1)(\alpha+2)x^2 + 2\beta(2-3\alpha)(1+\alpha)x + (2-\alpha)[(1+\alpha)(1-2\beta)+2\beta^2(1-\alpha)]$$

and

$$\phi''(x) = 6(\alpha+1)(\alpha+2)x + 2\beta(2-3\alpha)(1+\alpha).$$

Since $\phi'''(x) = 6(\alpha+1)(\alpha+2) > 0$, $\phi''(x)$ is increasing for all $x > 1$. So we have

$$\phi''(x) \ge \phi''(1) = 6\alpha^2(1-\beta) + 16\alpha + 12 + 4\beta + 2\alpha(1-\beta) > 0.$$

This implies that $\phi'(x)$ is increasing for $x>1$ and so

$$\phi'(x)\geq \phi'(1)=2(1+\alpha)(\alpha+2+2(1-\alpha\beta))+2\beta^2(1-\alpha)(2-\alpha)>0.$$

So $\phi(x)$ is also increasing for $x>1$ and hence,

$$\phi(x)\geq \phi(1)=4(1-\beta)(1+\alpha+\beta+\beta(1-\alpha))>0.$$

Therefore, $h'''(x)>0$ and so $h''(x)$ is increasing for $x>1$. Since $h''(x)$ is increasing in $(1,\infty)$ and

$$h''(1)=-2\alpha(1-\beta)[\alpha(1-\beta)+\beta]<0,$$

we see that $h''(x)$ has a unique zero in $(1,\infty)$, say $x=x_1$. Since $h'(1)=0$ and $h'(x)$ is increasing on $(x_1,\infty)$ and decreasing on $(1,x_1)$, we obtain that $h'(x)$ has a unique zero, say $x_2$ $(x_2>x_1)$ in $(1,\infty)$. Since $h(1)=0$, by the same argument we conclude that $h(x)$ has a unique zero, say $k=k(\alpha,\beta)>x_2$ in $(1,\infty)$. Thus $h(x)<0$ in $(1,k)$ and $h(x)>0$ in $(k,\infty)$, equivalently, $g'(x)$ is positive for $x\in(1,k)$ and negative for $x>k$. This shows that $g(x)$ assumes its maximum at $x=k$ and hence we have (2). Since $k$ is the zero of $h(x)$, it is the unique solution of the equation (3). Thus we have established (1). □

Remark 1. Here we calculate some bounds for $L(\alpha,\beta)$ and $k(\alpha,\beta)$ although these are not better estimates. Since $g(x)$ attains its maximum at $k>1$, we note that

$$L(\alpha,\beta)=g(k)>\lim_{x\to 1^+} g(x)=2\alpha(1-\beta).$$

Finally we observe that $g(x)$ satisfies the second order differential equation

$$A(x)g''(x)+B(x)g'(x)+C(x)g(x)=0,$$

where

$$\begin{aligned}
A(x) &= x(x-1)(x+1-2\beta)(x-\beta)^2,\\
B(x) &= 4x(x-\beta)^3+(1-\alpha)(x-1)(x+1-2\beta)(x-\beta)^2\\
&\quad -2x(x-1)(x+1-2\beta)(x-\beta),\\
C(x) &= 2(1-\alpha)(x-\beta)^3-2x(x-\beta)^2\\
&\quad -(1-\alpha)(x-1)(x+1-2\beta)(x-\beta)+2x(x-1)(x+1-2\beta).
\end{aligned}$$

This observation is to justify the close connection between these bounds and special functions.

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