MATHEMATICA, Tome 51 (74), N° 2, 2009, pp. 111–118

ON SOME STARLIKE MAPPINGS INVOLVING CERTAIN CONVOLUTION OPERATORS

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Abstract. This paper makes a modest contribution to the family of starlike univalent mappings in the open unit disk, by the introduction of some new subclasses of them via certain convolution operators. A new univalence condition is given with examples. Some basic characterizations of functions of the new subclasses are also mentioned.

MSC 2000. 30C45.
Key words. Analytic, univalent functions, starlikeness, convolution operator.

1. BACKGOUND

First geometry: A domain $D \subset \mathbb{C}$ is said to be starlike with respect to a point $\xi_0 \in D$ if the linear segment joining ξ_0 to every other point $\xi \in D$ lies entirely in D. In more picturesque language, every point of D is visible from ξ_0. If $\xi_0 = 0$, the origin, D is simply called starlike [3].

Let A denote the class of functions $f(z) = z + a_2 z^2 + \cdots$, which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A$ is called starlike if and only if f maps E onto a starlike domain. Analytically, f is starlike (see [7]) if and only if, for $z \in E$,

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0. \tag{1}$$

Let S^* denote the family of functions starlike in E. This family of univalent mappings has recieved so much attention in geometric functions theory, having found applications in many physical problems. In the sequel we employ convolution operators defined in [2] to identify new subclasses of this important family. A principal discovery in this paper is a new univalence condition in the open unit disk.

Let $f \in A$. For $g(z) = z + b_2 z^2 + \cdots \in A$, the convolution (or Hadamard product) of $f(z)$ and $g(z)$ (written as $(f * g)(z)$) is defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In [2] we defined the operators $L_n^\sigma : A \to A$ (using the convolution $*$) as follows:

$$L_n^\sigma f(z) = (\tau_\sigma * \tau_{\sigma,n}^{(-1)} * f)(z).$$

where

$$\tau_{\sigma,n}(z) = \frac{z}{(1-z)^{\sigma-(n-1)}}, \quad \sigma - (n-1) > 0,$$

$\tau_\sigma = \tau_{\sigma,0}$ and $\tau_{\sigma,n}^{(-1)}$ such that

$$(\tau_{\sigma,n} * \tau_{\sigma,n}^{(-1)})(z) = \frac{z}{1-z}$$

for a fixed real number σ and $n \in \mathbb{N}$. We noted that $L_0^\sigma f(z) = L_0^0 f(z) = f(z)$, $L_1^1 f(z) = zf'(z)$. Given $f \in A$, we have

$$L_n^\sigma f(z) = z + \sum_{k=2}^{\infty} \left\{ \prod_{j=0}^{n-1} \left(\frac{\sigma + k - 1 - j}{\sigma - j} \right) \right\} a_k z^k$$

$$= z + \sum_{k=2}^{\infty} \left\{ \frac{(\sigma + k - 1)!}{\sigma!} \frac{(\sigma - n)!}{(\sigma + k - 1 - n)!} \right\} a_k z^k \tag{2}$$

(see [2]).

Similarly we defined $l_n^\sigma : A \to A$ as follows:

$$l_n^\sigma f(z) = (\tau_\sigma^{(-1)} * \tau_{\sigma,n} * f)(z)$$

so that for $f \in A$ we have

$$L_n^\sigma(l_n^\sigma f(z)) = l_n^\sigma(L_n^\sigma f(z)) = f(z).$$

In this paper we further restrict σ and n by requiring $\sigma \geq n + 1$ in the definitions above. Thus the following relations hold:

$$(\sigma - n) L_{n+1}^\sigma f(z) = (\sigma - (n+1)) L_n^\sigma f(z) + z(L_n^\sigma f(z))' \tag{3}$$

from which we have

$$(\sigma - n)(L_{n+1}^\sigma f(z))' = (\sigma - n)(L_n^\sigma f(z))' + z(L_n^\sigma f(z))''. \tag{4}$$

Now we say:

DEFINITION. A function $f \in A$ belongs to the class S_n^σ if and only if

$$\mathrm{Re} \frac{L_{n+1}^\sigma f(z)}{L_n^\sigma f(z)} > \frac{\sigma - (n+1)}{\sigma - n}, \quad \sigma \geq n+1, \quad n \in \mathbb{N}.$$

Observing from (3) that

$$\frac{L_{n+1}^\sigma f(z)}{L_n^\sigma f(z)} = \frac{\sigma - (n+1)}{\sigma - n} + \frac{z(L_n^\sigma f(z))'}{(\sigma - n) L_n^\sigma f(z)}$$

and the fact that $L_0^\sigma f(z) = f(z)$ we have the following remarks.

REMARK 1. (a) $S_0^\sigma \equiv S^*$ and
(b) $f \in S_n^\sigma$ if and only if $L_n^\sigma f(z)$ is starlike.

Following from inclusion relations we will deduce that all members of the class S_n^σ are starlike univalent in E. We also give a univalence condition based on the inclusion relations and provide some examples. Furthermore we give some basic properties of the class, namely, integral representation, coefficient inequalities, closure under certain integral operators.

This paper is organised as follows: In the next section we give some preliminary lemmas while Section 3 contains the main results characterizing S_n^{σ}.

2. PRELIMINARY LEMMAS

Let P denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are regular in E and satisfy $\mathrm{Re}p(z) > 0$, $z \in E$.

LEMMA 1 ([1, 5]). *Let* $u = u_1 + u_2\mathrm{i}$, $v = v_1 + v_2\mathrm{i}$ *and* ψ *a complex-valued function satisfying:* (a) $\psi(u, v)$ *is continuous in a domain* Ω *of* $\mathbb{C}^2$, (b) $(1, 0) \in \Omega$ *and* $\mathrm{Re}\psi(1, 0) > 0$ *and* (c) $\mathrm{Re}\psi(u_2\mathrm{i}, v_1) \leq 0$ *when* $(u_2\mathrm{i}, v_1) \in \Omega$ *and* $2v_1 \leq -(1 + u_2^2)$. *If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *satisfies* $(p(z), zp'(z)) \in \Omega$ *and* $\mathrm{Re}\psi(p(z), zp'(z)) > 0$ *for* $z \in E$, *then* $\mathrm{Re}p(z) > 0$ *in* E.

LEMMA 2 ([4]). *Let* η *and* μ *be complex constants and* $h(z)$ *a convex univalent function in* E *satisfying* $h(0) = 1$, *and* $\mathrm{Re}(\eta h(z) + \mu) > 0$. *Suppose* $p \in P$ *satisfies the differential subordination:*

$$p(z) + \frac{zp'(z)}{\eta p(z) + \mu} \prec h(z), \quad z \in E. \tag{5}$$

If the differential equation:

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = h(z), \quad q(0) = 1 \tag{6}$$

has univalent solution $q(z)$ *in* E, *then* $p(z) \prec q(z) \prec h(z)$ *and* $q(z)$ *is the best dominant in* (5).

The formal solution of (6) is given as

$$q(z) = \frac{zF'(z)}{F(z)} \tag{7}$$

where

$$F(z)^{\eta} = \frac{\eta + \mu}{z^{\mu}} \int_0^z t^{\mu - 1} H(t)^{\eta} \mathrm{d}t$$

and

$$H(z) = z \exp\left(\int_0^z \frac{h(t) - 1}{t} \mathrm{d}t\right)$$

(see [6, 9]). The conditions for the univalence of the solution $q(z)$ of (6) (given by (7)) as given in [6] are that

LEMMA 3 ([6]). *Let* $\eta \neq 0$ *and* μ *be complex constants, and* $h(z)$ *regular in* E *with* $h'(0) \neq 0$, *then the solution* $q(z)$ *of* (6) *(given by* (7)*) is univalent in* E *if* (i) $\mathrm{Re}\{G(z) = \eta h(z) + \mu\} > 0$ *and* (ii) $Q(z) = zG'(z)/G(z)$ *and* $R(z) = Q(z)/G(z)$ *are both starlike in* E.

3. MAIN RESULTS

THEOREM 1. *Let* $\sigma \geq n+1$ *and* $h(z)$ *a convex univalent function in* E *satisfying* $h(0) = 1$, *and* $\mathrm{Re}(\sigma - (n+1) + h(z)) > 0$, $z \in E$. *Let* $f \in A$. *If* $\frac{L_{n+2}^{\sigma} f(z)}{L_{n+1}^{\sigma} f(z)} \prec h(z)$, *then* $\frac{L_{n+1}^{\sigma} f(z)}{L_{n}^{\sigma} f(z)} \prec h(z)$.

Proof. By Remark 1(b), it is sufficient to prove that: if

$$\frac{z(L_{n+1}^{\sigma} f(z))'}{L_{n+1}^{\sigma} f(z)} \prec h(z),$$

then

$$\frac{z(L_{n}^{\sigma} f(z))'}{L_{n}^{\sigma} f(z)} \prec h(z).$$

Now let

$$p = \frac{z(L_{n}^{\sigma} f(z))'}{L_{n}^{\sigma} f(z)}.$$

Then

$$z(L_{n}^{\sigma} f(z))'' + (L_{n}^{\sigma} f(z))' = p'(z) L_{n}^{\sigma} f(z) + p(z)(L_{n}^{\sigma} f(z))'$$

Using (4) we obtain

$$(\sigma - n) z(L_{n+1}^{\sigma} f(z))' = zp'(z) L_{n}^{\sigma} f(z) + zp(z)(L_{n}^{\sigma} f(z))' + (\sigma - (n+1)) z(L_{n}^{\sigma} f(z))'$$

so with (3) we have

$$\begin{aligned} \frac{z(L_{n+1}^{\sigma} f(z))'}{L_{n+1}^{\sigma} f(z)} &= \frac{zp'(z) L_{n}^{\sigma} f(z) + zp(z)(L_{n}^{\sigma} f(z))' + (\sigma - (n+1)) z(L_{n}^{\sigma} f(z))'}{(\sigma - (n+1)) L_{n}^{\sigma} f(z) + z(L_{n}^{\sigma} f(z))'} \\ &= \frac{zp'(z) + p(z)^2 + (\sigma - (n+1)) p(z)}{\sigma - (n+1) + p(z)} \\ &= p(z) + \frac{zp'(z)}{\sigma - (n+1) + p(z)}. \end{aligned} \tag{8}$$

Now take $\eta = 1$ and $\mu = \sigma - (n+1)$ in Lemma 2, the result follows. □

THEOREM 2. *Let* $\sigma \geq n+1$ *and* $h(z)$ *a convex univalent function in* E *satisfying* $h(0) = 1$, *and* $\mathrm{Re}(\sigma - (n+1) + h(z)) > 0$, $z \in E$. *Let* $f \in A$. *If* $f \in S_{n+1}^{\sigma}$, *then*

$$\frac{L_{n+1}^{\sigma} f(z)}{L_{n}^{\sigma} f(z)} \prec q(z)$$

where

$$q(z) = \frac{1 + \sum_{k=1}^{\infty} \frac{\sigma - n}{\sigma - n + k}(k+1)^2 z^k}{1 + \sum_{k=1}^{\infty} \frac{\sigma - n}{\sigma - n + k}(k+1) z^k} \tag{9}$$

and it is the best dominant.

Proof. By Remark 1(b) also, if $f \in S^{\sigma}_{n+1}$, then

$$\frac{z(L^{\sigma}_{n+1}f(z))'}{L^{\sigma}_{n+1}f(z)} \prec \frac{1+z}{1-z}.$$

So we have to prove that

$$\frac{z(L^{\sigma}_{n}f(z))'}{L^{\sigma}_{n}f(z)} \prec q(z).$$

By considering the differential equation

$$q(z) + \frac{zq'(z)}{\sigma-(n+1)+q(z)} = \frac{1+z}{1-z}$$

whose solution (using (7)) is given by (9), our result follows from Lemma 2 if we prove that $q(z)$ (given by (9)) is univalent in E. Now set $\eta = 1$, $\mu = \sigma - (n+1)$ and $h(z) = (1+z)/(1-z)$ in Lemma 3, we have

(i)

$$\mathrm{Re}G(z) = \mathrm{Re}[\mu + h(z)] > \mu \geq 0.$$

(ii)

$$Q(z) = \frac{zG'(z)}{G(z)} = \frac{2}{1+\mu}\frac{z}{(1+az)(1-z)}$$

where $a = (1-\mu)/(1+\mu)$, so that

$$\begin{aligned}\frac{zQ'(z)}{Q(z)} &= \frac{1+az^2}{(1+az)(1-z)}\\ &= \frac{1}{1-z} + \frac{1}{1+az} - 1.\end{aligned}$$

Thus $\mathrm{Re}zQ'(z)/Q(z) > \mu/2 > 0$. And finally we have

$$R(z) = \frac{Q(z)}{G(z)} = \frac{2}{1+\mu}\frac{z}{(1+az)^2}$$

so that $zR'(z)/R(z) = (1-az)/(1+az)$ with real part greater than zero. Thus $q(z)$ satisfies all conditions of Lemma 3, hence univalent in E. This completes the proof. □

Theorem 3.

$$S^{\sigma}_{n+1} \subset S^{\sigma}_{n}, \quad n \in \mathbb{N}.$$

Proof. Let $f \in S^{\sigma}_{n+1}$. From (8) above, define $\psi(p(z), zp'(z)) = p(z) + \frac{zp'(z)}{\sigma-(n+1)+p(z)}$ for $\Omega = [\mathbb{C} - \{-(\sigma-(n+1))\}] \times \mathbb{C}$. Obviously ψ satisfies the conditions (a) and (b) of Lemma 1. Now $\psi(u_2\mathrm{i}, v_1) = u_2\mathrm{i} + \frac{v_1}{\sigma-(n+1)+u_2\mathrm{i}}$ so that $\mathrm{Re}\psi(u_2\mathrm{i}, v_1) = \frac{(\sigma-(n+1))v_1}{(\sigma-(n+1))^2+u_2^2} \leq 0$ if $v_1 \leq -\frac{1}{2}(1+u_2^2)$. Hence by Lemma 1, we have $\mathrm{Re}\frac{z(L^{\sigma}_{n+1}f(z))'}{L^{\sigma}_{n+1}f(z)} > 0$ implies $\mathrm{Re}\frac{z(L^{\sigma}_{n}f(z))'}{L^{\sigma}_{n}f(z)} > 0$. By Remark 1(b)

$$\mathrm{Re}\frac{L^{\sigma}_{n+1}f(z)}{L^{\sigma}_{n}f(z)} > \frac{\sigma-(n+1)}{\sigma-n}.$$

This completes the proof. □

Corollary 1. *All functions in S_n^σ are starlike univalent in E.*

Following from the inclusion relations, setting $\sigma = 2$ and $n = 1$, we have the following important univalence condition.

Corollary 2. *Let $f \in A$ satisfy*

$$\mathrm{Re}\frac{2zf'(z) + z^2f''(z)}{f(z) + zf'(z)} > 0, \quad z \in E.$$

Then $f(z)$ is starlike univalent in E.

Example 1. The functions $f_j(z)$ $j = 1, 2, 3, 4.$, given by

$$f_1(z) = \frac{2[1-(1-z)\mathrm{e}^z]}{z}, \quad f_2(z) = \frac{2[1-(1+z)\mathrm{e}^{-z}]}{z},$$
$$f_3(z) = \frac{-2[z+\log(1-z)]}{z}, \quad f_4(z) = \frac{2[z-\log(1+z)]}{z}.$$

are starlike univalent in the open unit disk.

Proof. By direct computation we find that

$$\frac{2zf_j'(z) + z^2f_j''(z)}{f_j(z) + zf_j'(z)} = \begin{cases} 1+z & \text{if } j=1, \\ 1-z & \text{if } j=2, \\ \frac{1}{1-z} & \text{if } j=3, \\ \frac{1}{1+z} & \text{if } j=4. \end{cases}$$

Observe that the right hand side of the above equations are all functions in P, hence we have

$$\mathrm{Re}\frac{2zf_j'(z) + z^2f_j''(z)}{f_j(z) + zf_j'(z)} > 0, \quad j = 1,\ 2,\ 3,\ 4$$

so that $f_j \in S_1^2 \subset S^*$. The proof is complete. □

Theorem 4. *Functions in S_n^σ have integral representation:*

$$f(z) = l_n^\sigma\left\{z\exp\left(\int_0^z \frac{p(t)-1}{t}\mathrm{d}t\right)\right\}$$

for some $p \in P$.

Proof. Let $f \in S_n^\sigma$, then for some $p \in P$ we have

$$\frac{L_{n+1}^\sigma f(z)}{L_n^\sigma f(z)} = \frac{\sigma-(n+1)}{\sigma-n} + \frac{z(L_n^\sigma f(z))'}{(\sigma-n)L_n^\sigma f(z)}$$
$$= \frac{\sigma-(n+1)}{\sigma-n} + \frac{p(z)}{(\sigma-n)}.$$

Hence we have

$$\frac{z(L_n^\sigma f(z))'}{L_n^\sigma f(z)} = p(z).$$

Simple calculation now leads to

$$L_n^\sigma f(z) = z \exp\left(\int_0^z \frac{p(t)-1}{t}\mathrm{d}t\right).$$

Applying l_n^σ on both sides we have the representation. $\square$

If we choose $p(z) = (1+z)/(1-z)$, we obtain the leading example of the class S_n^σ, which is

$$k_n^\sigma(z) = z + \sum_{k=2}^{\infty}\left\{\frac{\sigma!}{(\sigma+k-1)!}\frac{(\sigma+k-1-n)!}{(\sigma-n)!}\right\} kz^k. \tag{10}$$

Next we investigate the closure property of the class S_n^σ under the Bernardi integral transformation:

$$F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\mathrm{d}t, \quad \gamma > -1. \tag{11}$$

The well known Libera integral corresponds to $\gamma = 1$.

Theorem 5. *The class S_n^σ is closed under F.*

Proof. From (11) we have

$$\gamma F(z) + zF'(z) = (\gamma+1)f(z). \tag{12}$$

If we apply L_n^σ on (12), noting from (2) that $L_n^\sigma(zF'(z)) = z(L_n^\sigma F(z))'$ we have

$$\frac{z(L_n^\sigma f(z))'}{L_n^\sigma f(z)} = \frac{(\gamma+1)z(L_n^\sigma F(z))' + z^2(L_n^\sigma F(z))''}{\gamma L_n^\sigma F(z) + z(L_n^\sigma F(z))'}.$$

Let $p(z) = z(L_n^\sigma f(z))'/L_n^\sigma f(z)$. Then by some calculation we find that

$$\frac{z(L_n^\sigma f(z))'}{L_n^\sigma f(z)} = p(z) + \frac{zp'(z)}{\gamma + p(z)} = \psi(p(z), zp'(z))$$

Define $\psi(p(z), zp'(z)) = p(z) + \frac{zp'(z)}{\gamma+p(z)}$ for $\Omega = [\mathbb{C} - \{-\gamma\}] \times \mathbb{C}$. Then, as in Theorem 2, ψ satisfies all the conditions of Lemma 1, hence $\frac{z(L_n^\sigma f(z))'}{L_n^\sigma f(z)} > 0$ implies $\mathrm{Re}\frac{z(L_n^\sigma F(z))'}{L_n^\sigma F(z)} > 0$ and by Remark 1(b) we have

$$\mathrm{Re}\frac{L_{n+1}^\sigma F(z)}{L_n^\sigma F(z)} > \frac{\sigma-(n+1)}{\sigma-n}$$

as required. $\square$

Theorem 6. *Let $f \in S_n^\sigma$. Then we have the inequalities*

$$|a_k| \le \frac{\sigma!}{(\sigma+k-1)!}\frac{(\sigma+k-1-n)!}{(\sigma-n)!}k, \quad k \ge 2.$$

The function $k_n^\sigma(z)$, given by (10), show that the inequalities are sharp.

Proof. It is known that for each $f \in S^*$, $|a_k| \le k$, $k \ge 2$. Thus by Remark 1(b), for each $f \in S_n^\sigma$ the coefficients of $L_n^\sigma f(z)$ satisfy $|a_k| \le k$, $k \ge 2$. Hence using (2) we have the inequalities. □

Acknowledgements. This work was carried out at the Centre for Advanced Studies in Mathematics, CASM, Lahore University of Management Sciences, Lahore, Pakistan during the author's postdoctoral fellowship at the Centre. The author is indebted to all staff of CASM for their hospitality, most especially Prof. Ismat Beg.

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Received March 31, 2008
Accepted February 22, 2009

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