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FUNCTIONS WITH STRONGLY β-θ-CLOSED GRAPHS

C. K. BASU, B. M. UZZAL AFSAN and S. S. MANDAL

Abstract. In this paper, by the use of β-open [1] sets, the notion of functions having strongly β-θ-closed graphs is being initiated. Several characterizations and basic properties of it are investigated. Important applications of this investigation are exhibited by establishing a new characterization of the concept β-closedness [7] and a sufficient condition for common fixed points of a family of functions having strongly β-θ-closed graphs.

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1. INTRODUCTION

In 1986, Monsef et al. [1] initiated the study of β-open sets, which is equivalent to semi-preopen sets due to Andrijević [4]. This notion has been studied extensively in recent years by a good many researchers, specially, on functions, covering properties and connectedness. It is worth to be mentioned some of recent research works related to β-open sets which are found in papers of Monsef et al. [1, 2], Andrijević [4, 5, 6], Basu and Ghosh [7, 8], Beceren and Noiri [9], Caldas and Jafari [11, 12], Duszyński [13], Halvac [16], Jafari and Noiri [17], Noiri [23], Popa and Noiri [24, 25, 26] and Tahiliani [27]. Keeping in mind the tremendous influences of functions having closed graphs on covering properties, we introduce the notion of strong β-θ-closedness of functions in terms of β-open sets and try to obtain its formulations and some of its basic properties. As applications, this idea is exploited to ultimately achieve a new characterization of the covering property β-closedness [7] and a sufficient condition for common fixed points of a family of functions having strongly β-θ-closed graphs.

2. PRELIMINARIES

Throughout this paper spaces (X, τ) and (Y, σ) (or simply X and Y) represent non-empty topological spaces and $\psi : (X, \tau) \to (Y, \sigma)$ (or $\psi : X \to Y$) denotes a function of a space (X, τ) into a space (Y, σ). The closure and the interior of a subset A of a space X are denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ respectively.

Definition 1. A subset A of a space X is called semi-open [19] if $A \subset \mathrm{cl}(\mathrm{int}(A))$, α-open [22] if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, pre-open [21] if $A \subset \mathrm{int}(\mathrm{cl}(A))$, β-open [1] or semi-preopen [4] if $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$.

The complement of a β-open set is called a β-closed set. A subset A of X is said to be β-regular [7] (=semi-preregular [23]) if it is both β-open as well as β-closed. The family of all semi-open (resp. α-open, pre-open, β-open, β-regular) subsets of (X,τ) is denoted by $SO(X)$ (resp. τ^α, $PO(X)$, $\beta O(X)$, $\beta R(X)$). It is well known that $\tau \subset \tau^\alpha = PO(X) \cap SO(X) \subset PO(X) \cup SO(X) \subset \beta O(X)$. None of the reverse implications is true in general. The family of all β-open (resp. β-regular, open) subsets of X containing $x \in X$ is denoted by $\beta O(X,x)$ (resp. $\beta R(X,x)$, $O(x)$). The graph of a function $\psi : X \to Y$, denoted by $G(\psi)$, is defined as the set $\{(x,\psi(x)) \in X \times Y : x \in X\}$.

Definition 2. (a) A space X is called extremally disconnected [22] if closure of each.

(b) A space X is called submaximal [10] if each dense subset of X is.

(c) A space X is called β-T_2 [20] if for points x and y in X with $x \neq y$ there exist disjoint β-open sets $U \in \beta O(X,x)$ and $V \in \beta O(X,y)$.

(d) A space X is called β-connected [3] if X can not be expressed as the union of two non-empty disjoint β-open sets.

Lemma 1. [4, 23] *The following hold, for a subset A of X*
(a) $\beta\text{-cl}(A) = A \cup \text{int}(\text{cl}(\text{int}(A)))$,
(b) $A \in \beta O(X)$ *if and only if* $\beta\text{-cl}(A) \in \beta R(X)$.

Definition 3. A point x of X is said to be in the β-θ-closure [7] (= sp-θ-closure [23]) of a subset A of X denoted by $\beta\text{-}\theta\text{-cl}(A)$, if $A \cap \beta\text{-cl}(V) \neq \emptyset$ for each $V \in \beta O(X,x)$. If $A = \beta\text{-}\theta\text{-cl}(A)$, then A is called β-θ-closed [7] (= sp-θ-closed [23]). The complement of a β-θ-closed (= sp-θ-closed) set is said to be a β-θ-open [7] (sp-θ-open [23]) set.

Lemma 2. [23] *Let A and B be any subsets of a space X. The following properties hold:*
(a) $\beta\text{-}\theta\text{-cl}(A) = \cap\{V : A \subset V \text{ and } V \in \beta R(X)\}$,
(b) $x \in \beta\text{-}\theta\text{-cl}(A)$ *if and only if* $A \cap U \neq \emptyset$ *for each* $U \in \beta R(X,x)$,
(c) *if* $A \subset B$ *then* $\beta\text{-}\theta\text{-cl}(A) \subset \beta\text{-}\theta\text{-cl}(B)$,
(d) $\beta\text{-}\theta\text{-cl}(\beta\text{-}\theta\text{-cl}(A)) = \beta\text{-}\theta\text{-cl}(A)$,
(e) *if* $A \in \beta O(X)$ *then* $\beta\text{-cl}(A) = \beta\text{-}\theta\text{-cl}(A)$,
(g) *intersection of an arbitrary family of β-θ-closed sets is β-θ-closed in X,*
(h) *A is β-θ-open if and only if for each $x \in A$ there exists an $U \in \beta R(X,x)$ such that $x \in U \subset A$.*

Definition 4. [7] A filter base $\mathcal{F}$ on X said to
(a) β-θ-adherent at x (written as $x \in \beta\text{-}\theta\text{-ad}\mathcal{F}$) if for each $U \in \beta O(X,x)$ and each $F \in \mathcal{F}$, $F \cap \beta\text{-cl}(U) \neq \emptyset$,
(b) β-θ-converge to x if for each $U \in \beta O(X,x)$, there exists an $F \in \mathcal{F}$ such that $F \subset \beta\text{-cl}(U)$.

The corresponding definitions of nets are obvious.

DEFINITION 5. [7] A subset S of a space X is called a β-closed set relative to X if every cover of S by β-open sets of X has a finite subfamily whose β-closures cover S.

If $S = X$ and S is β-closed set relative to X then X is called a β-closed space.

3. STRONGLY β-θ-CLOSED GRAPHS

DEFINITION 6. A function $\psi : X \to Y$ has a strongly β-θ-closed graph if for each $(x, y) \notin G(\psi)$, there exist an $U \in O(x)$ and a $V \in \beta O(Y, y)$ satisfying $(U \times \beta\text{-cl}(V)) \cap G(\psi) = \emptyset$.

LEMMA 3. *A function $\psi : X \to Y$ has a strongly β-θ-closed graph $G(\psi)$ if and only if for each $(x, y) \notin G(\psi)$, there exist an $U \in O(x)$ and a $V \in \beta O(Y, y)$ such that $\psi(U) \cap \beta\text{-cl}(V)) = \emptyset$.*

Proof. The proof is straight forward. □

THEOREM 1. *For a function $\psi : X \to Y$ following statements are equivalent:*
(a) *ψ has a strongly β-θ-closed graph,*
(b) *$\psi(x) = \cap\{\beta\text{-}\theta\text{-cl}(\psi(U)) : U \in O(x)\}$ for each $x \in X$.*

Proof. (a) ⇒ (b) Let the function ψ have a strongly β-θ-closed graph. Suppose for some $x \in X$, there is a point $y(\neq \psi(x))$ in Y such that $y \in \cap\{\beta\text{-}\theta\text{-cl}(\psi(U)) : U \in O(x)\}$. Then $y \in \beta\text{-}\theta\text{-cl}(\psi(U))$ for each $U \in O(x)$. So, for each $V \in \beta O(Y, y)$, $\beta\text{-}\theta\text{-cl}(V) \cap \psi(U) \neq \emptyset$. Hence $(U \times \beta\text{-cl}(V)) \cap G(\psi) \neq \emptyset$ — a contradiction. So, for each $x \in X$, $\psi(x) = \cap\{\beta\text{-}\theta\text{-cl}(\psi(U)) : U \in O(x)\}$.

(b) ⇒ (a) Suppose for the function $\psi : X \to Y$, $\psi(x) = \cap\{\beta\text{-}\theta\text{-cl}(\psi(U)) : U \in O(x)\}$ for each $x \in X$. If $(x, y) \notin G(\psi)$ then $y \neq \psi(x)$ and hence $y \notin \beta\text{-}\theta\text{-cl}(\psi(U))$ for some $U \in O(x)$. So, there is a $V \in \beta O(Y, y)$ such that $\psi(U) \cap \beta\text{-cl}(V) = \emptyset$ and hence $(U \times \beta\text{-cl}(V)) \cap G(\psi) = \emptyset$ for some $U \in O(x)$ and some $V \in \beta O(Y, y)$. Therefore ψ has a strongly β-θ-closed graph. □

DEFINITION 7. A space (X, τ) is β-O-regular if for each β-closed set F of X and $x \notin F$, there is an $U \in O(x)$ and a β-open set V containing F such that $V \cap U = \emptyset$.

THEOREM 2. *For a function $\psi : X \to Y$, consider the following statements:*
(a) *ψ has a strongly β-θ-closed graph $G(\psi)$,*
(b) *whenever a filter base $\mathcal{F} \to x$ in X and $\psi(\mathcal{F})$ β-θ-converges to y in Y, it follows that $y = \psi(x)$,*
(c) *whenever a net $x_\lambda \to x$ in X and $\psi(x_\lambda)$ β-θ-converges to y in Y, it follows that $y = \psi(x)$.*

Then (a) ⇒ (b) ⇒ (c)*. Moreover, if Y is β-O-regular, then* (c) ⇒ (a) *and hence the above statements are equivalent.*

Proof. (a) ⇒ (b) Let the function $\psi : X \to Y$ has a strongly β-θ-closed graph and let $\mathcal{F}$ be a filter base on X such that $\mathcal{F} \to x$ and $\psi(\mathcal{F})$ β-θ-converges to y. Suppose $y \neq \psi(x)$. Then $(x, y) \notin G(\psi)$. Therefore $O(x) \subset \mathcal{F}$

and $\{\beta\text{-cl}(V) : V \in \beta O(Y, y)\} \subset \psi(\mathcal{F})$. So, for each $U \in O(x)$ and each $V \in \beta O(Y, y)$ there exist $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$ such that $F_1 \subset U$ and $\psi(F_2) \subset \beta\text{-cl}(V)$. Hence there exists an $F_3 \in \mathcal{F}$ such that $F_3 \subset F_1 \cap F_2$ and satisfies $F_3 \subset U$ as well as $\psi(F_3) \subset \beta\text{-cl}(V)$. Hence $\emptyset \neq \psi(F_3) \subset \psi(U) \cap \beta\text{-cl}(V)$ and so $(U \times \beta\text{-cl}(V)) \cap G(\psi) \neq \emptyset$ — a contradiction. So $y = \psi(x)$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Suppose Y is β-O-regular. Let the given condition holds for the function $\psi : X \to Y$, but if possible, the function ψ does not have a strongly β-θ-closed graph. Then there exists $(x, y) \notin G(\psi)$ for which $(U \times \beta\text{-cl}(V)) \cap G(\psi) \neq \emptyset$ for each $U \in O(x)$ and each $V \in \beta O(Y, y)$. Consider the family $\mathcal{F} = \{F_{UV} = \{z \in U : (z, \psi(z)) \in (U \times \beta\text{-cl}(V)) \cap G(\psi)\} : U \in O(x)$ and $V \in \beta O(Y, y)\}$. Since Y is β-O-regular then $\mathcal{F}$ is clearly a filter base on X. But $\mathcal{F} \to x$ in X and $\psi(\mathcal{F})$ β-θ-converges to y and $y \neq \psi(x)$ — a contradiction. So ψ has a strongly β-θ-closed graph. ☐

Definition 8. Let $\{S_\alpha : \alpha \in D\}$ be a net of sets in X, where D is a directed set. The superior and inferior limits denoted respectively by $\text{limsup} S_\alpha$ and $\text{liminf} S_\alpha$ are defined as follows:

$\text{limsup} S_\alpha = \{x \in X :$ for any $U \in O(x)$ and for any $\alpha \in D$ there exists $\beta \in D$ with $\beta \geq \alpha$ such that $U \cap S_\beta \neq \emptyset\}$,

$\text{liminf} S_\alpha = \{x \in X :$ for any $U \in O(x)$ there exists an $\alpha \in D$ such that $U \cap S_\beta \neq \emptyset$ for all $\beta \geq \alpha\}$.

Theorem 3. *For a function $\psi : X \to Y$, consider the following statements:*

(a) *ψ has a strongly β-θ-closed graph,*

(b) *for any net (y_α) in Y which β-θ-converges to $y^\star \in Y$, $\text{limsup}\psi^{-1}(y_\alpha) \subseteq \psi^{-1}(y^\star)$,*

(c) *for any net (y_α) in Y which β-θ-converges to $y^\star \in Y$, $\text{liminf}\psi^{-1}(y_\alpha) \subseteq \psi^{-1}(y^\star)$.*

Then we have that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *Moreover, if Y is β-O-regular, then* (c) $\Rightarrow$ (a) *and hence the above statements are equivalent.*

Proof. (a) $\Rightarrow$ (b). Suppose $(y_\alpha)_{\alpha \in D}$ be a net in Y that β-θ-converges to $y^\star$ in Y. Let $x \in \text{limsup}\psi^{-1}(y_\alpha)$. If $x \notin \psi^{-1}(y^\star)$ then $y^\star \neq \psi(x)$ and hence $(x, y^\star) \notin G(\psi)$. Since ψ has a strongly β-θ-closed graph there exist an $U \in O(x)$ and $V \in \beta O(Y, y^\star)$ such that $\psi(U) \cap \beta\text{-cl}(V) = \emptyset$. Since $(y_\alpha)_{\alpha \in D}$ β-θ-converges to $y^\star$, there exists $\alpha_0 \in D$ such that $y_\alpha \in \cap\beta\text{-cl}(V)$ for all $\alpha \geq \alpha_0$. Now as $x \in \text{limsup}\psi^{-1}(y_\alpha)$, there exists $\alpha_1 \geq \alpha_0$ such that $U \cap \psi^{-1}(y_{\alpha_1}) \neq \emptyset$. Let $x_1 \in U \cap \psi^{-1}(y_{\alpha_1})$. Hence $\psi(x_1) = y_{\alpha_1} \in \beta\text{-cl}(V)$ and thus $\psi(U) \cap \beta\text{-cl}(V) \neq \emptyset$ — a contradiction. So $x \in \psi^{-1}(y^\star)$.

(b) $\Rightarrow$ (c) Since $\text{liminf}\psi^{-1}(y_\alpha) \subseteq \text{limsup}\psi^{-1}(y_\alpha)$, the proof follows immediately.

(c) $\Rightarrow$ (a) Suppose Y is β-O-regular. If possible let ψ has no strongly β-θ-closed graph. Then by Theorem 2, there exists a net $(x_\alpha)_{\alpha \in D}$ in X such that $(x_\alpha) \to x^\star \in X$, the net $(y_\alpha) = (\psi(x_\alpha))$ β-θ-converges to $y^\star \in Y$ and $y^\star \neq \psi(x^\star)$. Clearly, $x^\star \in \text{liminf}\psi^{-1}(y_\alpha)$ and hence by hypothesis (c), $x^\star \in \psi^{-1}(y^\star)$.

Thus $\psi(x^\star) = y^\star$ — a contradiction. Therefore ψ has a strongly β-θ-closed graph. $\square$

Definition 9. A function $\psi : X \to Y$ is called (θ, β)-continuous [7] if each filter base $\mathcal{F}$ on X satisfies $\psi(\mathrm{ad}\mathcal{F}) \subset \beta$-$\theta$-ad$\psi(\mathcal{F})$.

Equivalently, $\psi : X \to Y$ is (θ, β)-continuous if for each $x \in X$ and each $V \in \beta O(Y, \psi(x)$, there is an open set U containing x such that $\psi(U) \subset \beta$-cl(V).

Theorem 4. *If $\psi : X \to Y$ is (θ, β)-continuous and Y is β-T_2 then ψ has a strongly β-θ-closed graph.*

Proof. Let $(x, y) \notin G(\psi)$. Then $y \neq \psi(x)$. Hence there exists $V \in \beta O(Y, y)$ and $U \in \beta O(Y, \psi(x))$ such that $U \cap V = \emptyset$. This implies that $\psi(x) \notin \beta$-cl(V). Since β-cl(V) is β-regular, $Y - \beta$-cl(V) is also a β-regular set containing $\psi(x)$. Now as ψ is being (θ, β)-continuous, there is a $W \in O(x)$ such that $\psi(W) \subset Y - \beta$-cl$(V)$. Hence $\psi(W) \cap \beta$-cl$(V) = \emptyset$. Consequently, $(W \times \beta$-cl$(V)) \cap G(\psi) = \emptyset$ for some $W \in O(x)$ and some $V \in \beta O(Y, y)$. Therefore ψ has a strongly β-θ-closed graph. $\square$

Theorem 5. *If a surjective function $\psi : X \to Y$ has a strongly β-θ-closed graph then Y is β-T_2.*

Proof. Let y_1 and y_2 be any two distinct points in Y. As ψ is surjective, there is $x_1 \in X$ such that $y_1 = \psi(x_1)$ and hence $(x_1, y_2) \notin G(\psi)$. Since ψ has a strongly β-θ-closed graph there exist an $U \in O(x_1)$ and a $V \in \beta O(Y, y_2)$ such that $(U \times \beta$-cl$(V)) \cap G(\psi) = \emptyset$ i.e. $\psi(U) \cap \beta$-cl$(V) = \emptyset$. Since $Y - \beta$-cl$(V) \in \beta O(Y, y_1)$ and $V \in \beta O(Y, y_2)$ then Y is β-T_2. $\square$

Theorem 6. *A β-connected space X is β-T_2 if and only if the identity function $i : X \to X$ has a strongly β-θ-closed graph.*

Proof. The sufficiency part follows from Theorem 5.

Let X is β-T_2. Since X be β-connected then for any non-empty β-open set U, β-cl$(U) = X$ and hence the identity function $i : X \to X$ is obviously (θ, β)-continuous. Then by Theorem 4, $i : X \to X$ has a strongly β-θ-closed graph. $\square$

Theorem 7. *If an injection $\psi : X \to Y$ has a strongly β-θ-closed graph then X is T_1.*

Proof. Let x_1 and x_2 be any two distinct points in X. Since ψ is injective, $(x_2, \psi(x_1)) \notin G(\psi)$. As ψ has a strongly β-θ-closed graph, there is an $U \in O(x_2)$ and a $W \in \beta O(Y, \psi(x_1))$ satisfying $(U \times \beta$-cl$(W)) \cap G(\psi) = \emptyset$ i.e. $\psi(U) \cap \beta$-cl$(W) = \emptyset$ and hence $x_1 \notin U$. Similarly, we can obtain $V \in O(x_1)$ with $x_2 \notin V$. Hence X is T_1. $\square$

Corollary 1. *If a bijection $\psi : X \to Y$ has a strongly β-θ-closed graph then both X and Y are β-T_1.*

Proof. The proof follows from the facts that every β-T_2 space is β-T_1 and every T_1 space is β-T_1 and from the Theorem 5 and the Theorem 7. □

THEOREM 8. *Let $\phi, \psi : X \to Y$ be two functions such that ϕ is (θ, β)-continuous and ψ has a strongly β-θ-closed graph. Then the set $\{(x_1, x_2) \in X \times X : \phi(x_1) \neq \psi(x_2)\}$ is open in $X \times X$.*

Proof. To prove this theorem it is enough to prove that the set $E = \{(x_1, x_2) \in X \times X : \phi(x_1) = \psi(x_2)\}$ is closed in $X \times X$. Let $(x_1, x_2) \notin E$. Then $\phi(x_1) \neq \psi(x_2)$ and hence $(x_2, \phi(x_1)) \notin G(\psi)$. Since ψ has strongly β-θ-closed graph, then by Lemma 3, there exist an open set U containing x_2 and a $V \in \beta O(Y, \phi(x_1))$ such that $\psi(U) \cap \beta\text{-cl}(V) = \emptyset$. Since ϕ is (θ, β)-continuous, there exists an open set W containing x_1 such that $\phi(W) \subset \beta\text{-cl}(V)$. So we have $\psi(U) \cap \phi(W) = \emptyset$. Therefore $(W \times U) \cap E = \emptyset$ and hence E is closed in $X \times X$. □

COROLLARY 2. *If $\psi : X \to Y$ is a (θ, β)-continuous function where Y is β-T_2 then the set $\{(x_1, x_2) \in X \times X : \psi(x_1) \neq \psi(x_2)\}$ is open in $X \times X$.*

Proof. Since ψ is (θ, β)-continuous and Y is β-T_2 then by Theorem 4 ψ has a strongly β-θ-closed graph. Hence the result is followed from Theorem 8. □

THEOREM 9. *Let $\psi : X \to Y$ be a function having strongly β-θ-closed graph. If K is a subset β-closed relative to Y, then $\psi^{-1}(K)$ is closed in X.*

Proof. Let $x \notin \psi^{-1}(K)$. So $(x, y) \notin G(\psi)$ for each $y \in K$. As ψ has strongly β-θ-closed graph, then there exist an open set U_y containing x and a $V_y \in \beta O(Y, y)$ such that $\psi(U_y) \cap \beta\text{-cl}(V_y) = \emptyset$. Since K is β-closed relative to Y and $\{V_y : y \in K\}$ is a cover of K by β-open sets of Y, there exist $V_{y_1}, V_{y_2}, ..., V_{y_n}$ such that $K \subset \cup_{i=1}^{n} \beta\text{-cl}(V_{y_i})$. Let $U = \cap_{i=1}^{n} U_{y_i}$. Then clearly $\psi(U) \cap (\cup_{i=1}^{n} \beta\text{-cl}(V_{y_i})) = \emptyset$ and hence $\psi(U) \cap K = \emptyset$. Thus $x \in U \subset X - \psi^{-1}(K)$. So $\psi^{-1}(K)$ is closed in X. □

THEOREM 10. *Let $\psi : X \to Y$ be a function having strongly β-θ-closed graph where Y is either β-O-regular or submaximal and extremally disconnected. If K is a compact subset of X then $\psi(K)$ is closed in Y.*

Proof. The proof is analogous to the proof of the Theorem 9 and is thus omitted. □

4. APPLICATIONS

In the earlier section, we have derived several characterizations and properties of strongly β-θ-closed graphs of functions. In this section, applications of such notion we derive a new characterization of the covering property β-closedness [7] and a theorem that concerns on common fixed points of a family of functions having strongly β-θ-closed graphs.

DEFINITION 10. [7] A space X is said to be β-closed [7] if every cover of X by β-open sets has a finite subfamily whose β-closures cover X.

Theorem 11. [7] *The following are equivalent for a space X*
(a) *X is β-closed,*
(b) *each family of β-θ-closed sets with the finite intersection property has non-empty intersection,*
(c) *each filter base on X has at least β-θ-adherent point,*
(d) *each filter base on X with at most one β-θ-adherent point is β-θ-convergent,*
(e) *every maximal filter base β-θ-converges to some point in X.*

Theorem 12. *Let Y be a β-closed space. Then every function ψ from any space X to Y having strongly β-θ-closed graph is (θ, β)-continuous.*

Proof. In order to show that $\psi : X \to Y$ is (θ, β)-continuous it is sufficient to show Theorem 11 that for each $B \subset Y$, $\mathrm{cl}(\psi^{-1}(B)) \subset \psi^{-1}(\beta\text{-}\theta\text{-cl}(B))$. Let $x \in \mathrm{cl}(\psi^{-1}(B))$. If $x \in \psi^{-1}(B)$ then obviously $x \in \psi^{-1}(\beta\text{-}\theta\text{-cl}(B))$. If $x \notin \psi^{-1}(B)$ then there exists a filter base $\mathcal{F}$ containing $\psi^{-1}(B)$ such that $\mathcal{F} \to x$. Now as $\psi(\mathcal{F})$ is a filter base on the β-closed space Y, then by Theorem 11, we have $\emptyset \neq \beta\text{-}\theta\text{-ad}\psi(\mathcal{F}) \subset \beta\text{-}\theta\text{-cl}(B)$. We claim that $\beta\text{-}\theta\text{-ad}\psi(\mathcal{F}) \subset \{\psi(x)\}$. Indeed if $y \neq \psi(x)$ then $(x, y) \notin G(\psi)$. Since ψ has a strongly β-θ-closed graph then there are an $U \in O(x)$ and a $V \in \beta O(Y, y)$ satisfying $(U \times \beta\text{-cl}(V)) \cap G(\psi) = \emptyset$ i.e. $\psi(U) \cap \beta\text{-cl}(V) = \emptyset$. Since $\mathcal{F} \to x$ then there is an $F \in \mathcal{F}$ such that $F \subset U$. Therefore $\psi(F) \cap \beta\text{-cl}(V) = \emptyset$. So, $y \notin \beta\text{-}\theta\text{-cl}(\psi(F))$ and hence $y \notin \beta\text{-}\theta\text{-ad}\psi(F)$. Therefore $\beta\text{-}\theta\text{-ad}\psi(\mathcal{F}) \subset \{\psi(x)\}$. So $\{\psi(x)\} = \beta\text{-}\theta\text{-ad}\psi(\mathcal{F}) \subset \beta\text{-}\theta\text{-cl}(B)$ i.e. $\psi(x) \in \beta\text{-}\theta\text{-cl}(B)$ and hence $x \in \psi^{-1}(\beta\text{-}\theta\text{-cl}(B))$. Therefore ψ is (θ, β)-continuous. $\square$

Theorem 13. *A space (Y, τ) is β-closed if each function ψ from any space X to Y having strongly β-θ-closed graph is (θ, β)-continuous.*

Proof. Let for any space X, any function $\psi : X \to Y$ is (θ, β)-continuous whenever ψ has a strongly β-θ-closed graph. In order to show that Y is β-closed it suffices to show by virtue of Theorem 11 that each filter base on Y has a β-θ-adherent point. If not, then there exists a filter base Ω on Y such that $\beta\text{-}\theta\text{-ad}\Omega = \emptyset$. Let us choose and fix some $y_0 \in Y$. If we take collection $\tau_0 = \{A \subset Y : y_0 \in Y - A \text{ or } F \subset A \text{ for some } F \in \Omega\}$ then (Y, τ_0) is a topological space [18] and we write $(Y, \tau_0) = Y(y_0, \Omega)$. Clearly $\Omega \to y_0$ in $Y(y_0, \Omega)$. If $p \in Y$, $p \neq y_0$ and $\mathcal{F}$ is a filter base on $Y - \{p\}$ then $\mathcal{F}$ can not converge to p in $Y(y_0, \Omega)$, since $\{p\}$ is open. So whenever a filter base $\mathcal{F}$ on $Y - \{p\}$ converges to p then $p = y_0$. Also it is clear that $\Omega \subset \mathcal{G}$, whenever $\mathcal{G}$ is a filter base on $Y - \{y_0\}$ such that $\mathcal{G} \to y_0$ in $Y(y_0, \Omega)$. We define a function $\psi : Y(y_0, \Omega) \to (Y, \tau)$ as follows: $\psi(y) = y$ for each $y \in Y$. We shall show that ψ has strongly β-θ-closed graph. Suppose $(y, z) \notin G(\psi)$. Then $z \neq \psi(y) = y$. Consider $\mathcal{G} = \{V - \{y\} : V \text{ is an open set in } Y(y_0, \Omega) \text{ containing } y\}$. If $\mathcal{G}$ is a filter base, then $\mathcal{G} \to y$ in $Y(y_0, \Omega)$. So by the argument given above $y = y_0$ and $\Omega \subset \mathcal{G}$. Now $\beta\text{-}\theta\text{-ad}\psi(\mathcal{G}) = \beta\text{-}\theta\text{-ad}\mathcal{G} \subset \beta\text{-}\theta\text{-ad}\Omega$ (since $\Omega \subset \mathcal{G}) = \emptyset$. So $z \notin \beta\text{-}\theta\text{-ad}\psi(\mathcal{G})$. Therefore there exist an open set V in $Y(y_0, \Omega)$ containing y and a β-open set W in (Y, τ) containing z $(\neq \psi(y))$

such that $\psi(V - \{y\}) \cap \beta\text{-cl}(W) = \emptyset$. So $((V - \{y\}) \times \beta\text{-cl}(W)) \cap G(\psi) = \emptyset$ and hence $(V \times \beta\text{-cl}(W)) \cap G(\psi) = \emptyset$. If $\mathcal{G}$ is not a filter base, then $V = \{y\}$ for some open set V containing y in $Y(y_0, \Omega)$ and the rest is trivial. Therefore ψ has a strongly β-θ-closed graph. But ψ is not (θ, β)-continuous. In fact in the space $Y(y_0, \Omega)$, ad$\Omega = \{y_0\}$. Since $\psi(y_0) = y_0$ and β-θ-ad$\Omega = \emptyset$, we can not have $\psi(\text{ad}\Omega) \subset \beta\text{-}\theta\text{-ad}\psi(\Omega)$ — a contradiction. Hence every filter base Ω on (Y, τ) has a β-θ-adherent point. So (Y, τ) is β-closed. □

Corollary 3. *A space Y is β-closed if and only if each function from any space X into Y having strongly β-θ-closed graph is (θ, β)-continuous.*

Lemma 4. *Let Y be β-T_2. If $\psi : X \to Y$ has a strongly β-θ-closed graph and if $\phi : X \to Y$ is (θ, β)-continuous then the set $\{x \in X : \phi(x) \neq \psi(x)\}$ is an open subset of X.*

Proof. To prove this theorem it is enough to prove that the set $E = \{x \in X : \phi(x) = \psi(x)\}$ is a closed subset of X. Let $z \in \text{cl}(E) - E$. Then there exists a filter base Ω on E such that $\Omega \to z$. Since ϕ is (θ, β)-continuous then by Theorem 11, $\phi(\Omega)$ β-θ-converges to $\phi(z)$. Now as $\phi(F) = \psi(F)$ for each $F \in \Omega$, the filter base $\psi(\Omega)$ is also β-θ-converging to some point, say, y in Y. Since ψ has a strongly β-θ-closed graph then by Theorem 2 $y = \psi(z)$. But as Y is being β-T_2 and $\phi(\Omega)$ and $\psi(\Omega)$ are being the same filter base on Y, $\psi(z) = \phi(z)$. Therefore $z \in E$ — a contradiction. So E is closed in X. □

Theorem 14. *Let X be β-T_2 β-connected space. If $\psi : X \to X$ has a strongly β-θ-closed graph then the set of fixed points of ψ is a closed subset of X.*

Proof. As X is β-connected then for any non-empty β-open set V, $\beta\text{-cl}(V) = X$ [17] and hence the identity function $\phi : X \to X$ is obviously (θ, β)-continuous. So by Lemma 4, it follows that the set of fixed points of ψ i.e. $E_\psi = \{x \in X : \psi(x) = \phi(x) = x\}$ is closed in X. □

Theorem 15. *Let X be a β-T_2 semi-preregular* [23] *β-connected β-closed space. Also let $\mathcal{F}$ be a family of functions from X into itself with strongly β-θ-closed graph. If for each finite $\mathcal{F}_0 \subset \mathcal{F}$ there exists an $x \in X$ such that $\psi(x) = x$ for all $\psi \in \mathcal{F}_0$ then there exists an $x \in X$ such that $\psi(x) = x$ for all $\psi \in \mathcal{F}$.*

Proof. Since X is β-connected the identity function $\phi : X \to X$ is obviously (θ, β)-continuous. Then by Lemma 4, and by hypothesis, the family of closed sets $\Omega = \{E_\psi = \{x \in X : \psi(x) = \phi(x)\} : \psi \in \Omega\}$ has the finite intersection property. Let $\mathcal{G}$ be the filter base generated by Ω. Then as X is being β-closed, β-θ-ad$\mathcal{G} \neq \emptyset$. Now, $\emptyset \neq \beta\text{-}\theta\text{-ad}\mathcal{G} \subset \cap\{E_\psi : \psi \in \Omega\}$. Hence there exists at least one $x \in X$ such that $\psi(x) = \phi(x) = x$ for all $\psi \in \Omega$. □

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Department of Mathematics
West Bengal State University
Barasat, Kolkata-700126
North 24 Parganas, West Bengal, India
E-mail: ckbasu1962@yahoo.com

Department of Mathematics
Sripat Singh College
Jiaganj-742123, Murshidabad
West Bengal, India

Department of Mathematics
Krishnagar Women's College
Krishnagar-741101, Nadia
West Bengal, India