MATHEMATICA, Tome 51(74), N° 2, 2009, pp. 143–151

LOEWNER CHAINS AND QUASICONFORMAL EXTENSIONS OF HOLOMORPHIC MAPPINGS IN $\mathbb{C}^n$

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Abstract. In this paper, by using the method of Loewner chains, we give an univalence criterion which contains as particular cases some univalence criteria for holomorphic mappings in the unit ball of $\mathbb{C}^n$. Also, we obtain a sufficient condition for a normalized quasiregular mapping $f \in \mathcal{H}(B)$ to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

MSC 2000. 32H99, 30C45.

Key words. Loewner chains, Loewner differential equation, quasiregular mapping, quasiconformal mapping, quasiconformal extension.

1. INTRODUCTION AND PRELIMINARIES

Let $f : U \to \mathbb{C}$ be a complex valued function that is holomorphic and locally univalent in the open disc of the complex plane and satisfies the condition:

$$(1 - |z|^2)|zf''(z)/f'(z)| \leq q, \quad |z| < 1. \tag{1.1}$$

Becker [1] has shown that $q = 1$ implies that f is univalent in $|z| < 1$, and $q < 1$ implies that f can be extended to a quasiconformal homeomorphism of the whole plane onto itself. Becker's work is based on the theory of Loewner chains and the generalized Loewner differential equation. Pfaltzgraff [16], [17] obtained the n-dimensional ($n > 1$) generalizations of the previous results.

The problems of univalence criteria and quasiconformal extensions for quasiregular holomorphic mappings on the unit ball in $\mathbb{C}^n$ have been studied by P. Curt [3], [4], [5], [7], H. Hamada and G. Kohr [14], [15], P. Curt and G. Kohr [9], [10], [11], A.A. Brodskii [2].

In this paper we shall generalize the results due to J.A. Pfaltzgraff [16], [17], P. Curt [3], [4], [5], [7].

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w}_i$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let B denote the open unit ball in $\mathbb{C}^n$.

Let $\mathcal{H}(B)$ be the set of holomorphic mappings from B into $\mathbb{C}^n$. Also, let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \ \|z\| = 1\}.$$

By I we denote the identity in $\mathcal{L}(\mathbb{C}^n)$. A mapping $f \in \mathcal{H}(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

We say that a mapping $f \in \mathcal{H}(B)$ is K-quasiregular, $K \geq 1$, if

$$\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in B.$$

A mapping $f \in \mathcal{H}(B)$ is called quasiregular if is K-quasiregular for some $K \geq 1$. It is well known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 1.1. Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(x)\|^m \leq K|\det Df(x)| \text{ a.e. } , x \in G$$

where $Df(x)$ denotes the real Jacobian matrix of f and K is a constant.

Note that a K-quasiregular biholomorphic mapping is K^2-quasiconformal.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in \mathcal{H}(B)$ and $\|v(z)\| \leq \|z\|$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$.

Definition 1.2. A mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if the following conditions hold:
(i) $L(0, t) = 0$ and $L(\cdot, t) \in \mathcal{H}(B)$ for $t \geq 0$;
(ii) $L(\cdot, s) \prec L(\cdot, t)$ for $0 \leq s \leq t < \infty$.

A key role in our discussion is played by the n-dimensional version of the class of holomorphic functions on the unit disc with positive real part

$$\mathcal{N} = \{h \in \mathcal{H}(B) : \ h(0) = 0, \ \mathrm{Re}\, \langle h(z), z\rangle > 0, \ z \in B \setminus \{0\}\}$$

$$\mathcal{M} = \{h \in \mathcal{N}; \ Dh(0) = I\}.$$

I. Graham, H. Hamada, G. Kohr ([12], Theorem 1.10) and P. Curt, G. Kohr [8] proved that normalized univalent subordination chains satisfy the generalized Loewner differential equation.

By using an elementary change of variable, it is not difficult to reformulate the mentioned result in the case of nonnormalized subordination chains $L(z, t) = a(t)z + \ldots$, where $a : [0, \infty) \to \mathbb{C}$, $a(\cdot) \in C^1([0, \infty))$, $a(0) = 1$ and $\lim\limits_{t\to\infty} |a(t)| = \infty$.

Theorem 1.3. *Let $L(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ be a Loewner chain such that $L(z, t) = a(t)z + \ldots$, where $a \in C^1([0, \infty))$, $a(0) = 1$, and $\lim\limits_{t\to\infty} |a(t)| = \infty$. Then there exists a mapping $h = h(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ such that $h(\cdot, t) \in \mathcal{N}$ for $t \geq 0$, $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B$ and*

$$\frac{\partial L}{\partial z}(z, t) = DL(z, t)h(z, t), \ \text{a.e. } t \geq 0, \ \forall\, z \in B. \tag{1.2}$$

We shall use the following theorem to prove our results [6]. We mention that this result is a simplified version of Theorem 3 in [3] due to Theorem 1.2 in [12].

THEOREM 1.4. *Let $L(z,t) = a(t)z + \dots,$ be a function from $B \times [0,\infty)$ into $\mathbb{C}^n$ such that*

(i) *$L(\cdot,t) \in \mathcal{H}(B)$, for each $t \geq 0$*
(ii) *$L(z,t)$ is absolutely continuous of t, locally uniformly with respect to B.*

Let $h(z,t)$ be a function from $B \times [0,\infty)$ into $\mathbb{C}^n$ such that

(iii) *$h(\cdot,t) \in \mathcal{N}$ for each $t \geq 0$*
(iv) *$h(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in B$.*

Suppose $h(z,t)$ satisfies:

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t) \text{ a.e. } t \geq 0, \ \forall\, z \in B.$$

Further, suppose

(a) *$a(0) = 1$, $\lim\limits_{t\to\infty} |a(t)| = \infty$, $a(\cdot) \in C^1([0,\infty))$.*
(b) *There is a sequence $\{t_m\}_m$, $t_m > 0$, $t_m \to \infty$ such that*

$$\lim_{m\to\infty} \frac{L(z,t)}{a(t_m)} = F(z) \tag{1.3}$$

locally uniformly in B, where $F \in \mathcal{H}(B)$.

Then for each $t \geq 0$, $L(\cdot,t)$ is univalent on B.

Recently, P. Curt and G. Kohr [11] proved the following result that will be used in the next section.

THEOREM 1.5. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$, $L(z,t) = a(t)z + \dots,$ be a Loewner chain such that $a(\cdot) \in C^1[0,\infty)$, $a(0) = 1$ and $\lim\limits_{t\to\infty} |a(t)| = \infty$. Assume that the following conditions hold:*

(i) *There exists $K > 0$ such that $L(\cdot,t)$ is K-quasiregular for each $t \geq 0$.*
(ii) *There exist some constants $M > 0$ and $\beta \in [0,1)$ such that*

$$\|DL(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^\beta}, \quad z \in B,\ t \in [0,\infty) \tag{1.4}$$

(iii) *There exists a sequence $\{t_m\}_{m\in\mathbb{N}}$, $t_m > 0$, $\lim\limits_{m\to\infty} t_m = \infty$, and a mapping $F \in \mathcal{H}(B)$ such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a(t_m)} = F(z) \text{ locally uniformly on } B.$$

Further, assume that the mapping $h(z,t)$ defined by Theorem 1.4 satisfies the following conditions

(iv) *There exists a constant $C > 0$ such that*

$$C\|z\|^2 \le \mathrm{Re}\,\langle h(z,t), z\rangle, \quad z \in B,\ t \in [0,\infty) \tag{1.5}$$

(v) *There exists a constant $C_1 > 0$ such that*

$$\|h(z,t)\| \le C_1, \quad z \in B,\ t \in [0,\infty). \tag{1.6}$$

Then the function $f = L(\cdot, 0)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

2. MAIN RESULTS

THEOREM 2.1. *Let $f, g \in \mathcal{H}(B)$ be such that $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ and g is quasiregular in B. Also let $c \in \mathbb{C} \setminus \{-1\}$ with $|c| \le 1$ and let α be a real number, $\alpha \ge 2$.*

If the following conditions hold

$$\left\|(c+1)[Dg(z)]^{-1}Df(z) - \frac{\alpha}{2}I\right\| < \frac{\alpha}{2}, \quad z \in B \tag{2.1}$$

and

$$\Big\|\|z\|^{\alpha}\{(c+1)[Dg(z)]^{-1}Df(z) - I\} \tag{2.2}$$

$$+(1-\|z\|^{\alpha})[Dg(z)]^{-1}D^2g(z)(z,\cdot) + \left(1-\frac{\alpha}{2}\right)\Big\| < \frac{\alpha}{2},\ z \in B$$

then the function f is univalent on B.

Proof. We shall show that (2.1) and (2.2) enables us to embed f as the initial element $f(z) = L(z,0)$ of a suitable subordination chain.

We define

$$L(z,t) = f(\mathrm{e}^{-t}z) + \frac{1}{1+c}(\mathrm{e}^{\alpha t} - 1)\mathrm{e}^{-t}Dg(\mathrm{e}^{-t}z)(z), \quad t \ge 0,\ z \in B. \tag{2.3}$$

Since $a_1(t) = \dfrac{\mathrm{e}^{(\alpha-1)t}(1+c\mathrm{e}^{-\alpha t})}{1+c}$ we deduce that $a(t) \ne 0$, $a(0) = 1$, $\lim\limits_{t\to\infty} |a(t)| = \infty$ and $a(\cdot) \in C^1([0,\infty))$.

It is easy to check that:

$$L(z,t) = a(t)z + (\text{holomorphic term}) \text{ so } \lim_{t\to\infty}\frac{L(z,t)}{a(t)} = z$$

locally uniformly with respect to $z \in B$, and thus (1.3) holds with $F(z) = z$. Obviously L satisfies the absolute continuity requirements of Theorem 1.4.

From (2.3) we obtain:

$$DL(z,t) = \frac{1}{1+c}\mathrm{e}^{(\alpha-1)t}\frac{\alpha}{2}Dg(\mathrm{e}^{-t}z)\left\{I - \left(1-\frac{2}{\alpha}\right)I\right. \tag{2.4}$$

$$+\frac{2}{\alpha}\mathrm{e}^{-\alpha t}\{(c+1)[Dg(\mathrm{e}^{-t}z)]^{-1}Df(\mathrm{e}^{-t}z) - I\}$$

$$\left.+\frac{2}{\alpha}(1-\mathrm{e}^{-\alpha t})[Dg(\mathrm{e}^{-t}z)]^{-1}D^2g(\mathrm{e}^{-t}z)(\mathrm{e}^{-t}z,\cdot)\right\}.$$

If we let, for each fixed $(z,t)\in B\times[0,\infty)$, $E(z,t)$ the linear operator

$$E(z,t)=-\frac{2}{\alpha}\mathrm{e}^{-\alpha t}\{(c+1)[Dg(\mathrm{e}^{-z}z)]^{-1}Df(\mathrm{e}^{-t}z)-I\} \tag{2.5}$$
$$-\frac{2}{\alpha}(1-\mathrm{e}^{-\alpha t})[Dg(\mathrm{e}^{-t}z)]^{-1}D^2g(\mathrm{e}^{-t}z)(\mathrm{e}^{-t}z,\cdot)+I\left(1-\frac{2}{\alpha}\right)$$

then (2.4) becomes:

$$DL(z,t)=\frac{1}{1+c}\mathrm{e}^{(\alpha-1)t}\frac{\alpha}{2}Dg(\mathrm{e}^{-t}z)[I-E(z,t)] \tag{2.6}$$

Next, we shall show that for each $z\in B$ and $t\in[0,\infty)$, $I-E(z,t)$ is an invertible operator.

For $t=0$, $E(z,0)=\left(1-\frac{2}{\alpha}-\frac{2}{\alpha}c\right)I$, we have $I-E(z,t)=\frac{2}{\alpha}(1+c)I$ and since $1+c\neq 0$ it follows that $I-E(z,0)$ is an invertible operator.

For $t>0$, since $E(\cdot,t):\overline{B}\to\mathcal{L}(\mathbb{C}^n)$ is holomorphic, by using the weak maximum modulus theorem we obtain that $\|E(z,t)\|$ can have no maximum in B unless $\|E(z,t)\|$ is of constant value throughout $\overline{B}$.

If $z=0$ and $t>0$, since $\alpha\geq 2$, we have

$$\|E(0,t)\|=\left\|\left(1-\frac{2}{\alpha}-\frac{2}{\alpha}c\mathrm{e}^{-\alpha t}\right)I\right\|=\left|1-\frac{2}{\alpha}(1+c)\mathrm{e}^{-\alpha t}\right|<1. \tag{2.7}$$

Also, we have

$$\|E(z,t)\|\leq\max_{\|w\|=1}\|E(w,t)\|. \tag{2.8}$$

If we let now $u=\mathrm{e}^{-t}w$, where $\|w\|=1$, then $\|u\|=\mathrm{e}^{-t}$ and so

$$E(w,t)=-\frac{2}{\alpha}\|u\|^{\alpha}\{(c+1)[Dg(u)]^{-1}Df(u)-I\} \tag{2.9}$$
$$-\frac{2}{\alpha}(1-\|u\|^{\alpha})(Dg(u))^{-1}D^2g(u)(u,\cdot)+I\left(1-\frac{2}{\alpha}\right).$$

By using (2.2), (2.7), (2.8) and (2.9) we obtain:

$$\|E(z,t)\|<1,\quad t>0.$$

Hence for $t>0$, $I-E(z,t)$ is an invertible operator, too.

Further computations show that

$$\frac{\partial L}{\partial t}(z,t)=\frac{1}{1+c}Dg(\mathrm{e}^{-t}z)\mathrm{e}^{(\alpha-1)t}\frac{\alpha}{2}\Big[I+I\left(1-\frac{2}{\alpha}\right)$$
$$-\frac{2}{\alpha}\mathrm{e}^{-\alpha t}[(c+1)[Dg(\mathrm{e}^{-t}z)]^{-1}Df(\mathrm{e}^{-t}z)-I]$$
$$-\frac{2}{\alpha}(1-\mathrm{e}^{-\alpha t})[Dg(\mathrm{e}^{-t}z)]^{-1}D^2g(\mathrm{e}^{-t}z)(\mathrm{e}^{-t}z,\cdot)\Big](z)$$

and

$$\frac{\partial L}{\partial z}(z,t)=\frac{1}{1+c}\mathrm{e}^{(\alpha-1)t}\frac{\alpha}{2}Dg(\mathrm{e}^{-t}z)[I+E(z,t)](z). \tag{2.10}$$

In conclusion, by using (2.6) and (2.10) we obtain

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B.$$

Hence $L(z,t)$ satisfies the differential equation (1.1) for all $z \in B$ and $t \geq 0$ where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B. \tag{2.11}$$

It remains to show that the function defined by (2.11) satisfies the conditions of Theorem 1.4. Clearly $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$.

Furthermore, the inequality:

$$\|h(z,t) - z\|\| = \|\|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\| \cdot \|h(z,t) + z\| < \|h(z,t) + z\|$$

implies that Re $\langle h(z,t), z\rangle > 0$, $\forall\ z \in B \setminus \{0\}$, $t \geq 0$.

Since all assumptions of Theorem 1.4 are satisfied, it follows that the functions $L(\cdot,t)$ $(t \geq 0)$ are univalent in B.

In particular $f = L(\cdot, 0)$ is univalent in B. □

REMARK 2.2. If $\alpha = 2$, $f = g$ and $c = 0$, then Theorem 2.1 becomes the n-dimensional version of Becker's univalence criterion [17].

If $\alpha = 2$, $f = g$, then Theorem 2.1 becomes the n-dimensional version of Ahlfors and Becker's univalence criterion [3].

If $c = 0$ then Theorem 2.1 becomes Theorem 2 [4].

If $f = g$ and $c = 0$ then Theorem 2.1 becomes Theorem 2 [5].

Next, we present a sufficient condition for a normalized quasiregular holomorphic mapping on B to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

THEOREM 2.3. *Let $f, g \in \mathcal{H}(B)$ be such that $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ and g is quasiregular in B. Also let $a \geq 2$. If there is $q \in [0,1)$ such that $1 - \frac{2}{\alpha} \leq q < \frac{2}{\alpha}$,*

$$\frac{2}{\alpha}\left\|(c+1)[Dg(z)]^{-1}Df(z) - \frac{\alpha}{2}I\right\| \leq q < 1 \tag{2.12}$$

and

$$\frac{2}{\alpha}\Big\|\|z\|^{\alpha}\{(c+1)[Dg(z)]^{-1}Df(z) - I\} + (1 - \|z\|^{\alpha})[Dg(z)]^{-1}D^2g(z)(z,\cdot) + \left(1 - \frac{\alpha}{2}\right)I\Big\| \leq q < 1, \quad z \in B \tag{2.13}$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. We shall show that the conditions (2.12) and (2.13) enable us to embed f as the initial element $f(z) = L(z,0)$ of a suitable subordination chain.

We define L by (2.3). In the proof of the previous theorem we showed that L (defined by (2.3)) is a subordination chain which satisfies the generalized Loewner differential equation where the mapping h is defined by (2.11) and the mapping $E : B \times [0,\infty) \to \mathcal{L}(\mathbb{C}^n)$ is defined by (2.5).

Further, we shall show that $\|E(z,t)\| \le q$ for all $(z,t) \in B \times [0,\infty)$.

We have

$$\|E(z,0)\| = \frac{2}{\alpha}\left\|(c+1)[Dg(z)]^{-1}Df(z) - \frac{\alpha}{2}I\right\| \le q < 1, \quad z \in B,$$

by the condition (2.12). Next, fix $t \in (0,\infty)$. In view of the maximum principle for holomorphic mappings into complex Banach spaces, we obtain that

$$\|E(z,t)\| \le \max \|E(w,t)\|$$
$$= \frac{2}{\alpha}\max_{\|w\|=1}\Big\|\|we^{-t}\|^{\alpha}[(c+1)[Dg(we^{-t})]^{-1}Df(we^{-t}) - I]$$
$$+(1-\|we^{-t}\|^{\alpha})[Dg(we^{-t})]^{-1}D^2g(we^{-t})(we^{-t},\cdot) + I\left(1-\frac{\alpha}{2}\right)\Big\|, \quad z \in B.$$

We deduce from the condition (2.13) that

$$\|E(z,t)\| \le q < 1, \quad z \in B.$$

Therefore $\|E(z,t)\| \le q < 1$, $z \in B$, $t \in [0,\infty)$ and hence $I - E(z,t)$ is an invertible linear operator.

On the other hand, taking into account the conditions (2.12) and (2.13) in the hypothesis, we deduce that

$$(1-\|z\|^{\alpha})\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\|$$
$$\le q\frac{\alpha}{2} + \|z\|^{\alpha}\frac{\alpha}{2}q + (1-\|z\|^{\alpha})\left(\frac{\alpha}{2}-1\right)$$
$$= \|z\|^{\alpha}\left(q\frac{\alpha}{2} - \frac{\alpha}{2} + 1\right) + q\frac{\alpha}{2} + \frac{\alpha}{2} - 1$$
$$\le \max_{x\in[0,1]}\left\{x\left(q\frac{\alpha}{2} - \frac{\alpha}{2} + 1\right) + q\frac{\alpha}{2} + \frac{\alpha}{2} - 1\right\}$$
$$= \max\left\{q\frac{\alpha}{2} + \frac{\alpha}{2} - 1, q\alpha\right\} = q\alpha = 2\beta,$$

where $\beta = \frac{q\alpha}{2} < 1$.

Since $\alpha \ge 2$, we deduce from the above relation that

$$(1-\|z\|^2)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \le 2\beta. \tag{2.14}$$

From (2.14), by using a similar argument with that used in the proof of Theorem 2.1 [17] we obtain that there exists $M > 0$ such that

$$|\det Dg(z)| \le \frac{M}{(1-\|z\|)^{n\beta}}, \quad z \in B, \tag{2.15}$$

and hence

$$\|Dg(z)\| \le \frac{L}{(1-\|z\|)^{\beta}} \text{ where } L = \sqrt[n]{MK}. \tag{2.16}$$

It remains to prove that the mappings $L(\cdot, t)$, $t \geq 0$ are quasiregular. Since g is a quasiregular holomorphic mapping and the following inequality holds

$$1 - q \leq \|I - E(z,t)\| \leq 1 + q, \quad z \in B,\ t \geq 0,$$

by using (2.6) we easily obtain

$$\|DL(z,t)\| \leq \frac{\alpha}{2}\mathrm{e}^{(\alpha-1)t}\frac{1}{|1+c|}\|Dg(z\mathrm{e}^{-t})\| \cdot \|I - E(z,t)\| \tag{2.17}$$

$$\leq |a(t)|\frac{1+q}{1-q} \cdot \frac{L}{(1-\|z\|)^{\beta}} = \frac{L^*|a(t)|}{(1-\|z\|)^{\beta}}.$$

On the other hand, we have

$$\|DL(z,t)\|^n \leq \left(\frac{\alpha}{2}\right)^n \mathrm{e}^{n(\alpha-1)t}\frac{1}{|1+c|^n}\|Dg(z\mathrm{e}^{-t})\|^n(1+q)^n \tag{2.18}$$

$$\leq \left(\frac{\alpha}{2}\right)^n \mathrm{e}^{n(\alpha-1)t}\frac{1}{|1+c|^n}K|\det Dg(z\mathrm{e}^{-t})|(1+q)^n$$

$$\leq \left(\frac{1+q}{1-q}\right)^n K|\det DL(z,t)|, \quad z \in B,\ t \geq 0.$$

Since all the conditions of Theorem 1.5 are satisfies we obtain that the function $f(z) = L(z, \cdot)$ admits a quasiconformal extension defined on $\mathbb{R}^{2n}$. $\square$

Remark 2.4. If $\alpha = 2$, $c = 0$ and $f = g$ in Theorem 2.3, we obtain the n-dimensional version of the quasiconformal extension result due to Becker [17].

If $\alpha = 2$ and $f = g$ in Theorem 2.3, we obtain the n-dimensional version of the quasiconformal extension result due to Ahlfors and Becker [3].

If $\alpha = 2$ and $c = 0$ in Theorem 2.3, we obtain Theorem 2.1 of [7].

Acknowledgement. This work is supported by the Romanian Ministry of Education and Research Grant PN II-ID-524/2007.

REFERENCES

[1] Becker, J., *Löewnersche Differentialgleichung und quasikonform fortsetzbare schlichte Functionen*, J. Reine Angew. Math., **255** (1972), 23–43.

[2] Brodksii, A.A., *Quasiconformal extension of biholomorphic mappings*, In: Theory of Mappings and Approximation of Functions, 3–34, Naukova Durka, Kiew, 1983.

[3] Curt, P., *A generalization in n-dimensional complex space of Ahlfors and Becker's criterion for univalence*, Stud. Univ. Babeş-Bolyai Math., **39** (1994), 31–38.

[4] Curt, P., *A univalence criterion for holomorphic mappings in* $\mathbb{C}^n$, Mathematica (Cluj), **37(60)** (1995), 67–71.

[5] Curt, P., *A sufficient condition for univalence of holomorphic mappings in* $\mathbb{C}^n$, Mem. Sect. Ştiinţifice Iaşi, seria IV, **XIX** (1996), 49–53.

[6] Curt, P., Special Chapters of Geometric Function Theory of Several Complex Variables, Editura Albastră, Cluj-Napoca, 2001 (in Romanian).

[7] Curt, P., *Quasiconformal extensions of holomorphic maps in* $\mathbb{C}^n$, Mathematica (Cluj), **46(69)** (2004), 55–60.

[8] Curt, P. and Kohr, G., *Subordination chains and Loewner differential equations in several complex variables*, Ann. Univ. Marie Curie-Skłowska Sect. A, **57** (2003), 35–43.
[9] Curt, P. and Kohr, G., *Quasiconformal extensions and q-subordination chains in* $\mathbb{C}^n$, Mathematica (Cluj), **49(72)** (2007), 149–159.
[10] Curt, P. and Kohr, G., *The asymptotical case of certain quasiconformal extensions results for holomorphic mappings in* $\mathbb{C}^n$, Bull. Belg. Math. Soc. Simon Stevin, **14** (2007), 653–667.
[11] Curt, P. and Kohr, G., *Some remarks concerning quasiconformal extensions in several complex variables*, J. Inequal. Appl., 2008, 16 pages, Article ID690932.
[12] Graham, I., Hamada, H., Kohr, G., *Parametric representation of univalent mappings in several complex variables*, Canad. J. Math., **54** (2002), 324–351.
[13] Graham, I. and Kohr, G., Geometric Function Theory in One and Higher Dimensions, Marcel Dekker Inc., New York, 2003.
[14] Hamada, H. and Kohr, G., *Loewner chains and quasiconformal extension of holomorphic mappings*, Ann. Polon. Math., **81** (2003), 85–100.
[15] Hamada, H. and Kohr, G., *Quasiconformal extension of biholomorphic mappings in several complex variables*, J. Anal. Math., **96** (2005), 269–282.
[16] Pfaltzgraff, J.A., *Subordination chains and univalence of holomorphic mappings in* $\mathbb{C}^n$, Math. Ann., **210** (1974), 55–68.
[17] Pfaltzgraff, J.A., *Subordination chains and quasiconformal extension of holomorphic maps in* $\mathbb{C}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math., **1** (1975), 13–25.

Received October 30, 2008
Accepted June 5, 2009

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