MATHEMATICA, Tome 51 (74), N° 2, 2009, pp. 173–180

GEOMETRIC PROPERTIES OF A PARTICULAR FUNCTION

RÓBERT SZÁSZ and PÁL A. KUPÁN

Abstract. In this paper we will determine the radius of starlikeness and convexity of a particular function.

MSC 2000. 30C45.
Key words. Convexity, starlikeness, set of poles.

1. INTRODUCTION

Let $U(z_0, r) = \{z \in \mathbb{C} : \ |z - z_0| < r\}$ be the open disc with center z_0 and radius r. The particular disc $U(0, 1)$ will be denoted by U. Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc U and having the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots .$$

It is simple to prove that the function f_0 defined by the equality

$$f_0(z) = \frac{z^2}{\sin z}$$

belongs to the class $\mathcal{A}$. The class of starlike functions S^* is a subclass of $\mathcal{A}$ and consists of functions f for which the domain $f(U)$ is starlike with respect to 0. An analytic description of S^* is ([2], pp.8)

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0, \ z \in U \right\}.$$

A function $f \in \mathcal{A}$ belongs to the class K of convex functions if and only if $f(U)$ is a convex domain in $\mathbb{C}$. It is well-known (see [2], pp.8) that

$$K = \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in U \right\}.$$

We are going to determine

$$r_1 = \sup\left\{ r \in (0, \infty) : \ \frac{1}{r} f_0(rz) \text{ is in } \ S^* \right\}$$

and

$$r_2 = \sup\left\{ r \in (0, \infty) : \ \frac{1}{r} f_0(rz) \text{ belongs } \text{ to } K \right\}.$$

The real number r_1 is the radius of starlikeness and r_2 is the radius of convexity. These problems are equivalent to determine the largest $r_1, r_2 \in (0, \infty)$ so that

This work was supported by the Institut for Research Programs (KPI) of the Sapientia Foundation.

$f_0(U(0,r_1))$ is starlike with respect to 0 and that $f_0(U(0,r_2))$ is a convex domain, respectively.

REMARK 1. The analytic descriptions of S^* and K imply that

$$r_1 = \sup\left\{r \in (0,\infty) :\ \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ \text{for all } z \in U(0,r)\right\} \tag{1}$$

and

$$r_2 = \sup\left\{r \in (0,\infty) :\ \mathrm{Re}\big(1 + \frac{zf''(z)}{f'(z)}\big) > 0,\ \text{for all } z \in U(0,r)\right\}. \tag{2}$$

The aim of this paper is to determine the radius of starlikeness and convexity of the function f_0.

2. PRELIMINARIES

In order to prove the main result we need the following lemmas.

LEMMA 1. ([1], p. 200) (Chauchy's theorem) *Let f be a meromorphic function on $\mathbb{C}$ so that zero is a regular point for f and f has only simple poles. Let $P_f = \{\alpha_j \in \mathbb{C} :\ j \in N^*\}$ be the set of poles of the function f. Suppose that $(\Gamma_n)_{n\geq 1}$ is a sequence of simple rectifiable contours having the properties:*

(i) $0 \in \mathrm{Int}(\Gamma_n) \subset \mathrm{Int}(\Gamma_{n+1})$, *where* $\mathrm{Int}(\Gamma_n)$ *denotes the bounded domain determined by the contour* Γ_n.

(ii) $\lim\limits_{n\to\infty} d(0,\Gamma_n) = 0$, *where* $d(0,\Gamma_n) = \inf\{|z|, z \in \Gamma_n\}$.

(iii) *There exists* $A > 0$ *so that* $L(\Gamma_n) < Ad(0,\Gamma_n),\ n \in \mathbb{N}^*$.

(iv) *There exists* $B > 0$ *so that* $|f(z)| < B,\ \ z \in \Gamma_n,\ n \in \mathbb{N}^*$.

If $m(n)$ denotes the number of poles of the function f contained in the domain $\mathrm{Int}(\Gamma_n)$, then the following equality holds:

$$f(z) = f(0) + \lim_{n\to\infty} \sum_{j=1}^{m(n)} \mathrm{Res}(f,\alpha_j)\left(\frac{1}{z-\alpha_j} + \frac{1}{\alpha_j}\right).$$

The obtained series is uniformly convergent on every compact subset of $\mathbb{C} \setminus P_f$.

LEMMA 2. *If $v \in \mathbb{C},\ \ \alpha \in \mathbb{R}$ and $\alpha > |v|$, then*

$$\mathrm{Re}\left(\frac{v}{\alpha - v}\right) \geq \frac{-|v|}{\alpha + |v|}, \tag{3}$$

$$\mathrm{Re}\left(\frac{v}{\alpha + v}\right) \geq \frac{-|v|}{\alpha - |v|}. \tag{4}$$

Proof. Let $v = x + \mathrm{i}y$ and $m = |v| = \sqrt{x^2+y^2}$. Inequality (3) becomes

$$\frac{\alpha x - m^2}{\alpha^2 - 2\alpha x + m^2} \geq \frac{-m}{\alpha + m}$$

which is equivalent to

$$\alpha(\alpha-m)(m+x)\geq 0.$$

The proof of the second inequality is similar. □

LEMMA 3. *If* $\alpha,\beta\in\mathbb{R}$, $\alpha>\beta\geq\pi$ *and* $v\in\mathbb{C}$, $|v|<\frac{\pi}{2}$, $|\alpha-\beta|<\frac{\pi}{2}$ *then*

$$\mathrm{Re}\frac{(2\alpha-v-\beta)v}{(\alpha-v)(\beta-v)}\geq-\frac{(2\alpha+|v|-\beta)|v|}{(\alpha+|v|)(\beta+|v|)}.$$

Proof. The desiderated inequality is equivalent to

$$\mathrm{Re}\left[\frac{2|v|}{\beta+|v|}+\frac{2v}{\beta-v}-\left(\frac{|v|}{\alpha+|v|}+\frac{v}{\alpha-v}\right)\right]\geq 0.$$

For $v=x+\mathrm{i}y$ the inequality becomes

$$(x+\sqrt{x^2+y^2})\left[\frac{2\beta(\beta-\sqrt{x^2+y^2})}{(\beta+\sqrt{x^2+y^2})((\beta-x)^2+y^2)}-\right.$$
$$\left.\frac{\alpha(\alpha-\sqrt{x^2+y^2})}{(\alpha+\sqrt{x^2+y^2})((\alpha-x)^2+y^2)}\right]\geq 0.$$

Thus we only have to show that

$$\frac{2\beta(\beta-\sqrt{x^2+y^2})}{(\beta+\sqrt{x^2+y^2})((\beta-x)^2+y^2)}-\frac{\alpha(\alpha-\sqrt{x^2+y^2})}{(\alpha+\sqrt{x^2+y^2})((\alpha-x)^2+y^2)}>0. \tag{5}$$

Let $g\colon[\pi,\infty)\to\mathbb{R}$ be the function defined by

$$g(t)=\frac{t(t-\sqrt{x^2+y^2})}{(t+\sqrt{x^2+y^2})((t-x)^2+y^2)}.$$

The inequality (5) is equivalent to $\ln\left(\frac{g(\alpha)}{g(\beta)}\right)<\ln 2$. We have to discuss the case $\frac{g(\alpha)}{g(\beta)}>1$. The mean value theorem for the function $h(t)=\ln(g(t))$ implies that there is a point $c\in(\beta,\alpha)$ so that

$$\ln\left(\frac{g(\alpha)}{g(\beta)}\right)=(\alpha-\beta)\frac{g'(c)}{g(c)}=(\alpha-\beta)\left(\frac{1}{c}+\frac{1}{c-m}-\frac{1}{c+m}-\frac{2c-2x}{c^2-2cx+m^2}\right),$$

where $m=\sqrt{x^2+y^2}$. A simple calculation leads to

$$\begin{aligned}&\frac{1}{c}+\frac{1}{c-m}-\frac{1}{c+m}-\frac{2c-2x}{c^2-2cx+m^2}\\ =\ &\frac{1}{c}-\frac{2c^2(c-m-x)+2mc(2x-m)+2m^2(x-m)}{(c^2-m^2)(c^2-2cx+m^2)}<\frac{1}{c}.\end{aligned} \tag{6}$$

Relation (6) and the conditions $\pi<\beta<\alpha$, $|\alpha-\beta|<\frac{\pi}{2}$ imply that

$$\ln\left(\frac{g(\alpha)}{g(\beta)}\right)<(\alpha-\beta)\frac{1}{c}<\frac{\pi}{2}\frac{1}{\pi}=\frac{1}{2}<\ln 2.$$

□

LEMMA 4. *Let Γ_n be the quadrate determined by the vertexes $\pm n\pi \pm \mathrm{i}n\pi$, where n is a fixed natural number. The equation $z\cos z = 2\sin z$ has exactly $2n+1$ roots inside the quadrate Γ_n. Between this $2n+1$ roots two are pure imaginary and the others are real. Also, zero is a root . If $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$ denote the positive real roots then $\alpha_k \in (k\pi, (k+\frac{1}{2})\pi)$, $k = \overline{1, n-1}$, and the negative roots are $-\alpha_k$, $k = \overline{1, n-1}$.*

Proof. If z is a point on the side of the quadrate Γ_n, then

$$z = \pm n\pi + \mathrm{i}y,\ y \in [-n\pi, n\pi] \quad \text{or} \quad z = x \pm \mathrm{i}n\pi,\ x \in [-n\pi, n\pi].$$

In the first case

$$|z\cos z| = \sqrt{n^2\pi^2 + y^2}|\cos \mathrm{i}y| = \sqrt{n^2\pi^2 + y^2}\ \mathrm{ch}y$$

and

$$|2\sin z| = 2\mathrm{sh}y,$$

which means that

$$|z\cos z| > |2\sin z|.$$

In the second case $|z\cos z| = \sqrt{n^2\pi^2 + y^2}|\cos(x \pm \mathrm{i}n\pi)| \geq \sqrt{n^2\pi^2 + y^2}\ \mathrm{sh}n\pi$ and $|2\sin z| = 2|\sin(x \pm \mathrm{i}n\pi)| \leq 2\mathrm{ch}(\mathrm{n}\pi)$. It is easy to show that

$$\sqrt{n^2\pi^2 + y^2}\ \mathrm{sh}n\pi \geq \mathrm{ch}(\mathrm{n}\pi)$$

and so the inequality

$$|z\cos z| > |2\sin z|$$

holds true in the second case too. Rouche's theorem yields that the equations

$$z\cos z = 0 \ \text{ and } \ z\cos z - 2\sin z = 0$$

have the same number of roots inside the quadrate Γ_n and that the equation $z\cos z = 0$ has exactly 2n+1 roots in $\mathrm{Int}(\Gamma_n)$, where $\mathrm{Int}(\Gamma_n)$ denotes the domain bounded by the curve Γ_n.

If $z = x \in \mathbb{R}$ then the equation $z\cos z - 2\sin z = 0$ is equivalent to $\tan x = \frac{x}{2}$. This equation has exactly one simple root in every interval $(k\pi+\frac{\pi}{2}, k\pi+\frac{\pi}{3})$, $k = \overline{1, n-1}$, zero is a simple root too, and if $\alpha_k \in (k\pi + \frac{\pi}{2}, k\pi + \frac{\pi}{3}), k = \overline{1, n-1}$, are roots, then $-\alpha_k, k = \overline{1, n-1}$ are also roots.

In the case $z = \mathrm{i}y$, $y \in \mathbb{R}$, the equation $z\cos z - 2\sin z = 0$ becomes $\tanh y = \frac{y}{2}$. This equation has two real roots $\pm y_0$ with $y_0 \in (\frac{3}{2}, 2)$.

We finally obtain that the set of the roots of the equation $z\cos z - 2\sin z = 0$ is $\{\pm \mathrm{i}y_0 : y_0 \in (\frac{3}{2}, 2)\} \cup \{0\} \cup \{\pm\alpha_k : \alpha_k \in (k\pi + \frac{\pi}{2}, k\pi + \frac{\pi}{3}),\ k = \overline{1, n-1}\}$. □

LEMMA 5. *Let h be the function defined by*

$$h(z) = \frac{2\sin z + z^2 \sin z}{2z\sin z - z^2\cos z}.$$

Then there exists a real number $B > 0$ which does not depend on the natural number n so that

$$|h(z)| < B \quad \textit{for} \quad \textit{all} \ \ z \in \Gamma_n \ \textit{and} \ \ n \geq 1.$$

Proof. If z is a point on the side of the quadrate Γ_n, then

$$z = \pm n\pi + \mathrm{i}y,\ y \in [-n\pi, n\pi] \quad \text{or} \quad z = x \pm \mathrm{i}n\pi,\ x \in [-n\pi, n\pi].$$

We have in the first case

$$z = \pm n\pi + \mathrm{i}y \ \text{ and } \ |\cot z| = |\cot(\mathrm{i}y)| = \left|\frac{1+\mathrm{e}^{-2y}}{1-\mathrm{e}^{-2y}}\right| > 1, y \neq 0.$$

This implies that

$$|h(z)| = \left|\frac{2+z^2}{2z - z^2\cot z}\right| \leq \frac{1+\frac{2}{|z|^2}}{|\cot z| - \frac{2}{|z|}} \leq \frac{1+\frac{2}{|z|^2}}{1-\frac{2}{|z|}} < 6,\ \text{and}\ h(\pm n\pi) = 0.$$

The second case $z = x \pm \mathrm{i}n\pi$ leads to the relations

$$|\cot z| = \left|\frac{\mathrm{e}^{-2\pi}\mathrm{e}^{2\mathrm{i}x}+1}{\mathrm{e}^{-2\pi}\mathrm{e}^{2\mathrm{i}x}-1}\right| \geq \frac{1-\mathrm{e}^{-2\pi}}{1+\mathrm{e}^{-2\pi}}$$

and

$$|h(z)| = \left|\frac{2+z^2}{2z - z^2\cot z}\right| \leq \frac{1+\frac{2}{|z|^2}}{|\cot z| - \frac{2}{|z|}} \leq \frac{1+\frac{2}{|z|^2}}{|\cot z| - \frac{2}{|z|}} \leq \frac{1+\frac{2}{\pi^2}}{\frac{1-\mathrm{e}^{-2\pi}}{1+\mathrm{e}^{-2\pi}} - \frac{2}{\pi}}.$$

Put $B = \max\left\{6, \frac{1+\frac{2}{\pi^2}}{\frac{1-\mathrm{e}^{-2\pi}}{1+\mathrm{e}^{-2\pi}} - \frac{2}{\pi}}\right\}$. Then we conclude that

$$|h(z)| \leq B,\ \text{for all}\ z \in \Gamma_n,\ n \geq 1.$$

□

3. THE MAIN RESULT

THEOREM 1. *The radius of starlikeness of the function* $f_0(z) = \frac{z^2}{\sin z}$ *is the unique root* $r_1 \in (1,2)$ *of the equation*

$$2 - r\coth r = 0.$$

Proof. According to the equality (1) from Remark 1, we have to determine the largest $r_1 \in (0,\infty)$ so that

$$\mathrm{Re}\frac{zf_0'(z)}{f_0(z)} > 0 \quad \text{for} \quad \text{every}\ z \in U(0, r_1).$$

A simple calculation gives $\frac{zf_0'(z)}{f_0(z)} = 2 - z\cot z$. It is well-known that

$$z\cot z = 1 + \sum_{k=1}^{\infty} \frac{2z^2}{z^2 - k^2\pi^2}$$

and that the function series is uniformly convergent on every compact subset of $\mathbb{C} \setminus \{k\pi :\ k \in \mathbb{Z}\}$. This leads to

$$\mathrm{Re}\frac{zf_0'(z)}{f_0(z)} = 1 + \mathrm{Re}\sum_{k=1}^{\infty} \frac{2z^2}{k^2\pi^2 - z^2}.$$

If $\pi > |z|$ and $v = z^2$, then Lemma 2 implies that

$$\mathrm{Re}\frac{2z^2}{k^2\pi^2 - z^2} \geq \frac{-2|z|^2}{k^2\pi^2 + |z|^2}$$

and

$$\mathrm{Re}\frac{zf_0'(z)}{f_0(z)} = 1 + \mathrm{Re}\sum_{k=1}^{\infty}\frac{2z^2}{k^2\pi^2 - z^2} \geq 1 - \sum_{k=1}^{\infty}\frac{2|z|^2}{k^2\pi^2 + |z|^2} = \frac{\mathrm{i}|z|f_0'(\mathrm{i}|z|)}{f_0(\mathrm{i}|z|)}.$$

Equality holds in the last inequality from above if and only if $z = \mathrm{i}|z| = \mathrm{i}r$. This means that the largest $r_1 \in (0, \infty)$ for which the inequality $\mathrm{Re}\frac{zf'(z)}{f(z)} > 0$ is true for every $z \in U(0, r_1)$ is the root of the equation

$$1 - \sum_{k=1}^{\infty}\frac{2r^2}{k^2\pi^2 + r^2} = \frac{\mathrm{i}rf_0'(\mathrm{i}r)}{f_0(\mathrm{i}r)} = 0,$$

or, equivalently,

$$2 - r\ \mathrm{coth}r = 0.$$

An elementary study of the behavior of the function $\varphi\colon (0,2) \to \mathbb{R}$, $\varphi(r) = r\ \mathrm{coth}r - 2$ shows that it has a unique root $r_1 \in (1,2)$, where $r_1 = 1{,}915\ldots.$ □

Remark 2. Since $z_1 = \mathrm{i}r_1$ is the root of the derivative $f_0'(z)$, the function f_0 is not univalent on any disc $U(0,r),\ r > r_1$. This means that r_1 is simultaneously the radius of star-likeness and the radius of univalence of the function f_0.

Theorem 2. *The radius of the convexity of the function* $f_0(z) = \frac{z^2}{\sin z}$ *is the unique solution* $r_2 \in (0,1)$ *of the equation*

$$1 + \frac{2\mathrm{sinh}r - r^2\mathrm{sinh}r}{2\mathrm{sinh}r - r\mathrm{cosh}r} - 2r\mathrm{coth}r = 0.$$

Proof. It is simple to prove that the point $z_0 = 0$ is a removable singularity of the function $f_0(z) = \frac{z^2}{\sin z}$ and that $f_0 \in \mathcal{A}$. According to (2), the image of the disk $U(0, r_2)$ under the function f_0 is a convex domain if and only if

$$\mathrm{Re}\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right) > 0, \quad \text{for} \quad \text{every} \quad z \in U(0, r_2).$$

We have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = 1 + \frac{2\sin z + z^2\sin z}{2\sin z - z\cos z} - 2z\cot z.$$

Denote by

$$h(z) = \frac{2\sin z + z^2\sin z}{2z\sin z - z^2\cos z}.$$

Then Lemma 5 implies that the restriction of the function $h_1(z) = h(z) - \frac{2}{z}$ to the set $\cup_{n=1}^{\infty}\Gamma_n$ is bounded. It is easy to observe that zero is a regular point of the function h_1 and that $h_1(0) = 0$.

According to Lemma 4, the poles of the function h_1 in the domain $\mathrm{Int}\Gamma_n$ are simple and the set of poles is

$$\{\pm \mathrm{i}y_0 :\ y_0 \in (\frac{3}{2}, 2)\} \cup \{\pm\alpha_k :\ \alpha_k \in (k\pi + \frac{\pi}{2}, k\pi + \frac{\pi}{3}),\ k = \overline{1, n-1}\}.$$

Each condition of Lemma 1 is satisfied and we get that

$$h_1(z) = \frac{2z}{z^2 + y_0^2} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - \alpha_k^2}.$$

Using again the equality

$$z \cot z = 1 + \sum_{k=1}^{\infty} \frac{2z^2}{z^2 - k^2\pi^2},$$

it follows that

$$1 + \frac{z f_0''(z)}{f_0'(z)} = 1 + zh(z) - 2z\cot z = 1 + \frac{2z^2}{z^2 + y_0^2} + \sum_{k=1}^{\infty} \frac{2z^2(2\alpha_k^2 - k^2 - z^2)}{(k^2\pi^2 - z^2)(\alpha_k^2 - z^2)}.$$

Inequality (4) of Lemma 2, and Lemma 3 imply that

$$\mathrm{Re}\frac{2z^2}{z^2 + y_0^2} \geq -\frac{2|z|^2}{y_0^2 - |z|^2}$$

$$\mathrm{Re}\frac{2z^2(2\alpha_k^2 - k^2\pi^2 - z^2)}{(k^2\pi^2 - z^2)(\alpha_k^2 - z^2)} \geq \frac{-2|z|^2(2\alpha_k^2 - k^2\pi^2 + |z|^2)}{(k^2\pi^2 + |z|^2)(\alpha_k^2 + |z|^2)}.$$

Thus the following relations hold for every $z \in U(0, y_0)$

$$\mathrm{Re}\left(1 + \frac{z f_0''(z)}{f_0'(z)}\right) \geq 1 + \frac{-2|z|^2}{y_0^2 - |z|^2} - \sum_{k=1}^{\infty} \frac{2|z|^2(2\alpha_k^2 - k^2 + |z|^2)}{(k^2\pi^2 + |z|^2)(\alpha_k^2 + |z|^2)} = 1 + \frac{\mathrm{i}|z| f_0''(\mathrm{i}|z|)}{f_0'(\mathrm{i}|z|)}.$$

Equality occurs in the above inequality only if $z = \mathrm{i}|z| = \mathrm{i}r$. This means that the radius of the convexity is the smallest positive root of the equation

$$1 + \frac{-2r^2}{y_0^2 - r^2} - \sum_{k=1}^{\infty} \frac{2r^2(2\alpha_k^2 - k^2 + r^2)}{(k^2\pi^2 + r^2)(\alpha_k^2 + r^2)} = 0,$$

or, equivalently,

$$1 + \frac{2\sin(\mathrm{i}r) + (\mathrm{i}r)^2 \sin(\mathrm{i}r)}{2\sin(\mathrm{i}r) - \mathrm{i}r\cos(\mathrm{i}r)} - 2\mathrm{i}r\cot(\mathrm{i}r) = 0.$$

This can be rewritten in the form

$$1 + \frac{\sinh r - r^2 \sinh r}{2\sinh r - r\cosh r} - 2r\coth r = 0.$$

A simple study of the above equation shows that it has exactly one root $r_2 \in (0, 1)$, with $r_2 = 0,9361\ldots$ and $r_2 < \min\{y_0, \alpha_1\}$. □

COROLLARY 1. (a) *The radius of convexity of the function*

$$f_1(z) = \frac{z^2}{\sinh z}$$

it is also r_2.

(b) *The following inequalities hold for all* $z \in U(0, r_2)$

$$\frac{r_2^2}{\sin r_2} \geq \mathrm{Re}\frac{z^2}{\sin z} \geq -\frac{r_2^2}{\sin r_2},$$

and

$$\frac{r_2^2}{\sinh r_2} \geq \mathrm{Re}\frac{z^2}{\sinh z} \geq -\frac{r_2^2}{\sinh r_2}.$$

(c) *The largest value* $M > 0$ *for which the inequality* $\mathrm{Re}\frac{(Mz)^2}{\sin(Mz)} \geq -\frac{1}{2}$ *holds for all* $z \in U$ *is the positive real root of the equation* $\sin M = 2M^2$.

REFERENCES

[1] DUNCAN, J., *The Elements of Complex Analysis*, John Wiley and Sons, 1969.
[2] MILLER, S.S. and MOCANU, P.T., *Differential Subordinations. Theory and Applications*, Marcel Dekker, 2000.

Received August 26, 2008
Accepted October 8, 2008

Department of Mathematics
Sapientia University
Str. Sighişoarei 1c
547367 Corunca (Tg. Mureş), Romania
E-mail: szasz_robert2001@yahoo.com
E-mail: kupanp@ms.sapientia.ro