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ON MAPPING PROPERTIES OF LAYER POTENTIAL OPERATORS FOR BRINKMAN EQUATIONS ON LIPSCHITZ DOMAINS IN RIEMANNIAN MANIFOLDS

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Abstract. In this paper we present the main properties of layer potential operators for general Brinkman equations on Lipschitz domains in compact Riemannian manifolds. These properties are used to obtain an existence and solvability result in Sobolev-Slobodetski spaces for a transmission problem given in terms of two general Brinkman operators, when the solution is defined in two complementary Lipschitz or C^1 domains on a Riemannian manifold and satisfies prescribed transmission conditions at the interface between these domains.

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Key words. Brinkman operator, Lipschitz and C^1 domains, Riemannian manifold, transmission problem, layer potential operators.

1. INTRODUCTION

The importance of potential theory in the treatment of boundary value problems for elliptic differential operators is well known. Fabes, Kenig and Verchota [3] extended the method of layer potential operators developed in the treatment of the Stokes system on Euclidean smooth domains to Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, and solved the corresponding Dirichlet problem with L^2 boundary data. Mitrea and Taylor developed the potential theory for elliptic operators on Lipschitz domains in Riemannian manifolds and treated the corresponding boundary value problems by using the layer potential methods (see the series of papers [14]-[17] and also the references therein). In [16] Mitrea and Taylor studied the Stokes system and the corresponding layer potential operators on arbitrary Lipschitz domains in a smooth compact Riemannian manifold, and extended the results in [3] obtained for the Stokes system on Euclidean Lipschitz domains. They treated the L^2-Dirichlet problem for the Stokes system and also more regular versions of it, by using a technique based on single-layer potentials. Dindoš and Mitrea analyzed in [2] the Poisson problem for the Stokes system on C^1 or even Lipschitz domains in a smooth compact Riemannian manifold and with data in Sobolev or Besov spaces, by using a layer potential approach. They extended this approach to operators with variable-coefficients, and, in addition, to the Poisson problem for the stationary, nonlinear Navier-Stokes equations on Riemannian manifolds. Variable

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coefficient transmission problems and spectral theory for singular integral operators on Lipschitz domains on non-smooth manifolds have been treated in [12]. Kohr and Wendland developed in [11] a layer potential approach to show existence and uniqueness to a transmission problem for a Brinkman-coupled system in Lipschitz domains in $\mathbb{R}^3$ and in some Sobolev spaces. Other applications of layer potential methods to boundary value problems for the Stokes or Brinkman operators can be consulted in [9, 10, 18]. Transmission problems for Stokes and Brinkman operators on Lipschitz and C^1 domains in Riemannian manifolds have been treated in [7, 8], by employing layer potential techniques. The purpose of this paper is to extend the results in [7, 8] in the setting of more general Sobolev spaces.

2. PRELIMINARIES

In this section we briefly review some basic results for partial differential equations in compact boundaryless Riemannian manifolds.

2.1. Differential operators in Riemannian manifolds. Let (M, g) be a compact boundaryless oriented manifold of dimension $m \geq 2$, which is equipped with a smooth Riemannian metric tensor

$$g = \sum_{j,k=1}^{m} g_{jk} dx_j \otimes dx_k := g_{jk} dx_j \otimes dx_k,^{1}$$

and let (g^{jk}) be the inverse to (g_{jk}). The volume element in M is given by $\mathrm{dVol} = \sqrt{g} dx_1 \dots dx_m$, where $g := \det(g_{jk})$. The tangent and cotangent bundles are $TM = \bigcup_{p \in M} T_p M$ and $T^*M = \bigcup_{p \in M} T_p^* M$, respectively, and $\mathfrak{X}(M)$ is the space of smooth vector fields on M, i.e., the space $C^\infty(M, TM)$ of smooth sections of TM. One differential forms on M consist of smooth sections of $\Lambda^1 TM = T^*M$. For simplicity, the space of smooth sections of $\Lambda^1 TM$, namely $C^\infty(M, \Lambda^1 TM)$, is also denoted by $\Lambda^1 TM$. One may naturally identify $\Lambda^1 TM$ with $\mathfrak{X}(M)$, and $\Lambda^1 TM$ carries the pointwise inner product

$$\langle dx_j, dx_k \rangle = g^{jk}, \quad \langle X, Y \rangle = X_j g^{jk} Y_k, \tag{2.1}$$

where[2] $X = X^k \partial_k \in TM$ is identified with the one-form $X_r dx_r = X^k g_{kr} dx_r$, and $X_r = g_{kr} X^k$, $X^k = g^{kj} X_j$. The exterior derivative and co-derivative operators are $d : C^\infty(M) \to C^\infty(M, \Lambda^1 TM)$, $\delta : C^\infty(M, \Lambda^1 TM) \to C^\infty(M)$, where $d = \partial_j dx_j$, and $\delta = d^*$, respectively. By ∇ we denote the Levi-Civita connection on M. For $X \in \mathfrak{X}(M)$, the symmetric part of the tensor field

[1]Hereafter Einstein's index summation convention is used.

[2]In (2.1) the notation $\langle \cdot, \cdot \rangle$ is used for the pointwise inner product of both vector fields and one-forms, but, later, the same notation will also be used for the pairing between a vector space and its dual. In what follows, $H^s(X, \Lambda^1 TM|_X)$, for some $X \subset M$, is simply denoted by $H^s(X, \Lambda^1 TM)$. Also $\langle \cdot, \cdot \rangle_X$, for some $X \subset M$, stands for the pairing between a Sobolev space $H^s(X, \Lambda^1 TM)$, or $\tilde{H}^{-s}(X, \Lambda^1 TM)$ $(s > 0)$, and its dual.

$\nabla X : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$, $(\nabla X)(Y,Z) = \langle \nabla_Y X, Z\rangle$, is denoted by Def X and is called the *deformation* of X,

$$(2.2)\qquad (\mathrm{Def}\ X)(Y,Z) = \frac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall\ Y,Z \in \mathfrak{X}(M).$$

A *Killing field* is a vector field $X \in \mathfrak{X}(M)$, which satisfies Def $X = 0$. Hereafter we assume that *the manifold M has no nontrivial Killing fields* (see [16]).

2.2. The Stokes and Brinkman operators on Riemannian manifolds. By OPS^{ℓ}_{cl} one denotes the class of classical pseudodifferential operators of order ℓ. The symbol $p(x,\xi)$ of such an operator $P(x,D)$ admits an asymptotic expansion of the form $p(x,\xi) \sim p_\ell(x,\xi) + p_{\ell-1}(x,\xi) + \cdots$, where $p_k(x,\xi)$ is smooth in x and ξ, positively homogeneous of degree k in $\xi \in \mathbb{R}^m$. The term $\sigma^0_P(x,\xi) := p_\ell(x,\xi)$ is called the *principal symbol* of $P(x,D)$ (for more details on pseudodifferential operators on smooth manifolds see [6, 19, 20, 21]).

For $s \geq 0$, the Sobolev space $H^s(M) := W^{s,2}(M)$ of scalar functions on M is obtained by lifting $H^s(\mathbb{R}^m) := \{(\mathbb{I} - \triangle)^{-s/2} f : f \in L^2(\mathbb{R}^m)\}$ via a partition of unity on M and pullback on corresponding local charts, and $H^0(M) = L^2(M)$. The spaces $H^s(M)$ and $H^{-s}(M)$ are dual to each other with respect to the $L^2(M)$-duality. Similarly, $H^s(M, \Lambda^1 TM) := H^s(M) \otimes \Lambda^1 TM$ is the space of one-forms whose local representations have coefficients in $H^s(M)$.

Let us now consider the second-order partial differential operator

$$(2.3)\qquad L : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad L := 2\mathrm{Def}^*\mathrm{Def} = -\triangle + \mathrm{d}\delta - 2\mathrm{Ric},$$

where Def^* is the adjoint of Def, $\triangle := -(\mathrm{d}\delta + \delta d)$ is the Hodge Laplacian and Ric is the Ricci tensor. This operator is elliptic and extends to a Fredholm operator of index 0, $L : H^1(M, \Lambda^1 TM) \to H^{-1}(M, \Lambda^1 TM)$. Also let us denote by $P \in OPS^0_{\mathrm{cl}}(\Lambda^1 TM, \Lambda^1 TM)$ a self-adjoint and non-negative operator with respect to the $L^2(M, \Lambda^1 TM)$ - inner product $\langle \cdot, \cdot \rangle$, i.e.

$$(2.4)\qquad \langle Pu, w\rangle = \langle u, Pw\rangle, \quad \langle Pu, u\rangle \geq 0 \ \text{ for all } \ u, w \in L^2(M, \Lambda^1 TM).$$

The operator $B_P : C^\infty(M, \Lambda^1 TM) \times C^\infty(M) \to C^\infty(M, \Lambda^1 TM) \times C^\infty(M)$,

$$(2.5)\qquad B_P := \begin{pmatrix} L & \mathrm{d} \\ \delta & 0 \end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_P & \mathrm{d} \\ \delta & 0 \end{pmatrix}, \quad L_P := L + P,$$

called the *general Brinkman operator*, is elliptic in the sense of Agmon-Douglis-Nirenberg (see [7, 8]). It extends to a Fredholm operator of the same index zero $B_P : H^1(M, \Lambda^1 TM) \times L^2(M) \to H^{-1}(M, \Lambda^1 TM) \times L^2(M)$. For $P = 0$ one gets the *Stokes operator* B_0.

2.3. Sobolev spaces on Lipschitz domains and on their boundaries. Let $\Omega_+ := \Omega \subset M$ be a Lipschitz domain and assume that $\Omega_- := M \setminus \overline{\Omega}$ is connected. The sets $\Omega_\pm$ are Lipschitz domains. Let $\mathrm{Tr}^\pm$ be the non-tangential boundary trace operators on $\partial\Omega$, $(\mathrm{Tr}^\pm u)(x) := \lim\limits_{\gamma_\pm(x) \ni y \to x} u(y)$, $x \in \partial\Omega$, where

$\gamma_\pm(x) \subseteq \Omega_\pm$ are appropriate non-tangential approach regions (see e.g. [14]). For $s \geq 0$, let us consider the Sobolev spaces

$$H^s(\Omega_\pm) := \{f|_{\Omega_\pm} : f \in H^s(M)\}, \quad \tilde{H}^s(\Omega_\pm) := \{f \in H^s(M) : \mathrm{supp} f \subseteq \overline{\Omega}_\pm\},$$

and denote by $H^{-s}(\Omega_\pm) = \left(\tilde{H}^s(\Omega_\pm)\right)^*$ the dual of the space $\tilde{H}^s(\Omega_\pm)$ with respect to the $L^2(\Omega_\pm)$-duality. Also, $H^s(\Omega_\pm, \Lambda^1 TM|_{\Omega_\pm}) := H^s(\Omega_\pm) \otimes \Lambda^1 TM|_{\Omega_\pm}$, $\tilde{H}^s(\Omega_\pm, \Lambda^1 TM|_{\Omega_\pm}) := \tilde{H}^s(\Omega_\pm) \otimes \Lambda^1 TM|_{\Omega_\pm}$ are the Sobolev spaces of 1-forms, and $H^{-s}(\Omega_\pm, \Lambda^1 TM) := (\tilde{H}^s(\Omega_\pm, \Lambda^1 TM))^*$ is the dual of $\tilde{H}^s(\Omega_\pm, \Lambda^1 TM)$. Also, for $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, consider the spaces

$$\begin{aligned}
&\tilde{H}^{-1+\beta}(\Omega_\pm, \Lambda^1 TM) = \{\mathbf{f} \in H^{-1+\beta}(M, \Lambda^1 TM) : \mathrm{supp}\ \mathbf{f} \subseteq \overline{\Omega}_\pm\},\\
&H^{1+\beta}(\Omega_\pm, \mathcal{L}_P) = \big\{(\mathbf{u}, \pi, \mathbf{f}) : \mathbf{u} \in H^{1+\beta}(\Omega_\pm, \Lambda^1 TM), \pi \in H^\beta(\Omega_\pm),\\
&\mathbf{f} \in \tilde{H}^{-1+\beta}(\Omega_\pm, \Lambda^1 TM) \text{ such that } \mathcal{L}_P(\mathbf{u}, \pi) = \mathbf{f}|_{\Omega_\pm},\ \delta\mathbf{u} = 0 \text{ in } \Omega_\pm\big\},
\end{aligned} \tag{2.6}$$

where $\mathcal{L}_P(\mathbf{u}, p) := L\mathbf{u} + P\mathbf{u} + dp$. Note that $H^{1+\beta}(\Omega_\pm, \mathcal{L}_P)$ is a normed space with respect to the norm

$$\|(\mathbf{u}, \pi, \mathbf{f})\|_{H^{1+\beta}(\Omega_\pm, \mathcal{L}_P)} := \|\mathbf{u}\|_{H^{1+\beta}(\Omega_\pm, \Lambda^1 TM)} + \|\pi\|_{H^\beta(\Omega_\pm)} + \|\mathbf{f}\|_{H^{-1+\beta}(M, \Lambda^1 TM)}.$$

For $r \in [0, 1]$, by $H^r(\partial\Omega)$ and $H^r(\partial\Omega, \Lambda^1 TM)$ one denotes the boundary Sobolev spaces of scalar functions and 1-forms, respectively, on $\partial\Omega$, and by $H^{-r}(\partial\Omega)$ and $H^{-r}(\partial\Omega, \Lambda^1 TM)$ their dual spaces with respect to the L^2-duality. The trace operators have the property below (see e.g. [1, 2, 5, 18]):

Lemma 2.1. *For every $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, the restriction operator to the boundary, $C^\infty(\overline{\Omega}_\pm, \Lambda^1 TM) \to H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1 TM)$, $u \mapsto u|_{\partial\Omega_\pm}$, extends to a linear and bounded operator $\mathrm{Tr}^\pm : H^s(\Omega_\pm, \Lambda^1 TM) \to H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1 TM)$, which is onto and has a bounded right inverse $\mathcal{Z}^\pm : H^{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1 TM) \to H^s(\Omega_\pm, \Lambda^1 TM)$. For $s > \frac{3}{2}$, $\mathrm{Tr}^\pm : H^s(\Omega_\pm, \Lambda^1 TM) \to H^1(\partial\Omega_\pm, \Lambda^1 TM)$ is also bounded.*

2.4. The conormal derivative operator on Lipschitz boundaries. Let $r \in [0, 1]$ and $\nu \in H^{-r}(\partial\Omega, \Lambda^1 TM)$ be the outward unit conormal to $\partial\Omega$, which is defined with respect to the $L^2(\partial\Omega, \Lambda^1 TM)$-inner product and the outward unit normal field $\mathbf{n} \in L^\infty(\partial\Omega, TM)$. Note that $\mathbf{n}$ is defined a.e., with respect to the surface element $\mathrm{d}\sigma$, on $\partial\Omega$. The next result extends the notion of the conormal derivative operator, given by Mitrea and Wright in [18] for the Stokes system on Lipschitz domains in $\mathbb{R}^n$, to the Brinkman system on Lipschitz domains in Riemannian manifolds (see also [2, 7, 8, 16]):

Lemma 2.2. *For any $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, the conormal derivative operator*

$$\mathbf{t}^\pm : H^{1+\beta}(\Omega_\pm, \mathcal{L}_P) \to H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \tag{2.7}$$

$$\begin{aligned}\pm\langle \mathbf{t}^{\pm}(\mathbf{u},\pi,\mathbf{f}),\Phi\rangle_{\partial\Omega} := {} & 2\int_{\Omega_\pm}\langle \mathrm{Def}\ \mathbf{u},\mathrm{Def}\ (\mathcal{Z}^{\pm}\Phi)\rangle \mathrm{dVol}\\ & + \int_{\Omega_\pm}\langle P\mathbf{u},\mathcal{Z}^{\pm}\Phi\rangle \mathrm{dVol}\\ & + \int_{\Omega_\pm}\langle \pi,\delta(\mathcal{Z}^{\pm}\Psi)\rangle \mathrm{dVol} - \langle \mathbf{f},\mathcal{Z}^{\pm}\Phi\rangle_{\Omega_\pm},\end{aligned} \tag{2.8}$$

$\forall\ \Phi \in H^{\frac{1}{2}-\beta}(\partial\Omega,\Lambda^1TM)$, is well defined and bounded. Also, the Green formula

$$\begin{aligned}\pm\langle \mathbf{t}^{\pm}(\mathbf{u},\pi,\mathbf{f}),\mathrm{Tr}^{\pm}\ \mathbf{v}\rangle_{\partial\Omega} - 2\int_{\Omega_\pm}\langle \mathrm{Def}\ \mathbf{u},\mathrm{Def}\ \mathbf{v}\rangle \mathrm{dVol} - \int_{\Omega_\pm}\langle P\mathbf{u},\mathbf{v}\rangle \mathrm{dVol}\\ = \int_{\Omega_\pm}\langle \pi,\delta\mathbf{v}\rangle \mathrm{dVol} - \langle \mathbf{f},\mathbf{v}\rangle_{\Omega_\pm}\end{aligned} \tag{2.9}$$

holds for all $(\mathbf{u},\pi,\mathbf{f}) \in H^{1+\beta}(\Omega_\pm,\mathcal{L}_P)$ and $\mathbf{v} \in H^{1-\beta}(\Omega_\pm,\Lambda^1TM)$.

Proof. Let us observe that all duality pairings in the right-hand side of (2.8) are well-defined. This shows that $\mathbf{t}^{\pm}(\mathbf{u},\pi,\mathbf{f}) \in H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$ and, in addition,[3]

$$\|\mathbf{t}^{\pm}(\mathbf{u},\pi,\mathbf{f})\|_{H^{-\frac{1}{2}+\beta}(\partial\Omega_\pm,\Lambda^1TM)} \le c\|(\mathbf{u},\pi,\mathbf{f})\|_{H^{1+\beta}(\Omega_\pm,\mathcal{L}_P)}$$

with some constant $c > 0$ and for every $(\mathbf{u},\pi,\mathbf{f}) \in H^{1+\beta}(\Omega_\pm,\mathcal{L}_P)$. This finishes the proof of the well-posedness and boundedness of the operator (2.7)-(2.8). The Green formula (2.9) can be obtained with arguments similar to those in the proof of [8, Lemma 2.2]. □

3. INVERTIBILITY OF THE BRINKMAN OPERATOR

The operator $B_P : H^1(M,\Lambda^1TM)\times L^2(M) \to H^{-1}(M,\Lambda^1TM)\times L^2(M)$, given by (2.5), has kernel $\{0\}\times\mathbb{R}$ and range $H^{-1}(M,\Lambda^1TM)\times L^2_*(M)$, where $L^2_*(M) := H^0_*(M)$ and, for $s \ge 0$, $H^s_*(M) := \{q \in H^s(M) : \langle q,1\rangle = 0\}$. In addition, the restriction of B_P to $H^1(M,\Lambda^1TM)\times L^2_*(M)$, denoted by B^0_P, is an injective Fredholm operator of index zero, and hence, it is invertible (see [7, 8]). Also, the second order pseudodifferential operator

$$L_P = 2\mathrm{Def}^*\mathrm{Def} + P : H^1(M,\Lambda^1TM) \to H^{-1}(M,\Lambda^1TM), \tag{3.1}$$

is invertible. Then we have the following results (see [8]):

LEMMA 3.1. *The operator $\Upsilon_P : L^2_*(M) \to L^2_*(M)$, $\Upsilon_P := \delta L_P^{-1}d$, is invertible, and the inverse of $B^0_P : H^1(M,\Lambda^1TM)\times L^2_*(M) \to H^{-1}(M,\Lambda^1TM)\times$*

[3]The notation $\mathbf{t}^{\pm}_P$ for the conormal derivative operator is more suggestive than $\mathbf{t}^{\pm}$, but, in the sequel, we shall omit the subscript P whenever obvious from the context. Also, we use the notation $\mathbf{t}^{\pm}(\mathbf{u},p)$ instead of $\mathbf{t}^{\pm}(\mathbf{u},p,\mathbf{0})$ whenever $\mathcal{L}_P(\mathbf{u},p) = \mathbf{0}$ in $\Omega_\pm$.

$L^2_*(M)$ *is* $(B^0_P)^{-1} : H^{-1}(M, \Lambda^1 TM) \times L^2_*(M) \to H^1(M, \Lambda^1 TM) \times L^2_*(M)$,

$$(B^0_P)^{-1} := \begin{pmatrix} \mathfrak{A}_P & \mathfrak{B}_P \\ \mathfrak{C}_P & \mathfrak{D}_P \end{pmatrix}, \tag{3.2}$$

where the pseudodifferential operators $\mathfrak{A}_P,\ \mathfrak{B}_P,\ \mathfrak{C}_P,\ \mathfrak{D}_P$ *are defined as*

$$\mathfrak{A}_P := L_P^{-1} - L_P^{-1}\mathrm{d}\,\Upsilon_P^{-1}\,\delta L_P^{-1}, \quad \mathfrak{B}_P := L_P^{-1}\mathrm{d}\Upsilon_P^{-1}, \tag{3.3}$$

$$\mathfrak{C}_P := \Upsilon_P^{-1}\delta L_P^{-1}, \quad \mathfrak{D}_P := -\,\Upsilon_P^{-1}\,. \tag{3.4}$$

Note that $\mathfrak{A}_P \in OPS^{-2}_{\mathrm{cl}}$, $\mathfrak{B}_P$, $\mathfrak{C}_P \in OPS^{-1}_{\mathrm{cl}}$ and $\mathfrak{D}_P \in OPS^0_{\mathrm{cl}}$, and the principal symbols of these operators are given, in local coordinates, by

$$\begin{aligned} \left(\sigma^0_{\mathfrak{A}_P}\right)_{jk} &= \frac{1}{|\xi|^2}\delta_{jk} - \frac{1}{|\xi|^4}\xi_j g^{k\ell}\xi_\ell,\ \left(\sigma^0_{\mathfrak{B}_P}\right)_j = \frac{i}{|\xi|^2}\xi_j, \\ \left(\sigma^0_{\mathfrak{C}_P}\right)_\ell &= \frac{i}{|\xi|^2}g^{\ell k}\xi_k,\ \sigma^0_{\mathfrak{D}_P} = 2 \end{aligned} \tag{3.5}$$

for any $\xi \in \mathbb{R}^m \setminus \{0\}$, where δ_{jk} is the Kronecker symbol. In addition, in the distributional sense, we have on M

$$L_P\mathfrak{A}_P + \mathrm{d}\mathfrak{C}_P = \mathbb{I}, \quad \delta\mathfrak{A}_P = 0, \tag{3.6}$$

where $\mathbb{I}$ is the identity operator on $H^{-1}(M, \Lambda^1 TM)$. Moreover,

$$\begin{aligned} \mathfrak{A}_P - \mathfrak{A}_0 &= -\mathfrak{A}_0 P \mathfrak{A}_P \in OPS^{-4}_{\mathrm{cl}}(\Lambda^1 TM, \Lambda^1 TM), \\ \mathfrak{C}_P - \mathfrak{C}_0 &= -\mathfrak{C}_0 P \mathfrak{A}_P \in OPS^{-3}_{\mathrm{cl}}(\Lambda^1 TM, \mathbb{R}), \end{aligned} \tag{3.7}$$

where the matrix operator

$$(B^0_0)^{-1} := \begin{pmatrix} \mathfrak{A}_0 & \mathfrak{B}_0 \\ \mathfrak{C}_0 & \mathfrak{D}_0 \end{pmatrix} \tag{3.8}$$

is the inverse of the operator B^0_0, which defines the Stokes system. Let us now denote by $\mathcal{G}_P(x,y)$ and $\Pi_P(x,y)$ the Schwartz kernels of the operators $\mathfrak{A}_P$ and $\mathfrak{C}_P$, respectively. Correspondingly, let $\mathcal{G}(x,y)$ and $\Pi(x,y)$ be the Schwartz kernels of Φ_0 and Ψ_0. From (3.6) one obtains on M:

$$(L_x + P_x)\mathcal{G}_P(x,y) + d_x\Pi_P(x,y) = \mathrm{Dirac}_y(x), \quad \delta_x\mathcal{G}_P(x,y) = 0, \tag{3.9}$$

where Dirac_y is the Dirac distribution with mass at y, i.e., $\mathcal{G}_P(x,y)$ and $\Pi_P(x,y)$ determine the fundamental solution of the general Brinkman system on M. In addition using the method of Fourier transform and (3.5), one obtains in local coordinates[4] (see [2])

$$
\begin{aligned}
(\mathcal{G}_P)_{rs}(x,y) =& \frac{1}{4\pi}\Big\{ - g_{rs}(x)\log\left[(e_0(x-y,y))^{1/2}\right] \\
& + (e_0(x-y,y))^{-1/2}(x_\theta - y_\theta)(x_\eta - y_\eta)g_{r\theta}(x)g_{s\eta}(x)\Big\} \\
& + \{\text{lower order pseudohomogeneous terms}\} \text{ if } m=2,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
(\mathcal{G}_P)_{rs}(x,y) =& \frac{1}{m-2}(e_0(x-y,y))^{-(m-2)/2}g_{rs}(x) \\
& + (e_0(x-y,y))^{-m/2}(x_\theta - y_\theta)(x_\eta - y_\eta)g_{r\theta}(x)g_{s\eta}(x) \\
& + \{\text{lower order pseudohomogeneous terms}\} \text{ if m} \geq 3,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
(\Pi_P)_s(x,y) =& 2(e_0(x-y,y))^{-m/2}(x_\tau - y_\tau)g_{s\tau}(x) \\
& + \{\text{lower order pseudohomogeneous terms}\} \text{ if m} \geq 2,
\end{aligned} \tag{3.12}
$$

where $e_0(x-y,y) := g_{rs}(y)(x_r - y_r)(x_s - y_s)$.

4. LAYER POTENTIAL OPERATORS

Let us present the principal properties of layer potential operators for the Brinkman system. As in the previous section, $\Omega \subset M$ is a Lipschitz domain.

4.1. The single-layer potential operator for the Brinkman system. For $r \in [0,1]$ and $\mathbf{f} \in H^{-r}(\partial\Omega, \Lambda^1 TM)$, the single-layer potential $\mathbf{V}_{P;\partial\Omega}\mathbf{f}$ is the $\Lambda^1 TM$-valued function given on $M \setminus \partial\Omega$ by

$$(\mathbf{V}_{P;\partial\Omega}\mathbf{f})(x) := \langle \mathcal{G}_P(x,\cdot), \mathbf{f}\rangle_{\partial\Omega}, \quad x \in M \setminus \partial\Omega. \tag{4.1}$$

The corresponding pressure potential is given by

$$\mathcal{P}^s_{P;\partial\Omega}\mathbf{f} : M \setminus \partial\Omega \to \mathbb{R}, \quad (\mathcal{P}^s_{P;\partial\Omega}\mathbf{f})(x) := \langle \Pi_P(x,\cdot), \mathbf{f}\rangle_{\partial\Omega}, \quad x \in M \setminus \partial\Omega. \tag{4.2}$$

Taking into account (3.9), it follows that the layer potentials $\mathbf{V}_{P;\partial\Omega}\mathbf{f}$ and $\mathcal{P}^s_{P;\partial\Omega}\mathbf{f}$ satisfy on $M \setminus \partial\Omega$ the equations

$$\delta(\mathbf{V}_{P;\partial\Omega}\mathbf{f}) = 0, \quad (L+P)\mathbf{V}_{P;\partial\Omega}\mathbf{f} + \mathrm{d}\mathcal{P}^s_{P;\partial\Omega}\mathbf{f} = 0. \tag{4.3}$$

Also, the non-tangential boundary traces $\mathcal{V}^{\pm}_{P;\partial\Omega}\mathbf{f} := \mathrm{Tr}^{\pm}(\mathbf{V}_{P;\partial\Omega}\mathbf{f})$ of $\mathbf{V}_{P;\partial\Omega}\mathbf{f}$ on $\partial\Omega$ are well-defined (see [16]). In addition, in view of the decompositions

$$(\mathcal{G}_P)_{rs}(x,y) = \mathcal{G}_{rs}(x,y) + (\mathcal{G}_{P,0})_{rs}(x,y), \ (\Pi_P)_s(x,y) = \Pi_s(x,y) + (\Pi_{P,0})_s(x,y)$$

and relations (3.10)-(3.12), one finds that the main properties of the operators (4.1)-(4.2) are provided by those of the corresponding operators for the Stokes system. Consequently, from [16, Proposition 3.3, Theorem 3.1], [2, Theorem 2.1], [13, 1], we get:

[4]Recall that $m = \dim(M)$.

THEOREM 4.1. *For every $r \in [0,1]$, the operator*

$$\mathbf{V}_{P;\partial\Omega} : H^{-r}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{3}{2}-r}(\Omega_\pm, \Lambda^1 TM),$$

is well-defined and bounded. Also,

$$\mathrm{Tr}^+(\mathbf{V}_{P;\partial\Omega}\mathbf{f}) = \mathrm{Tr}^-(\mathbf{V}_{P;\partial\Omega}\mathbf{f}) := \mathcal{V}_{P;\partial\Omega}\mathbf{f},\ \forall\ \mathbf{f} \in H^{-r}(\partial\Omega, \Lambda^1 TM), \tag{4.4}$$

$$\mathbf{V}_{P;\partial\Omega}\nu = 0,\ \mathcal{P}^s_{P;\partial\Omega}\nu = c^{\pm}_{P;\partial\Omega} \in \mathbb{R} \text{ in } \Omega_\pm. \tag{4.5}$$

In addition, the single-layer potential operator

$$\mathcal{V}_{P;\partial\Omega} : H^{-r}(\partial\Omega, \Lambda^1 TM) \to H^{1-r}(\partial\Omega, \Lambda^1 TM) \tag{4.6}$$

is well-defined and bounded.

THEOREM 4.2. *For $\beta \in \left[0, \frac{1}{2}\right)$ the kernel of the layer potential operator*

$$\mathcal{V}_{P;\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$$

is $\mathrm{Ker}\mathcal{V}_{P;\partial\Omega} = \mathbb{R}\nu$, *where* $\mathbb{R}\nu := \{c\nu : c \in \mathbb{R}\}$.

Proof. This result follows from [8, Theorem 5.2] and the (compact) imbedding property $H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \hookrightarrow H^{-\frac{1}{2}}(\partial\Omega, \Lambda^1 TM)$. □

THEOREM 4.3. ([16]) *For any $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ the operator*

$$\mathcal{V}_{0;\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$$

is Fredholm with index zero and induces an invertible operator

$$\begin{aligned} [\mathcal{V}_{0;\partial\Omega}] : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\mu, \\ [\mathcal{V}_{0;\partial\Omega}]([\mathbf{g}]) := [\mathcal{V}_{0;\partial\Omega}\mathbf{g}], \end{aligned} \tag{4.7}$$

where $\mu \in H^1(\partial\Omega, \Lambda^1 TM) \hookrightarrow H^{\frac{1}{2}\pm\beta}(\partial\Omega, \Lambda^1 TM)$ satisfies the condition

$$\langle \nu, \mu \rangle_{\partial\Omega} = 1. \tag{4.8}$$

THEOREM 4.4. *For any $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ the modified layer potential operator*[5] *for the Stokes system*[6] $\tilde{\mathcal{V}}_{\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ *given by* $\tilde{\mathcal{V}}_{\partial\Omega} := \mathcal{V}_{\partial\Omega} + \langle \cdot, \mu \rangle_{\partial\Omega}\mu_{\partial\Omega}$ *is invertible.*

Proof. Let us first show that $\mathrm{Ker}\tilde{\mathcal{V}}_{\partial\Omega} = \{0\}$. Indeed, from [2, Theorem 2.1] and (4.8) we find that $\tilde{\mathcal{V}}_{\partial\Omega}\mathbf{g} = 0 \Leftrightarrow \langle \mathbf{g}, \mu \rangle_{\partial\Omega} = 0$ and $\mathcal{V}_{\partial\Omega}\mathbf{g} = 0$, and hence $\mathbf{g} = \beta\nu$, $\beta \in \mathbb{R}$, and $\beta\langle \nu, \mu \rangle_{\partial\Omega} = 0$, i.e., $\mathbf{g} = 0$. Now, let $\mathbf{G} \in H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ and $\tilde{\mathbf{G}} := \mathbf{G} - \langle \nu, \mathbf{G} \rangle_{\partial\Omega}\mu_{\partial\Omega} \in H^{\frac{1}{2}+\beta}_\nu(\partial\Omega, \Lambda^1 TM)$, where

$$H^{\frac{1}{2}+\beta}_\nu(\partial\Omega, \Lambda^1 TM) := \{\mathbf{h} \in H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) : \langle \nu, \mathbf{h} \rangle_{\partial\Omega} = 0\}.$$

[5]For $P = 0$, we use the notations $\mathbf{V}_{0;\partial\Omega} := \mathbf{V}_{\partial\Omega}$, $\mathcal{V}_{0;\partial\Omega} := \mathcal{V}_{\partial\Omega}$, $\mathcal{P}^s_{0;\partial\Omega} := \mathcal{P}^s_{\partial\Omega}$.

[6]The notation $\langle \cdot, \cdot \rangle_{H^s(X,\Lambda^1 TM)}$, for some $X \subset M$, stands for the inner product on the Hilbert space $H^s(X, \Lambda^1 TM)$, $s \in \mathbb{R}$.

Also let $[\tilde{\mathbf{G}}] := \tilde{\mathbf{G}} + \mathbb{R}\mu_{\partial\Omega} \in H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\mu_{\partial\Omega}$. In view of Theorem 4.3, there exists a unique equivalence class $[\mathbf{g}_0] \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu$ such that $[\mathcal{V}_{\partial\Omega}]([\mathbf{g}_0]) = [\tilde{\mathbf{G}}]$, i.e., $[\mathcal{V}_{\partial\Omega}\mathbf{g}_0] = [\tilde{\mathbf{G}}]$. Hence, $\mathcal{V}_{\partial\Omega}\mathbf{g}_0 - \tilde{\mathbf{G}} = \alpha_0\mu$, where $\alpha_0 \in \mathbb{R}$. Since $\langle \nu, \mathcal{V}_{\partial\Omega}\mathbf{g}_0\rangle_{\partial\Omega} = 0$ and $\langle \nu, \tilde{\mathbf{G}}\rangle_{\partial\Omega} = 0$ one finds $\alpha_0 = 0$. Thus, we have obtained an element $\mathbf{g}_0 \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ such that $\mathcal{V}_{\partial\Omega}\mathbf{g}_0 = \tilde{\mathbf{G}}$, and consider

$$\mathbf{g} := \mathbf{g}_0 - \langle \mathbf{g}_0, \mu\rangle_{\partial\Omega}\nu_{\partial\Omega} + \langle \nu, \mathbf{G}\rangle_{\partial\Omega}\nu \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM).$$

Then one gets $\tilde{\mathcal{V}}_{\partial\Omega}\mathbf{g} = \mathbf{G}$, i.e., $\tilde{\mathcal{V}}_{\partial\Omega}$ is onto. □

THEOREM 4.5. *If either $\Omega \subset M$ is a Lipschitz domain and* $\dim(M) = 2, 3$, *or Ω is a C^1-domain and* $\dim(M) \geq 2$, *then for any $\beta \in \left[0, \frac{1}{2}\right)$ the modified layer potential operator $\tilde{\mathcal{V}}_{P;\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$*

$$\tilde{\mathcal{V}}_{P;\partial\Omega} := \mathcal{V}_{P;\partial\Omega} + \langle \cdot, \mu\rangle_{\partial\Omega}\mu \tag{4.9}$$

is invertible, where $\mu \in H^1(\partial\Omega, \Lambda^1 TM)$ satisfies the condition (4.8).

Proof. $\tilde{\mathcal{V}}_{P;\partial\Omega} = \tilde{\mathcal{V}}_{\partial\Omega} + \mathcal{V}_{P,0;\partial\Omega}$ is invertible since $\tilde{\mathcal{V}}_{\partial\Omega}$ is Fredholm with index 0, the operator $\mathcal{V}_{P,0;\partial\Omega} : H^{-1/2+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{1/2+\beta}(\partial\Omega, \Lambda^1 TM)$, $\mathcal{V}_{P,0;\partial\Omega} = \mathcal{V}_{P;\partial\Omega} - \mathcal{V}_{\partial\Omega}$ is compact (see Theorem 4.7) and $\mathrm{Ker}\tilde{\mathcal{V}}_{P;\partial\Omega} = \{0\}$. □

4.2. The double-layer potential operator for the Brinkman system. For $\mathbf{h} \in H^r(\partial\Omega, \Lambda^1 TM)$, $r \in [0, 1]$, let $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ and $\mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h} : M \setminus \partial\Omega \to \mathbb{R}$ be the double-layer potential and its associated pressure, given on $M \setminus \partial\Omega$ by

$$(\mathbf{W}_{P;\partial\Omega}\mathbf{h})(x) := \int_{\partial\Omega} \big\langle \Pi_P^{\top}(y, x)\nu(y) - 2\big[(\mathrm{Def}_y\ \mathcal{G}_P(x, \cdot))\nu\big](y), \mathbf{h}(y)\big\rangle \mathrm{d}\sigma(y)$$

$$(\mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h})(x) := \int_{\partial\Omega} \langle -2[(\mathrm{Def}_y\ \Pi_P(x, \cdot))\nu](y) - \Xi_P(x, y)\nu(y), \mathbf{h}(y)\rangle \mathrm{d}\sigma(y).$$

The layer potentials $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ and $\mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h}$ satisfy on $M \setminus \partial\Omega$ the equations

$$(L + P)\mathbf{W}_{P;\partial\Omega}\mathbf{h} + \mathrm{d}\mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h} = 0, \quad \delta\mathbf{W}_{P;\partial\Omega}\mathbf{h} = 0. \tag{4.10}$$

Also, the non-tangential boundary traces $\mathbf{W}^{\pm}_{P;\partial\Omega}\mathbf{h} := \mathrm{Tr}^{\pm}(\mathbf{W}_{P;\partial\Omega}\mathbf{h})$ are well-defined (see [2]), and the principal value of $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ for a.e. $x \in \partial\Omega$ is

$$(\mathbf{K}_{P;\partial\Omega}\mathbf{h})(x) := \mathrm{p.v.}\int_{\partial\Omega} \big\langle \Pi_P^{\top}(y, x)\nu(y) - 2\big[(\mathrm{Def}_y\ \mathcal{G}_P(x, \cdot))\nu\big](y), \mathbf{h}(y)\big\rangle \mathrm{d}\sigma(y).$$

Moreover, the main singularity in the kernel $(S_P)_{ks\ell}(x, y)(\nu)_\ell(x)$ of the layer potential $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ is provided by the main singularity in the kernel of the potential $\mathbf{W}_{0;\partial\Omega}\mathbf{h}$ for the Stokes system, i.e., in local coordinates (see [2, 7]):

$$\begin{aligned}
&(S_P)_{ks\ell}(x, y)\nu_\ell(x) = S_{ks\ell}(x, y)\nu_\ell(x) + (S_{P,0})_{ks\ell}(x, y)\nu_\ell(x) \\
&= -\, m e_0(x - y, y)^{-(m+2)/2}\nu_j(x)(x_j - y_j)(x_\theta - y_\theta)(x_\eta - y_\eta)g_{\ell\theta}(x)g_{s\eta}(x) \\
&\quad + \{\text{lower order pseudohomogeneous terms}\}.
\end{aligned}$$

Thus, the properties of the double-layer potential for Brinkman' system follows from those of the double-layer potential for Stokes' system (see [2, 13, 16, 1]):

THEOREM 4.6. *Let $\Omega \subset M$ be a Lipschitz domain and let $r \in [0,1]$. Then*

$$\mathbf{W}_{P;\partial\Omega} : H^r(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+r}(\Omega, \Lambda^1 TM)$$

is well-defined and bounded. For $\mathbf{h} \in H^r(\partial\Omega, \Lambda^1 TM)$, $\mathbf{f} \in H^{-r}(\partial\Omega, \Lambda^1 TM)$:

$$\mathrm{Tr}^+(\mathbf{W}_{P;\partial\Omega}\mathbf{h}) = \Big(\frac{1}{2}\mathbb{I} + \mathbf{K}_{P;\partial\Omega}\Big)\mathbf{h}, \quad \mathrm{Tr}^-(\mathbf{W}_{P;\partial\Omega}\mathbf{h}) = \Big(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{P;\partial\Omega}\Big)\mathbf{h}, \tag{4.11}$$

$$\begin{aligned} \mathbf{t}^+(\mathbf{V}_{P;\partial\Omega}\mathbf{f}, \mathcal{P}^s_{P;\partial\Omega}\mathbf{f}) &= \Big(-\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{P;\partial\Omega}\Big)\mathbf{f}, \\ \mathbf{t}^-(\mathbf{V}_{P;\partial\Omega}\mathbf{f}, \mathcal{P}^s_{P;\partial\Omega}\mathbf{f}) &= \Big(\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{P;\partial\Omega}\Big)\mathbf{f}, \end{aligned} \tag{4.12}$$

where $\mathbf{K}^*_{P;\partial\Omega}$ is the formal transpose of $\mathbf{K}_{P;\partial\Omega}$, i.e., for a.e. $x \in \partial\Omega$

$$(\mathbf{K}^*_{P;\partial\Omega}\mathbf{f})(x) = \mathrm{p.v.}\int_{\partial\Omega} \langle -2[(\mathrm{Def}_x \mathcal{G}_P(\cdot, y))\nu](x) + \Pi_P(x,y)\nu(x), \mathbf{f}(y)\rangle \mathrm{d}\sigma(y).$$

In addition, for $\beta \in \big[0, \frac{1}{2}\big)$, $\mathbf{h} \in H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ and a.e. on $\partial\Omega$:

$$\mathbf{t}^+(\mathbf{W}_{P;\partial\Omega}\mathbf{h}, \mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h}) := \mathbf{D}^+_{P;\partial\Omega}\mathbf{h}, \ \mathbf{t}^-(\mathbf{W}_{P;\partial\Omega}\mathbf{h}, \mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h}) := \mathbf{D}^-_{P;\partial\Omega}\mathbf{h},$$

$$\mathbf{D}^+_{P;\partial\Omega}\mathbf{h} - \mathbf{D}^-_{P;\partial\Omega}\mathbf{h} \in \mathbb{R}\nu. \tag{4.13}$$

The following layer potential operators are well-defined and bounded:

$$\begin{aligned} \mathbf{K}_{P;\partial\Omega} &: H^r(\partial\Omega, \Lambda^1 TM) \to H^r(\partial\Omega, \Lambda^1 TM) \\ \mathbf{K}^*_{P;\partial\Omega} &: H^{r-1}(\partial\Omega, \Lambda^1 TM) \to H^{r-1}(\partial\Omega, \Lambda^1 TM) \\ \mathbf{D}^\pm_{P;\partial\Omega} &: H^r(\partial\Omega, \Lambda^1 TM) \to H^{r-1}(\partial\Omega, \Lambda^1 TM). \end{aligned} \tag{4.14}$$

Proof. We have to show only the relation (4.13), since the others can be obtained as those for the Stokes system in [2, 16]. For a double-layer potential and its associated pressure, $\mathbf{u} := \mathbf{W}_{P;\partial\Omega}\mathbf{h}$ and $\pi := \mathcal{P}^{\mathrm{d}}_{P;\partial\Omega}\mathbf{h}$, one has the layer potential representation formula

$$\mathbf{u}_+ = -\mathbf{V}_{P;\partial\Omega}(\mathbf{t}^+(\mathbf{u},\pi)) + \mathbf{W}_{P;\partial\Omega}(\mathrm{Tr}^+\mathbf{u}) \text{ in } \Omega_+,$$

where $\mathbf{u}_+ := \mathbf{u}|_{\Omega_+}$. A similar representation formula is valid for $\mathbf{u}$ in Ω_-. By applying the non-tangential boundary trace operators $\mathrm{Tr}^\pm$ to these formulas and using the jump relations (4.11), one obtains

$$\frac{1}{4}\mathbf{h} = -\mathcal{V}_{P;\partial\Omega}(\mathbf{t}^\pm(\mathbf{u},\pi)) + \mathbf{K}_{P;\partial\Omega}\mathbf{K}_{P;\partial\Omega}\mathbf{h} \ \text{ a.e. on } \ \partial\Omega. \tag{4.15}$$

Consequently, $\mathbf{t}^+(\mathbf{u},\pi) - \mathbf{t}^-(\mathbf{u},\pi) \in \mathrm{Ker}\mathcal{V}_{P;\partial\Omega}$, where, in view of Theorem 4.2, $\mathrm{Ker}\mathcal{V}_{P;\partial\Omega} = \mathbb{R}\nu$. This shows the desired result. □

We use the notations $\mathbf{D}_{P;\partial\Omega} := \mathbf{D}^+_{P;\partial\Omega}$, $\mathbf{W}_{0;\partial\Omega} := \mathbf{W}_{\partial\Omega}$ and $\mathbf{K}_{0;\partial\Omega} := \mathbf{K}_{\partial\Omega}$.

4.3. Compactness of the complementary layer potential operators.

THEOREM 4.7. *If $\Omega \subset M$ is a Lipschitz domain and $m = 2, 3$, then for any $\beta \in \left[0, \frac{1}{2}\right)$ the following complementary layer potential operators are compact:*

$$\mathcal{V}_{P,0;\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM),\ \mathcal{V}_{P,0;\partial\Omega} = \mathcal{V}_{P;\partial\Omega} - \mathcal{V}_{\partial\Omega}$$

$$\mathbf{K}_{P,0;\partial\Omega} : H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM),\ \mathbf{K}_{P,0;\partial\Omega} = \mathbf{K}_{P;\partial\Omega} - \mathbf{K}_{\partial\Omega}$$

$$\mathbf{K}^*_{P,0;\partial\Omega} : H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM), \mathbf{K}^*_{P,0;\partial\Omega} = \mathbf{K}^*_{P;\partial\Omega} - \mathbf{K}^*_{\partial\Omega}$$

$$\mathbf{D}_{P,0;\partial\Omega} : H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM) \to H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM), \mathbf{D}_{P,0;\partial\Omega} = \mathbf{D}_{P;\partial\Omega} - \mathbf{D}_{0;\partial\Omega}.$$

If Ω is a C^1-domain and $m \geq 2$, the result is still valid for any $\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Proof. This follows by using similar arguments to those in [8, Theorem 5.7]; see also [16, Theorem 3.1], [2, Proposition 3.1]. □

5. TRANSMISSION BOUNDARY VALUE PROBLEM

Let $\beta \in \left[0, \frac{1}{2}\right)$ and $\Omega_+ := \Omega \subset M$, $\Omega_- = M \setminus \overline{\Omega}$ be Lipschitz domains. Let us consider consider $\mathbf{F}_+ \in \tilde{H}^{-1+\beta}(\Omega_+, \Lambda^1 TM)$, $\mathbf{F}_- \in \tilde{H}^{-1+\beta}(\Omega^-, \Lambda^1 TM)$, $\mathbf{G} \in H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$, $\mathbf{H} \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1 TM)$ such that

$$\langle \nu, \mathbf{G} \rangle_{\partial\Omega} = 0. \tag{5.1}$$

Let $P_+, P_- \in OPS^0_{\mathrm{cl}}(\Lambda^1 TM, \Lambda^1 TM)$ be self-adjoint on $L^2(M, \Lambda^1 TM)$, which satisfy the condition

$$\langle P_\pm \mathbf{w}, \mathbf{w} \rangle \geq 0 \ \text{ a.e. on } M,\ \forall\ \mathbf{w} \in L^2(\Omega_+, \Lambda^1 TM) \cup L^2(\Omega_-, \Lambda^1 TM). \tag{5.2}$$

Consequently, $P_\pm$ are non-negative on $L^2(\mathcal{O}, \Lambda^1 TM)$ for any open subset $\mathcal{O}$ of M. In addition, we assume that there exists an open subset $\mathcal{O}_{\partial\Omega} \subset M$ such that $\partial\Omega \subset \mathcal{O}_{\partial\Omega}$ and

$$\int_{\mathcal{O}_{\partial\Omega}} \langle P_+ \mathbf{v}, \mathbf{v} \rangle \mathrm{dVol} \geq 0, \forall\ \mathbf{v} \in L^2(\mathcal{O}_{\partial\Omega}, \Lambda^1 TM); \int_{\mathcal{O}_{\partial\Omega}} \langle P_+ \mathbf{v}, \mathbf{v} \rangle \mathrm{dVol} = 0$$

imply that $\mathbf{v} = 0$, i.e., P_+ is positive definite on $L^2(\mathcal{O}_{\partial\Omega}, \Lambda^1 TM)$. Note that the first condition follows directly from (5.2). For example, one may choose P_+ as $\chi^2 \mathbb{I}$, $\chi > 0$.

Let us now consider the transmission problem for the general Brinkman operators B_{P_+} and B_{P_-}, consisting of the continuity and Brinkman equations:

$$\delta \mathbf{v}_+ = 0, \quad L\mathbf{v}_+ + P_+ \mathbf{v}_+ + dp_+ = \mathbf{F}_+|_{\Omega_+} \text{ in } \Omega_+ \tag{5.3}$$

$$\delta \mathbf{v}_- = 0, \quad L\mathbf{v}_- + P_- \mathbf{v}_- + dp_- = \mathbf{F}_-|_{\Omega^-} \text{ in } \Omega_- \tag{5.4}$$

and the transmission conditions:

$$\mathrm{Tr}^+ \mathbf{v}_+ - \mathrm{Tr}^- \mathbf{v}_- = \mathbf{G}, \ \ \mathbf{t}^+(\mathbf{v}_+, p_+, \mathbf{F}_+) - \mathbf{t}^-(\mathbf{v}_-, p_-, \mathbf{F}_-) = \mathbf{H} \ \text{ on } \ \partial\Omega. \tag{5.5}$$

THEOREM 5.1. *For $\beta \in \left[0, \frac{1}{2}\right)$ the transmission problem* (5.3)-(5.5) *has at most one solution $((\mathbf{v}_+, p_+), (\mathbf{v}_-, p_-))$ (p_+ and p_- are unique up to a constant) with $(\mathbf{v}_+, p_+, \mathbf{F}_+) \in H^{1+\beta}(\Omega_+, \mathcal{L}_{P_+})$, $(\mathbf{v}_-, p_-, \mathbf{F}_-) \in H^{1+\beta}(\Omega_-, \mathcal{L}_{P_-})$.*

Proof. This follows with arguments similar to those for [8, Theorem 7.1]. $\square$

5.1. Existence result for the transmission problem. Next, we use a layer potential method to show the existence of the solution to the transmission problem (5.3)-(5.5) in an L^2-space, when either $\Omega \subset M$ is Lipschitz a domain and $\dim(M) = 2, 3$, or $\Omega \subset M$ is a C^1-domain and $\dim(M) \geq 2$. To show this result in the space $H^{1+\beta}(\Omega_+, \mathcal{L}_{P_+}) \times H^{1+\beta}(\Omega_-, \mathcal{L}_{P_-})$, where $\beta \in \left[0, \frac{1}{2}\right)$, consider the potential representations (see also [7, 8] for $\beta = 0$):

$$\mathbf{v}_+ = \mathbf{W}_{P_+;\partial\Omega}\mathbf{h} + \mathbf{V}_{P_+;\partial\Omega}\mathbf{f} + \mathbf{U}_{\Omega_+}\mathbf{F}_+|_{\Omega_+} \ \text{ in } \ \Omega_+, \tag{5.6}$$

$$p_+ = \mathcal{P}^{\mathrm{d}}_{P_+;\partial\Omega}\mathbf{h} + \mathcal{P}^{s}_{P_+;\partial\Omega}\mathbf{f} + \mathcal{Q}_{\Omega_+}\mathbf{F}_+|_{\Omega_+} \ \text{ in } \ \Omega_+, \tag{5.7}$$

$$\mathbf{v}_- = \mathbf{W}_{P_-;\partial\Omega}\mathbf{h} + \mathbf{V}_{P_-;\partial\Omega}\mathbf{f} + \mathbf{U}_{\Omega^-}\mathbf{F}_-|_{\Omega_-} \ \text{ in } \ \Omega_-, \tag{5.8}$$

$$p_- = \mathcal{P}^{\mathrm{d}}_{P_-;\partial\Omega}\mathbf{h} + \mathcal{P}^{s}_{P_-;\partial\Omega}\mathbf{f} + \mathcal{Q}_{P_-;\Omega_-}\mathbf{F}_-|_{\Omega^-} + \lambda\langle \mathbf{f}, \mu\rangle_{\partial\Omega} \ \text{ in } \ \Omega_-, \tag{5.9}$$

where $\mathbf{h} \in H^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1TM)$, $\mathbf{f} \in H^{-\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1TM)$ are unknown densities, $\mu \in H^1(\partial\Omega, \Lambda^1TM)$ satisfies the condition (4.8), and $\lambda \in \mathbb{R}$ is a constant. The choice of this constant will be specified later. Also, $\mathbf{U}_{\Omega_\pm}\mathbf{g}$ and $\mathcal{Q}_{\Omega_\pm}\mathbf{g}$ are the Newtonian potentials

$$\int_{\Omega_\pm} \langle \mathcal{G}_{P_\pm}(x, y), \mathbf{g}(y)\rangle \mathrm{dVol}(y), \quad \int_{\Omega_\pm} \langle \Pi_{P_\pm}(x, y), \mathbf{g}(y)\rangle \mathrm{dVol}(y), \quad x \in \Omega_\pm.$$

In particular,

$$L_{P_\pm}\mathbf{U}_{\Omega_\pm} + \mathrm{d}\mathcal{Q}_{\Omega_\pm} = \mathbf{I}, \quad \delta\mathbf{U}_{\Omega_\pm} = 0.$$

Now, using the relations (4.4) and (4.11), which provide the non-tangential boundary traces of single- and double-layer potentials on both sides of $\partial\Omega$, and the first of the transmission conditions (5.5), one obtains the equation:

$$\begin{aligned} &\mathbf{h} + \mathbf{K}_{P_+,P_-;\partial\Omega}\mathbf{h} + \mathcal{V}_{P_+,P_-;\partial\Omega}\mathbf{f} = \\ &\qquad \mathbf{G} - \mathrm{Tr}^+(\mathbf{U}_{\Omega_+}\mathbf{F}_+|_{\Omega_+}) + \mathrm{Tr}^-(\mathbf{U}_{\Omega_-}\mathbf{F}_-) \text{ a.e. on } \partial\Omega, \end{aligned} \tag{5.10}$$

where $\mathbf{K}_{P_+,P_-;\partial\Omega} := \mathbf{K}_{P_+,0;\partial\Omega} - \mathbf{K}_{P_-,0;\partial\Omega}$, $\mathcal{V}_{P_+,P_-;\partial\Omega} = \mathcal{V}_{P_+,0;\partial\Omega} - \mathcal{V}_{P_-,0;\partial\Omega}$. On the other hand, by imposing the second transmission condition in (5.5) to the potential representations (5.6) and (5.8), one gets the equation

$$\begin{aligned} &-\mathbf{f} + \mathbf{K}^*_{P_+,P_-;\partial\Omega}\mathbf{f} - \lambda\langle \mathbf{f}, \mu\rangle_{\partial\Omega}\nu + \mathfrak{H}_{P_+,P_-;\partial\Omega}\mathbf{h} = \\ &\mathbf{H} - \mathbf{t}^+(\mathbf{U}_{\Omega_+}\mathbf{F}_+|_{\Omega_+}, \mathcal{Q}_{\Omega_+}\mathbf{F}_+|_{\Omega_+}) + \mathbf{t}^-(\mathbf{U}_{\Omega_-}\mathbf{F}_-|_{\Omega_-}, \mathcal{Q}_{\Omega_-}\mathbf{F}_-|_{\Omega_-}) \text{ on } \partial\Omega, \end{aligned} \tag{5.11}$$

where $\mathbf{t}^\pm(\mathbf{U}_{\Omega_\pm}\mathbf{F}_\pm|_{\Omega_\pm}, \mathcal{Q}_{\Omega_\pm}\mathbf{F}_\pm|_{\Omega_\pm})$ is the conormal derivative operator on $\partial\Omega$ due to the Newtonian potential $\mathbf{U}_{\Omega_\pm}\mathbf{F}_\pm|_{\Omega_\pm}$. Also,

$$\mathbf{K}^*_{P_+,P_-;\partial\Omega} := \mathbf{K}^*_{P_+;\partial\Omega} - \mathbf{K}^*_{P_-;\partial\Omega}$$

and

$$\begin{aligned} \mathfrak{H}_{P_+,P_-;\partial\Omega} &:= \mathbf{D}^+_{P_+;\partial\Omega} - \mathbf{D}^-_{P_-;\partial\Omega} \\ &= (\mathbf{D}^+_{P_+;\partial\Omega} - \mathbf{D}^+_{0;\partial\Omega}) + (\mathbf{D}^+_{P_-;\partial\Omega} - \mathbf{D}^-_{P_-;\partial\Omega}) - (\mathbf{D}^+_{P_-;\partial\Omega} - \mathbf{D}^+_{0;\partial\Omega}) \end{aligned}$$

are compact operators on $H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$ and $H^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)$, respectively (see (4.13) and Theorem 4.7). Consequently, the transmission problem (5.3)-(5.5) reduces to the system of Fredholm integral equations of the second kind (5.10) and (5.11), which has the equivalent matrix form

$$(\mathcal{I}_{\mathcal{X}}-\mathcal{A}_{P_+,P_-})\mathfrak{u}=\mathfrak{w}\quad \text{in}\quad \mathcal{X}, \tag{5.12}$$

where $\mathcal{I}_{\mathcal{X}}$ is the identity operator on the Hilbert space

$$\mathcal{X}=H^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)\times H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM), \tag{5.13}$$

and $\mathcal{A}_{P_+,P_-}:\mathcal{X}\to\mathcal{X}$ is the matrix operator

$$\mathcal{A}_{P_+,P_-}:=\left(\begin{array}{cc} -\mathbf{K}_{P_+,P_-;\partial\Omega} & -\mathcal{V}_{P_+,P_-;\partial\Omega}\\ \mathfrak{H}_{P_+,P_-;\partial\Omega} & \mathbf{K}^*_{P_+,P_-;\partial\Omega}-\lambda\langle\mathbf{f},\mu\rangle_{\partial\Omega}\nu\end{array}\right).$$

Also, $\mathfrak{u}:=(\mathbf{h},\mathbf{f})^\top\in\mathcal{X}$ is the column matrix of unknowns, and

$$\mathfrak{w}:=\left(\begin{array}{c}\mathbf{G}-\mathrm{Tr}^+(\mathbf{U}_{\Omega_+}\mathbf{F}_+|_{\Omega_+})+\mathrm{Tr}^-(\mathbf{U}_{\Omega^-}\mathbf{F}_-|_{\Omega^-})\\ -\mathbf{H}+\mathbf{t}^+(\mathbf{U}_{\Omega_+}\mathbf{F}_+|_{\Omega_+},\mathcal{Q}_{\Omega_+}\mathbf{F}|_{\Omega_+})-\mathbf{t}^-(\mathbf{U}_{\Omega^-}\mathbf{F}_-|_{\Omega^-},\mathcal{Q}_{\Omega^-}\mathbf{F}|_{\Omega^-})\end{array}\right)\in\mathcal{X}$$

Moreover, $\mathfrak{w}\in\mathcal{X}_\nu$, where $\mathcal{X}_\nu$ is the space

$$\mathcal{X}_\nu:=H_\nu^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)\times H^{-\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM), \tag{5.14}$$

$$H_\nu^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM)=\{\mathbf{g}\in H^{\frac{1}{2}+\beta}(\partial\Omega,\Lambda^1TM):\langle\nu,\mathbf{g}\rangle_{\partial\Omega}=0\}. \tag{5.15}$$

In addition, $\mathcal{X}_\nu$ is an invariant subspace of $\mathcal{A}_{P_+,P_-}$. Since the operator $\mathcal{A}_{P_+,P_-}$ is compact, its restriction $\mathcal{X}_\nu\to\mathcal{X}_\nu$, $(\mathbf{r},\mathbf{q})\mapsto\mathcal{A}_{P_+,P_-}(\mathbf{r},\mathbf{q})$, is also compact. Consequently, $\mathcal{I}_{\mathcal{X}_\nu}-\mathcal{A}_{P_+,P_-}:\mathcal{X}_\nu\to\mathcal{X}_\nu$ is a Fredholm operator of index zero, i.e., the existence of the solution $(\mathbf{h},\mathbf{f})^\top$ to the equation (5.12) in the Hilbert space $\mathcal{X}_\nu$ is equivalent with its uniqueness in the same space. According to this property, let us consider the homogeneous equation

$$(\mathcal{I}_{\mathcal{X}_\nu}-\mathcal{A}_{P_+,P_-})\mathfrak{u}^0=0,\ \ \mathfrak{u}^0:=(\mathbf{h}^0,\mathbf{f}^0)^\top\in\mathcal{X}_\nu. \tag{5.16}$$

By using an arbitrary solution $(\mathbf{h}^0,\mathbf{f}^0)^\top\in\mathcal{X}_\nu$ to this equation, we consider the layer potential representations $(\mathbf{u}_+,q_+)$ and $(\mathbf{u}_-,q_-)$, given by

$$\begin{aligned}\mathbf{u}_+&=\mathbf{W}_{P_+;\partial\Omega}\mathbf{h}^0+\mathbf{V}_{P_+;\partial\Omega}\mathbf{f}^0,\\ q_+&=\mathcal{P}^{\mathrm{d}}_{P_+;\partial\Omega}\mathbf{h}^0+\mathcal{P}^{s}_{P_+;\partial\Omega}\mathbf{f}^0 \text{ in } M\setminus\partial\Omega,\\ \mathbf{u}_-&=\mathbf{W}_{P_-;\partial\Omega}\mathbf{h}^0+\mathbf{V}_{P_-;\partial\Omega}\mathbf{f}^0 \text{ in } M\setminus\partial\Omega,\\ q_-&=\mathcal{P}^{\mathrm{d}}_{P_-;\partial\Omega}\mathbf{h}^0+\mathcal{P}^{s}_{P_-;\partial\Omega}\mathbf{f}^0+\lambda\langle\mathbf{f}^0,\mu\rangle_{\partial\Omega,\Lambda^1TM}\nu \text{ in } M\setminus\partial\Omega.\end{aligned}$$

In view of the Green formula (2.9) and the relations $\mathrm{Tr}^+\mathbf{u}_+ = \mathrm{Tr}^-\mathbf{u}_-$ and $\mathbf{t}^+(\mathbf{u}_+, q_+) = \mathbf{t}^-(\mathbf{u}_-, q_-)$ on $\partial\Omega$, one obtains the equality

$$\int_{\Omega_+} \big\{2\langle \mathrm{Def}\ \mathbf{u}_+, \mathrm{Def}\ \mathbf{u}_+\rangle + \langle P_+\mathbf{u}_+, \mathbf{u}_+\rangle\big\}\mathrm{dVol} = \\ -\int_{\Omega_-} \big\{2\langle \mathrm{Def}\ \mathbf{u}_-, \mathrm{Def}\ \mathbf{u}_-\rangle + \langle P_-\mathbf{u}_-, \mathbf{u}_-\rangle\big\}\mathrm{dVol}.$$

Thus, one gets the equations $\mathrm{Def}\ \mathbf{u}_+ = 0$ in Ω_+, $\mathrm{Def}\ \mathbf{u}_- = 0$ in Ω_-, and

$$\int_{\Omega_+} \langle P_+\mathbf{u}_+, \mathbf{u}_+\rangle\mathrm{dVol} = 0, \quad \int_{\Omega_-} \langle P_-\mathbf{u}_-, \mathbf{u}_-\rangle\mathrm{dVol} = 0.$$

As in the proof of [8, Theorem 7.1], one gets a constant $c_0 \in \mathbb{R}$ such that

$$\mathbf{u}_+ = 0 \text{ and } q_+ = c_0 \text{ in } \Omega_+, \quad \mathbf{u}_- = 0 \text{ and } q_- = c_0 \text{ in } \Omega_-. \tag{5.17}$$

Now, using (4.4), (4.11) and (5.17), one finds that

$$\mathrm{Tr}^-\mathbf{u}_+ = -\mathbf{h}^0,\ \mathrm{Tr}^+\mathbf{u}_- = \mathbf{h}^0 \text{ on } \partial\Omega. \tag{5.18}$$

In addition, one has

$$\mathbf{t}^+(\mathbf{u}_+, q_+) - \mathbf{t}^-(\mathbf{u}_+, q_+) = -\mathbf{f}^0 + \lambda_+\nu \text{ on } \partial\Omega,$$

where $\mathbf{D}^+_{P_+;\partial\Omega}\mathbf{h}^0 - \mathbf{D}^-_{P_+;\partial\Omega}\mathbf{h}^0 := \lambda_+\nu \in \mathbb{R}\nu$, and, hence,

$$\mathbf{t}^-(\mathbf{u}_+, q_+) = (c_0 - \lambda_+)\nu + \mathbf{f}^0 \text{ on } \partial\Omega. \tag{5.19}$$

Also, in view of $\mathbf{D}^+_{P_-;\partial\Omega}\mathbf{h}^0 - \mathbf{D}^-_{P_-;\partial\Omega}\mathbf{h}^0 := \lambda_-\nu \in \mathbb{R}\nu$, one finds that

$$\mathbf{t}^+(\mathbf{u}_-, q_-) = (c_0 + \lambda_-)\nu - \mathbf{f}^0 \text{ on } \partial\Omega. \tag{5.20}$$

Further, the Green formula (2.9) and (5.18) and (5.20) yield

$$\int_{\Omega_+} \big\{2\langle \mathrm{Def}\ \mathbf{u}_-, \mathrm{Def}\ \mathbf{u}_-\rangle + \langle P_-\mathbf{u}_-, \mathbf{u}_-\rangle\big\}\mathrm{dVol} = C_{0,P_-}\langle \nu, \mathbf{h}^0\rangle_{\partial\Omega} - \langle \mathbf{f}^0, \mathbf{h}^0\rangle_{\partial\Omega},$$

where $C_{0,P_-} := c_0 + \lambda_-$. Similarly, with $C_{0,P_+} := c_0 - \lambda_+$, one has

$$\int_{\Omega_-} \big\{2\langle \mathrm{Def}\ \mathbf{u}_+, \mathrm{Def}\ \mathbf{u}_+\rangle + \langle P_+\mathbf{u}_+, \mathbf{u}_+\rangle\big\}\mathrm{dVol} = C_{0,P_+}\langle \nu, \mathbf{h}^0\rangle_{\partial\Omega} + \langle \mathbf{f}^0, \mathbf{h}^0\rangle_{\partial\Omega}.$$

Since $\mathbf{h}^0 \in H_\nu^{\frac{1}{2}+\beta}(\partial\Omega, \Lambda^1TM)$, both integrals on $\Omega_\pm$ from above vanish, and hence

$$\int_{\Omega_-} \big\{2\langle \mathrm{Def}\ \mathbf{u}_+, \mathrm{Def}\ \mathbf{u}_+\rangle + \langle P_+\mathbf{u}_+, \mathbf{u}_+\rangle\big\}\mathrm{dVol} = 0,$$

which, by means of the positive definiteness of P_+ on $L^2(\mathcal{O}_{\partial\Omega}, \Lambda^1TM)$, yields that $\mathrm{Tr}^-\mathbf{u}_+ = 0$ on $\partial\Omega$. According to [2, Theorem 5.6], [8], one has $\mathbf{u}_+ = 0$ and $q_+ = c_+ \in \mathbb{R}$ in Ω_-. The first of these relations and (5.18) imply $\mathbf{h}^0 = 0$ on $\partial\Omega$. Further, from (5.18) one gets $\mathrm{Tr}^+\mathbf{u}_- = \mathbf{0}$ on $\partial\Omega$, and hence $\mathbf{u}_- = 0$, $q_- = c_- \in \mathbb{R}$ in Ω_+. These relations and (5.20) imply $\mathbf{f}^0 = c\nu$ on $\partial\Omega$, where $c := c_0 + \lambda_- - c_- \in \mathbb{R}$. Now, let us assume that $\lambda \neq -1 + C_{P_+,P_-;\partial\Omega}$, where the constant λ appears in (5.9) and $C_{P_+,P_-;\partial\Omega} \in \mathbb{R}$ such that

$$\mathbf{K}^*_{P_+,P_-;\partial\Omega}\nu = C_{P_+,P_-;\partial\Omega}\nu \text{ on } \partial\Omega. \tag{5.21}$$

Then, by using the homogeneous version of the equation (5.11), as well as the relation $\mathbf{h}^0 = 0$ on $\partial\Omega$, one gets $c = 0$, i.e. $\mathbf{f}^0 = 0$ on $\partial\Omega$. Consequently, the equation (5.16) has only the trivial solution in the space $\mathcal{X}_\nu$, and hence one obtains the following existence and uniqueness result:

THEOREM 5.2. *Let* $\beta \in \left[0, \frac{1}{2}\right)$ *and* $\lambda \neq -1+C_{P_+,P_-;\partial\Omega}$, *where* $C_{P_+,P_-;\partial\Omega} \in \mathbb{R}$ *is given by* (5.21). *If either* $\dim(M) = 2, 3$ *and* $\Omega \subset M$ *is a Lipschitz domain, or* $\dim(M) \geq 2$ *and* Ω *is a* C^1*-domain, then the system* (5.10)-(5.11) *has a unique solution* $(\mathbf{h}, \mathbf{f})^\top \in \mathcal{X}_\nu$, *where* $\mathcal{X}_\nu$ *is defined in* (5.14). *In addition, the potential representations* (5.6)-(5.9), *provided by* $\mathbf{h}$ *and* $\mathbf{f}$, *determine the unique solution* $((\mathbf{v}_+, p_+), (\mathbf{v}_-, p_-))$ *of the transmission problem* (5.3)-(5.5) (p_+, p_- *are unique up to a constant*), *such that* $(\mathbf{v}_+, p_+, \mathbf{F}_+) \in H^{1+\beta}(\Omega_+, \mathcal{L}_{P_+})$ *and* $(\mathbf{v}_-, p_-, \mathbf{F}_-) \in H^{1+\beta}(\Omega_-, \mathcal{L}_{P_-})$, *where* $H^{1+\beta}(\Omega_\pm, \mathcal{L}_{P_\pm})$ *are defined in* (2.6).

REMARK 5.3. For $\beta = 0$, one obtains the result in [8, Theorem 8.1], and, in addition, if $P_+ = \chi^2\mathbb{I}$ and $P_- = 0$, where $\chi > 0$, one obtains [7, Theorem 6.1].

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