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REMARKS ON INDUCTION OF G-ALGEBRAS AND SKEW GROUP ALGEBRAS

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Abstract. In the first section we give a pointed group version of a result of Dade on Green theory. Related to this, in the second section we consider an H-algebra B, where H is a subgroup of a finite group G. For the skew group algebra $B*H$, we prove that its induction to G in the sense of Puig is isomorphic to the skew group algebra over G of the induction, in the sense of Turull, of B to G.

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1. PRELIMINARIES

Let $\mathcal{O}$ be a discrete valuation, and let A be an $\mathcal{O}$-algebra with identity, finitely generated as an $\mathcal{O}$-module. Let G be a finite group acting as automorphisms of A, hence A is a G-algebra. For any $a \in A$, and $g \in G$ we will denote by ${}^g a$ the action of g on the element a.

For any subgroup H of G, denote by

$$A^H = \{a \in A \mid {}^g a = a \text{ for all } g \in H\},$$

the subalgebra of fixed elements of A by the action of H. Observe that by restriction, A is a H-algebra. Obviously this subalgebra contains the identity of the bigger algebra. For two subgroups K and H of G such that $K \subseteq H$, we have the relative trace map

$$\mathrm{Tr}_K^H : A^K \to A^H, \ \ \mathrm{Tr}_K^H(a) = \sum_{g \in [H/K]} {}^g a.$$

We denoted by $[H/K]$ a set of representatives of the right cosets H/K. It is clearly, a well-defined map which is a additive group homomorphism. If $b \in A^H$ then $\mathrm{Tr}_K^H(ab) = \mathrm{Tr}_K^H(a)b$ and $\mathrm{Tr}_K^H(ba) = b\,\mathrm{Tr}_K^H(a)$ which implies that the image $A_K^H := \mathrm{Tr}_K^H(A^K)$ is a two-sided ideal of A^H.

We are going to need the following definitions and remarks, hence for the sake of completeness we just state them here, for further details the reader is referred to [4].

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Definition 1. A pointed group on the G-algebra A is a pair (H, α), where H is a subgroup of G and α is a point on A^H, i.e. a conjugacy class of a primitive idempotent $i \in A^H$; we shall use the notation H_α for a pointed group.

Remark 1. There is an partial order relation denoted "$\le$" which can be defined on the set of pointed groups of the G-algebra A. Two subgroups K_β, H_α satisfy $K_\beta \le H_\alpha$, if $K \le H$ and for every $i \in \alpha$ there exists $j \in \beta$ such that $j = iji$.

Definition 2. A pointed group P_γ is called a defect pointed group of H_α if and only if P_γ is a minimal pointed group such that $\alpha \subseteq \mathrm{Tr}_P^H(A^P \gamma A^P)$. The last condition is equivalent to the following statement: for every $i \in \alpha$ there exists $j \in \gamma$ such that $i = \mathrm{Tr}_P^H(ajb)$ for some $a, b \in A^P$. We also say that H_α is projective relative to P_γ.

Remark 2. Without specifying the point γ one can equivalently define P to be the defect of the pointed group H_α, that is, P is minimal such that $\alpha \subseteq A_P^H$. One easily shows that these two definitions are equivalent.

2. A POINTED GROUP VERSION OF A RESULT IN GREEN THEORY

Let e be an idempotent of A satisfying:

1) If $g \in G$ and ${}^g e \neq e$, then ${}^g e e = 0$;
2) For all $a \in A^G$ we have $ea = ae$;

Let $G_e = \{g \in G \mid {}^g e = e\}$ be the subgroup of G fixing e under the conjugation action. Condition 1) implies that

$$c := \mathrm{Tr}_{G_e}^G(e) = \sum_{g \in [G/G_e]} {}^g e$$

is an idempotent of A^G, and using 2) we see that c is central in A^G. Since G_e fixes e we have $(eAe)^{G_e} = eA^{G_e}e$.

Proposition 1. *With the above notations, the map*

$$cA^G \to eA^{G_e}e, \ a \mapsto ae = ea$$

is a ring isomorphism.

The inverse map sends any $b \in eA^{G_e}e$ into $tr_{G_e}^G(b) \in cA^G$.

Since this is the exact restatement of a result in [1, Section 4], we leave the proof out of this paper.

Because e is fixed by G_e we deduce that eAe is a G_e-algebra. The next result is the pointed group version of [1, 4.9].

Proposition 2. *Let P_γ be a defect pointed group of $(G_e)_\beta$ on eAe. Then $P_{\gamma'}$ is a defect pointed group of G_α on cA^G. Moreover, the point α is the correspondent of β with respect to the above isomorphism and γ' is a point of cA^P.*

Proof. By definition $P_\gamma \leq (G_e)_\beta$ is minimal such that

$$\beta \subseteq \mathrm{Tr}_P^{G_e}(eA^P e) = e \cdot \mathrm{Tr}_P^{G_e}(A^P)e = eA_P^{G_e}e.$$

It follows that for every $i \in \beta$ there exists $w \in eA^P e$ such that

$$i = e\,\mathrm{Tr}_P^{G_e}(w)e = \mathrm{Tr}_P^{G_e}(w).$$

It follows that

$$j := \mathrm{Tr}_{G_e}^G(i) = \mathrm{Tr}_P^G(w) \in cA^G,$$

and moreover j is a primitive idempotent of cA^G satisfying $j = cj$. We may take α to be a point of cA^G containing j, hence $\alpha = \mathrm{Tr}_{G_e}^G(\beta)$. Since $ewe = w$, it follows

$$w = ew = cew = cw \in cA^P,$$

and because $j = \mathrm{Tr}_P^G(w)$, where $w \in cA^P$, we deduce that

$$\alpha \subseteq \mathrm{Tr}_P^G(cA^P) = (cA)_P^G.$$

The pointed group G_α is projective relative to P, hence there exists $\gamma^{'}$ a such that G_α is projective relative to $P_{\gamma'}$.

Suppose there would exist a pointed group R_ϵ on cA such that $R_\epsilon \leq P_{\gamma'}$. Then we would have $R \leq P \leq G_e$, and by [4, Exercise 13.5, p. 109], for the points β and γ there would exist a point $\epsilon^{'}$ such that $R_{\epsilon'} \leq P_\gamma$, which contradicts the minimality of P_γ. □

3. INDUCTION AND SKEW GROUP ALGEBRAS

Let H be a subgroup of a finite group G, and consider an H-algebra A. We use the definition of induction of A as in [5, Section 8]. The induction of A from H to G is

$$\mathrm{Ind}_H^G(A) = \mathcal{O}G \otimes_{\mathcal{O}H} A,$$

where an element $g \otimes a \in \mathcal{O}G \otimes_{\mathcal{O}H} A$ is denoted by ${}^g a$, and for $b \in \mathrm{Ind}_H^G(A)$ and $g \in G$ the element ${}^g b$ is the result of g acting on b. If $a, b \in A$ and $g_1, g_2 \in G$, the multiplication in this algebra is given by:

$$({}^{g_1}a)({}^{g_2}b) = \begin{cases} {}^g(ab) & \text{if } g = g_1 = g_2; \\ 0 & \text{if } g_1 H \neq g_2 H. \end{cases}$$

As noted in [3, 4.3], this is a particular case of the induction of crossed products introduced in [2].

Consider the map

$$\psi : G \to \mathrm{Aut}_{\mathcal{O}}(\mathrm{Ind}_H^G(A)), \quad g \mapsto \psi(g)(a) := {}^g a.$$

If $a \in \mathrm{Ind}_H^G(A)$ then $a = g \otimes a^{'}$ for some $g \in G$ and some $a^{'} \in A$, hence $a = \psi(g)(a^{'})$ and this means ψ is surjective. For $a \in \mathrm{Ind}_H^G(A)$ such that

$\psi(g)(a) = g \otimes a = 0$, it clearly follows that $a = 0$, hence $\psi(g)$ is injective. Even more, for $g \in G$ and $a, b \in \mathrm{Ind}_H^G(A)$ we have

$$\psi(g)(ab) = {}^g(ab) = {}^g(a){}^g(b) = \psi(g)(a)\psi(g)(b).$$

We have shown that for any $g \in G$, $\psi(g)$ is an automorphism of $\mathrm{Ind}_H^G(A)$ which is clearly $\mathcal{O}$-linear.

Let $g_1, g_2 \in G$ and $a \in \mathrm{Ind}_H^G(A)$. We have

$$\psi(g_1g_2)(a) = {}^{g_1g_2}a = {}^{g_1}({}^{g_2}a) = {}^{g_1}(\phi(g_2)(a)) = (\psi(g_1) \circ \psi(g_2))(a),$$

hence ψ is a group homomorphism which endows $\mathrm{Ind}_H^G(A)$ with a structure of a G-algebra.

Now let B be an H-algebra over $\mathcal{O}$ and consider the skew group algebra $S := B * H$ of B and H. Let $A = \mathrm{Ind}_H^G(B)$ be the above induced algebra, and denote by $R := A * G$ the skew group algebra of A over G. The algebra R has a natural structure of interior G-algebra given by

$$G \to R^*,\ g \mapsto 1 \cdot g = g \cdot 1,$$

in the same manner S has a structure of interior H-algebra.

We may view the elements of R as pairs of the form $a \cdot g = (a, g) = (g^{'} \otimes b, g)$ where $b \in B$, $g \in G$ and $g^{'} \in [G/H]$. The subset of R consisting of elements in which $g^{'} = 1$ and $g \in H$ is a subalgebra of R isomorphic to S. Identifying S with that subalgebra, the action of G on S is defined in the same way as the action of G on the elements of A.

There is another type of induction which is due to Puig and which can be applied to the interior H-algebra S, namely $\mathcal{O}G \otimes_{\mathcal{O}H} S \otimes_{\mathcal{O}H} \mathcal{O}G$. Recall that its algebra structure is given by

$$(g \otimes s \otimes g^{'})(g_1 \otimes s_1 \otimes g_1^{'}) = \begin{cases} g \otimes s \cdot g^{'}g_1 \cdot s_1 \otimes g_1^{'} & \text{if } g^{'}g_1 \in H \\ 0 & \text{if } g^{'}g_1 \notin H, \end{cases}$$

where $g, g^{'}, g_1, g_1^{'} \in G$ and $s, s_1 \in S$. The interior G-algebra structure is given by $g \cdot (x \otimes s \otimes y) = gx \otimes s \otimes y$ and $(x \otimes s \otimes y) \cdot g = x \otimes s \otimes yg$ for all $g, x, y \in G$ and $s \in S$. Observe that the induction of S is completely determined by elements in B and by sets of representatives of the left, respectively right cosets of H in G. We have the following result.

Theorem 1. *The map*

$$\varphi : \mathcal{O}G \otimes_{\mathcal{O}H} S \otimes_{\mathcal{O}H} \mathcal{O}G \to R, \quad g \otimes s \otimes f \mapsto g \cdot s \cdot f,$$

where $g, f \in G$ and $s \in S$, is an isomorphism of G-graded G-interior algebras, and the diagram

$$\begin{array}{ccc} \mathcal{O}G \otimes_{\mathcal{O}H} S \otimes_{\mathcal{O}H} \mathcal{O}G & \longrightarrow & R \\ \uparrow & \nearrow & \\ \mathcal{O}G & & \end{array}$$

of G-graded G-interior algebras is commutative.

Proof. For $x, y \in [G/H]$ and $b \in B$, the map φ sends $x \otimes b \otimes y$ to ${}^{x}b \cdot (xy)$. It is clear that φ is a well-defined map, since for other representatives of the right respectively left cosets $x^{'}$, $y^{'}$ we have

$$\begin{aligned} \varphi(x^{'} \otimes b \otimes y^{'}) &= \varphi(1x^{'} \otimes b \otimes 1y^{'}) \\ &= \varphi(x(x^{'})^{-1}x^{'} \otimes b \otimes y(y^{'})^{-1}y^{'}) \\ &= \varphi(x \otimes b \otimes y). \end{aligned}$$

Let us show that φ is indeed a morphism of algebras. Let $x \otimes b \otimes y$ and $x^{'} \otimes b^{'} \otimes y^{'}$ be two elements of the Puig's induction of S. Then by definition we have

$$(x \otimes b \otimes y)(x^{'} \otimes b^{'} \otimes y^{'}) = \begin{cases} x \otimes b \cdot yx^{'} \cdot b^{'} \otimes y^{'} & \text{if } yx^{'} \in H \\ 0 & \text{if } yx^{'} \notin H. \end{cases}$$

Note that in our case $yx^{'} \in H$ is equivalent to $yx^{'} = 1$. Then

$$\begin{aligned} \varphi((x \otimes b \otimes y)(x^{'} \otimes b^{'} \otimes y^{'})) &= \begin{cases} \varphi(x \otimes b \cdot yx^{'} \cdot b^{'} \otimes y^{'}) & \text{if } yx^{'} \in H \\ 0 & \text{if } yx^{'} \notin H \end{cases} \\ &= \begin{cases} {}^{x}(bb^{'}) \cdot (xy^{'}) & \text{if } yx^{'} = 1 \\ 0 & \text{if } yx^{'} \neq 1. \end{cases} \end{aligned}$$

On the other hand, $\varphi(x \otimes b \otimes y) = {}^{x}b \cdot (xy)$, and $\varphi(x^{'} \otimes b^{'} \otimes y^{'}) = {}^{x^{'}}b^{'} \cdot (x^{'}y^{'})$, hence by applying the definition of the product in R the definition of the product in A we get

$$\begin{aligned} \varphi(x \otimes b \otimes y)\varphi(x^{'} \otimes b^{'} \otimes y^{'}) &= {}^{x}b \cdot (xy) \cdot {}^{x^{'}}b^{'} \cdot (x^{'}y^{'}) \\ &= {}^{x}b^{(xy)x^{'}}b^{'} \cdot (xyx^{'}y^{'}) \\ &= \begin{cases} {}^{x}(bb^{'}) \cdot (xyx^{'}y^{'}) & \text{if } x = xyx^{'} \\ 0 & \text{if } xH \neq xyx^{'}H \end{cases} \\ &= \begin{cases} {}^{x}(bb^{'}) \cdot (xy^{'}) & \text{if } 1 = yx^{'} \\ 0 & \text{if } 1 \neq yx^{'}. \end{cases} \end{aligned}$$

Now let $g \in G$. Then

$$\begin{aligned}\varphi(g \cdot (x \otimes b \otimes y)) &= \varphi(gx \otimes b \otimes y) = {}^{gx}b \cdot (gxy)\\ &= (1 \cdot g)({}^{x}b \cdot (xy))\\ &= g \cdot \varphi(x \otimes b \otimes y),\end{aligned}$$

and

$$\begin{aligned}\varphi((x \otimes b \otimes y) \cdot g) &= \varphi(x \otimes b \otimes yg) = {}^{x}b \cdot (xyg)\\ &= ({}^{x}b \cdot (xy))(1 \cdot g) = \varphi(x \otimes b \otimes y) \cdot g.\end{aligned}$$

So φ is indeed a morphism of interior G-algebras. Note that in the above equalities

$$1 = \sum_{g \in [G/H]} g \otimes 1_B = \sum_{g \in [G/H]} {}^{g}1_B$$

is the identity of A and multiplying this identity by ${}^{x}b$ on either side the product is different from zero exactly when $g = x$.

In order to check the surjectivity of φ we consider $a \cdot g = {}^{g^{'}}b \cdot g \in R$ where $g, g^{'} \in G$ and $b \in B$. Let $x^{'}, x \in [G/H]$ be two representatives such that $g^{'} = x^{'}h^{'}$ and $g = hx$ for some $h^{'}, h \in H$. Then denoting by $b^{'}$ the element of A being ${}^{h^{'}}b$, we have

$$\begin{aligned}a \cdot g &= {}^{x^{'}}b^{'} \cdot (hx) = ({}^{x^{'}}b^{'} \cdot h)(1 \cdot x)\\ &= \varphi(x^{'} \otimes b^{'} \otimes h) \cdot x\\ &= \varphi(x^{'} \otimes b^{'} \cdot h \otimes 1) \cdot x\\ &= \varphi(x^{'} \otimes b^{'} \cdot h \otimes x),\end{aligned}$$

hence φ is surjective.

If $\sum_{x,y \in [G/H]} x \otimes b_{x,y} \otimes y \in \mathrm{Ker}(\varphi)$, then

$$\varphi(\sum_{x,y \in [G/H]} x \otimes b_{x,y} \otimes y) = \sum_{x,y \in [G/H]} {}^{x}b_{x,y} \cdot (xy) = 0.$$

Consider $a \in A$ and invertible element, then for any $g \in G$ the element ${}^{g}a$ is invertible. Fix $x^{'}, y^{'} \in [G/H]$ and multiply the above equality with ${}^{x^{'}}a \cdot 1$ on the left and with ${}^{(y^{'})^{-1}}a \cdot 1$ on the right. One obtains ${}^{x^{'}}a^{x^{'}}b_{x^{'},y^{'}}{}^{x^{'}}a = 0$, which means $ab_{x^{'},y^{'}}a = 0$ hence $b_{x^{'},y^{'}} = 0$. Thus φ is injective and the theorem is proven. □

Remark 3. a) Since we identified S with its isomorphic subalgebra of R, viewing $s = b \cdot h$, then the product $g \cdot s \cdot f$ is ${}^{g}b \cdot hf$, where ${}^{g}b \in A$.

b) One can easily verify that $\varphi(1) = 1$, and that for $g = xh \in G$ with $x \in [G/H]$ and $h \in H$, we have

$${}^{xh}b = g \otimes b = x \otimes {}^{h}b = {}^{x}({}^{h}b).$$

In order to clarify the choice of φ, observe that

$$g \cdot s \cdot f = g \cdot (\sum_{h \in H} b_h \cdot h) \cdot f = \sum_{h \in H} {}^g b_h \cdot ghf,$$

and for another element $g' \cdot s' \cdot f' = \sum_{h \in H} {}^{g'} b'_h \cdot g' h f'$, one gets

$$(g \cdot s \cdot f)(g' \cdot s' \cdot f') = (\sum_{h \in H} {}^g b_h \cdot ghf)(\sum_{h \in H} {}^{g'} b'_h \cdot g' h f')$$

On the other hand,

$$\varphi(g \otimes s \cdot fg' \cdot s' \otimes f') = \begin{cases} (\sum_{h \in H} {}^g b_h \cdot ghf)(\sum_{h \in H} {}^{g'} b'_h \cdot g' h f') & \text{if } fg' \in H \\ 0 & \text{if } fg' \notin H. \end{cases}$$

The product $(g \cdot s \cdot f)(g' \cdot s' \cdot f')$ contains sums of elements of the form

$${}^g b_h {}^{ghfg'} b'_{h'} \cdot (ghfg' h' f')$$

which are zero if $gH \neq ghfg'H$ that is $fg' \notin H$.

c) R has a G-algebra structure induced by its interior structure, namely if $\phi : G \to R^*$ is the homomorphism giving the interior structure, then

$$\psi : G \to \mathrm{Aut}(R), \quad \psi(g) = \mathrm{Inn}(\phi(g)),$$

where for $a \in R$, $\psi(g)(a) = g \cdot a \cdot g^{-1} := {}^g a$, gives R an G-algebras structure. The same argument works for the interior G-algebra $\mathcal{O}G \otimes_{\mathcal{O}H} S \otimes_{\mathcal{O}H} \mathcal{O}G$, which becomes a G-algebra by

$${}^g(x \otimes s \otimes y) = gx \otimes s \otimes yg^{-1}.$$

The isomorphism in the theorem is actually an isomorphism of G-algebras, in other words R and $\mathcal{O}G \otimes_{\mathcal{O}H} S \otimes_{\mathcal{O}H} \mathcal{O}G$ are isomorphic as G-algebras. Indeed, for $g \in G$ and $x \otimes s \otimes y \in \mathcal{O}G \otimes_{\mathcal{O}H} S \otimes_{\mathcal{O}H} \mathcal{O}G$ we obtain

$$\begin{aligned} f({}^g(x \otimes s \otimes y)) &= f(gx \otimes s \otimes yg^{-1}) = gx \cdot s \cdot yg^{-1} \\ &= g(x \cdot s \cdot y)g^{-1} = {}^g f(x \otimes s \otimes y). \end{aligned}$$

d) Let $c = \sum_{g \in [G/G_e]} {}^g e$ be the G-invariant idempotent constructed in the second paragraph. Then c is the identity of the algebra $(cAc) * G = c(A * G)c$. These two algebras are in particular crossed products of A and G, and of cAc and G respectively. The idempotent e is the identity, hence a central idempotent of $e(A * G_e)e = (eAe) * G_e$. By using the uniqueness of the induction as presented in [2], we may write

$$c(A * G)c = \mathrm{Ind}_{G_e}^G (e(A * G_e)e).$$

If $B = \mathrm{Ind}_{G_e}^{G}(eAe)$ is the induction to G in the sense of Turull of the algebra $e(A * G_e)e$, by using the above theorem we have

$$\begin{aligned} c(A*G)c &= \mathrm{Ind}_{G_e}^{G}(e(A*G_e)e) \\ &= \mathcal{O}G \otimes_{\mathcal{O}G_e} e(A*G_e)e \otimes_{\mathcal{O}G_e} \mathcal{O}G \\ &\simeq B*G. \end{aligned}$$

The equality $(cAc) * G = c(A * G)c$ forces the isomorphism

$$cAc \simeq B = \mathcal{O}G \otimes_{\mathcal{O}G_e} eAe,$$

hence the G-algebra cAc is the Turull induction to G of the G_e-algebra eAe.

REFERENCES

[1] Dade, E.C., *Block extensions*, Illinois J. Math., **17** (1973), 198–272.
[2] Klasen, W. and Schmid, P., *Induced crossed products*. Comm. Algebra, **18** (1990), 2573–2586.
[3] Marcus, A., *Derived invariance of Clifford classes*, J. Group Theory, **12** (2009), 83–94.
[4] Thévenaz, J., *G-Algebras and Modular Representation Theory*, Clarendon Press, Oxford 1995.
[5] Turull, A., *Reduction theorems for Clifford classes*, J. Group Theory, **9** (2006), 27–47.

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