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THE FARTHEST POINT PROBLEM IN NON-ARCHIMEDEAN NORMED SPACES

MOHAMMAD SAL MOSLEHIAN, ASSADOLLAH NIKNAM and SEDDIGHEH SHADKAM

Abstract. We study the farthest point mapping in non-Archimedean normed spaces. We prove that a uniquely remotal subset M in a non-Archimedean normed space X is singleton if for some Chebyshev center c and some $|\alpha| < 1$ the equality $q_M(\alpha c + (1-\alpha)q_M(c)) = q_M(c)$ holds. We show that M is singleton if and only if $\|x - q_M(x)\| = \|y - q_M(y)\|$ implies that $q_M(x) = q_M(y)$. We also prove that if X, Y are non-Archimedean normed spaces and $Z = X \times Y$ is equipped with the norm $\|(x, y)\| = \max\{|x|, |y|\}$, then all uniquely remotal sets in $(Z, \|.\|)$ are singletons.

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1. INTRODUCTION

A non-Archimedean field is a field K equipped with a function (valuation) $|.|$ from K into $[0, \infty)$ such that for each $r, s \in K$ the following relations hold $|rs| = |r||s|$, $|r + s| \leq \max\{|r|, |s|\}$, and $|r| = 0$ if and only if $r = 0$. An example of a non-Archimedean valuation is the mapping $|.|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.

In 1897 Hensel [3] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a number integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = \|x - y\|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field; see [9]. During the last three decades p-adic numbers have gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings (cf. [4]).

Now let X be a vector space over a scalar field K with a non-Archimedean valuation $|.|$. A function $\|.\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|rx\| = |r|\,\|x\| \quad (r \in K, x \in X)$,

(iii) the strong triangle inequality $\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X)$.

Then $(X, \|.\|)$ is called a non-Archimedean normed space. Throughout the paper we assume that X is a non-Archimedean normed space over a non-Archimedean filed K satisfying

$$\|X\| := \{\|x\| : x \in X\} = \{|r| : r \in K\}, \tag{1.1}$$

see [6, 7]. Let X be a real normed space and M be a non-empty bounded subset of X. The mapping $Q_M : X \to 2^M$ defined by

$$Q_M(x) = \{q_M(x) \in M : \|x - q_M(x)\| = \sup_{t \in M} \|x - t\|\}$$

is called the farthest point mapping of M. We call M a remotal (uniquely remotal) set if for each $x \in X$ the set $Q_M(x)$ is not empty (is singleton). A Chebyshev center of M in X is an element c in X satisfying

$$r(M) := \sup_{t \in M} \|c - t\| = \inf_{x \in X} \sup_{t \in M} \|x - t\|.$$

In fact, $r(M)$, the so-called Chebyshev radius of M, is the smallest ball in X containing M. The space X is said to admit centers whenever any non-empty bounded subset of X has at least one center; see [5].

We study the farthest point mapping in a non-Archimedean normed spaces. Using the strategies of [1, 2, 8], we prove that a uniquely remotal subset M in a non-Archimedean normed space X is singleton if for some Chebyshev center c and some $|\alpha| < 1$ the equality

$$q_M(\alpha c + (1 - \alpha) q_M(c)) = q_M(c)$$

holds. We show that M is singleton if and only if $\|x - q_M(x)\| = \|y - q_M(y)\|$ implies that $q_M(x) = q_M(y)$. We also prove that if X, Y are non-Archimedean normed spaces and $Z = X \times Y$ is equipped with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$, then all uniquely remotal sets in $(Z, \|.\|)$ are singletons.

2. THE MAIN RESULTS

LEMMA 2.1. *Let X be a non-Archimedean normed space and let M be a remotal subset of X. If $|\alpha| \geq 1$ then $q_M(\alpha x + (1 - \alpha) q_M(x)) = q_M(x)$.*

Proof. Let $t \in M$. Then

$$\begin{aligned} \|\alpha x + (1 - \alpha) q_M(x) - t\| &= \|q_M(x) - t + \alpha(x - q_M(x))\| \\ &\leq \max\{\|q_M(x) - t\|, \|\alpha(x - q_M(x))\|\}. \end{aligned}$$

If $\|q_M(x) - t\| \leq \|\alpha(x - q_M(x))\|$, then

$$\|\alpha x + (1 - \alpha) q_M(x) - t\| \leq \|\alpha(x - q_M(x))\| = \|\alpha x + (1 - \alpha) q_M(x) - q_M(x)\|.$$

If $\|\alpha(x-q_M(x))\| \le \|q_M(x)-t\|$, then

$$\begin{aligned}
\|\alpha x+(1-\alpha)q_M(x)-t\| &\le \|q_M(x)-t\| \\
&= \|q_M(x)-x+x-t\| \\
&\le \max\{\|x-t\|, \|x-q_M(x)\|\} \\
&= \|x-q_M(x)\| \\
&\le |\alpha|\,\|x-q_M(x)\| \\
&= \|\alpha x+(1-\alpha)q_M(x)-q_M(x)\|.
\end{aligned}$$

Hence $q_M(\alpha x+(1-\alpha)q_M(x)) = q_M(x)$. □

LEMMA 2.2. *Let X be a non-Archimedean normed space and let M be a remotal subset of X. If c is a Chebyshev center and if for $|\alpha|<1$ the equality $q_M(\alpha c+(1-\alpha)q_M(c)) = q_M(c)$ holds, then M is singleton .*

Proof. We know that

$$\begin{aligned}
\|c-q_M(c)\| &= \inf_{x\in X}\|x-q_M(x)\| \\
&\le \|\alpha c+(1-\alpha)q_M(c)-q_M(\alpha c+(1-\alpha)q_M(c))\| \\
&= \|\alpha c+(1-\alpha)q_M(c)-q_M(c)\| \\
&= \|\alpha(c-q_M(c)\| \\
&= |\alpha|\,\|c-q_M(c)\|.
\end{aligned}$$

If $c\ne q_M(c)$, then $1\le|\alpha|$, a contradiction. So $c=q_M(c)$. Thus

$$\sup_{t\in M}\|c-t\| = 0,$$

whence $M=\{c\}$. □

THEOREM 2.3. *If c is a Chebyshev center in a non-Archimedean normed space X and there exist x,α such that $|\alpha|>1$ and $\alpha x+(1-\alpha)q_M(x)=c$, then M is singleton.*

Proof. By Lemma 2.1, $q_M(x) = q_M(\alpha x+(1-\alpha)q_M(x)) = q_M(c)$. Since $|\frac{1}{\alpha}|<1$, we have

$$\begin{aligned}
q_M\left(\frac{1}{\alpha}c+(1-\frac{1}{\alpha})q_M(c)\right) &= q_M\left(\frac{1}{\alpha}(\alpha x+(1-\alpha)q_M(x))+(1-\frac{1}{\alpha})q_M(c)\right) \\
&= q_M(x)=q_M(c).
\end{aligned}$$

It follows from Lemma 2.2 that M is singleton. □

EXAMPLE 2.4. Let X be the field of rational numbers endowed with the 2-adic valuation. If $M=\{\frac{1}{2}\}$ we have the following cases:

Case (i) $x=2^m\frac{p}{q}$, where $m\ge1$ and $(p,2)=(q,2)=1$. Since $p\cdot2^{m+1}-q$ is an odd integer, we conclude that

$$\left|2^m\frac{p}{q}-\frac{1}{2}\right|_2 = \left|\frac{p\cdot2^{m+1}-q}{2q}\right|_2 = 2.$$

Case (ii) $x = 2^{-m}\frac{p}{q}$, where $m > 1$ and $(p, 2) = (q, 2) = 1$. Since $p-q\cdot(2^{m-1})$ is an odd integer, we have that

$$\left|\frac{p}{2^m q} - \frac{1}{2}\right|_2 = \left|\frac{p - q\cdot 2^{m-1}}{2^m q}\right|_2 = 2^m.$$

Case (iii) $x = \frac{p}{2q}$ where $(p, 2) = (q, 2) = 1$. Since $p - q$ is an even integer, we have that

$$\left|\frac{p}{2q} - \frac{1}{2}\right|_2 = \left|\frac{p-q}{2q}\right|_2 = 0 \text{ or } 2^r \text{ for some } r \leq 0.$$

Case (iv) $x = \frac{p}{q}$ where $(p, 2) = (q, 2) = 1$. Since $2p - q$ is an odd integer, we have that

$$\left|\frac{p}{q} - \frac{1}{2}\right|_2 = \left|\frac{2p-q}{2q}\right|_2 = 2.$$

Hence $x = \frac{1}{2}$ is a Chebyshev center.

It is clear that Theorem 2.3 holds for $c = \frac{1}{2}$ with $x = \frac{1}{2}$ and $\alpha = \frac{1}{4}$. Note that $|\alpha| = 4 > 1$ and $q_M(x) = \frac{1}{2}$.

THEOREM 2.5. *Let M be a uniquely remotal set of a non-Archimedean normed space X. Then M is singleton if and only if $\|x-q_M(x)\| = \|y-q_M(y)\|$ implies $q_M(x) = q_M(y)$ for every $x, y \in X$.*

Proof. If M is singleton then $q_M(x) = q_M(y)$, for every $x, y \in X$, therefore $\|x - q_M(x)\| = \|y - q_M(y)\| \Rightarrow q_M(x) = q_M(y)$. Now suppose that the implication $\|x-q_M(x)\| = \|y-q_M(y)\| \Rightarrow q_M(x) = q_M(y)$ holds. We will show that M is singleton. If M is not singleton, there exist x, y such that $q_M(x) \neq q_M(y)$, thus $\|x - q_M(x)\| \neq \|y - q_M(y)\|$. We assume that $\|x - q_M(x)\| < \|y - q_M(y)\|$. Let $k \in K$ such that

$$|k| = \frac{\|y - q_M(y)\|}{\|x - q_M(x)\|} > 1.$$

This follows easily from (1.1) and the multiplicative property of $|.|$ on K.
If $z = q_M(x)+k(x-q_M(x))$ then $\|z-q_M(x)\| = |k|\,\|x-q_M(x)\| = \|y-q_M(y)\|$. Also,

$$\|y-q_M(y)\| = \|z-q_M(x)\| \leq \|z-q_M(z)\| \leq \max\{\|z-x\|, \|x-q_M(z)\|\}. \tag{2.1}$$

Since $\|z - x\| = \|(x - q_M(x))(k - 1)\| = |k - 1|\,\|x - q_M(x)\|$ and $|k - 1| \leq \max\{|k|, |1|\} = |k|$, we get

$$\|z - x\| = |k - 1|\,\|x - q_M(x)\| \leq |k|\,\|x - q_M(x)\| = \|y - q_M(y)\|. \tag{2.2}$$

If $\|z - x\| < \|x - q_M(z)\|$, then inequality (2.1) yields

$$\begin{aligned}\|y - q_M(y)\| = \|z - q_M(x)\| &\leq \|z - q_M(z)\| \leq \|x - q_M(z)\| \\ &\leq \|x - q_M(x)\| < \|y - q_M(y)\|,\end{aligned}$$

a contradiction. Hence $\|z-x\| \geq \|x-q_M(z)\|$. Now inequalities (2.1) and (2.2) yield

$$\|y-q_M(y)\| = \|z-q_M(z)\| = \|z-q_M(x)\|, \tag{2.3}$$

whence $q_M(y) = q_M(x) = q_M(z)$ (in fact the first equality of (2.3) implies $q_M(y) = q_M(z)$, and the second equality of (2.3) and the assumption that M is uniquely remotal imply that $q_M(x) = q_M(z)$). □

The following lemma can be proved in a straightforward way and we omit its proof.

LEMMA 2.6. *Let $(X, \|.\|)$, $(Y, \|.\|)$ be arbitrary non-Archimedean normed spaces. Then $Z = X \times Y$, endowed with the norm $\|(x,y)\| = \max\{\|x\|, \|y\|\}$, is also a non-Archimedean normed space over the non-Archimedean field K.*

LEMMA 2.7. *Let $Z = X \times Y$ and $\emptyset \neq M \subset Z$ be a uniquely remotal bounded set. If $p_1, p_2 \in Z$, $p_i = (x_i, y_i)$ $(i = 1,2)$, $q_M(p_i) = z_i = (a_i, b_i)$ $(i = 1,2)$, $C_1 = \|p_1 - q_M(p_1)\| = \|x_1 - a_1\|$ and $C_2 = \|p_2 - q_M(p_2)\| = \|y_2 - b_2\|$, then $z_1 = z_2$.*

Proof. Assume $C_1 \geq C_2$, and set $\lambda = \frac{C_1}{C_2}$. Let $k \in K$ be such that $|k| = \lambda \geq 1$. We set

$$p_3 = (x_3, y_3) = kp_2 + (1-k)z_2,$$

thus

$$\|p_3 - z_2\| = \|k(p_2 - z_2)\| = |k|\,\|(p_2 - z_2)\| = C_1 \tag{2.4}$$

and

$$\begin{aligned}\|p_3 - p_2\| &= \|(1-k)(p_2 - z_2)\| = |1-k|\,\|p_2 - z_2\| \\ &\leq \max\{|k|, 1\}\,\|p_2 - z_2\| = |k|\,\|p_2 - z_2\| = C_1.\end{aligned}$$

It follows that

$$\begin{aligned}\|p_3 - q_M(p_3)\| &\leq \max\{\|p_3 - p_2\|, \|p_2 - q_M(p_3)\|\} \\ &\leq \max\{\|p_3 - p_2\|, \|p_2 - q_M(p_2)\|\} \\ &\leq \max\{C_1, C_2\} = C_1. \qquad (2.5)\end{aligned}$$

Then (2.4) and (2.5) yield $q_M(p_3) = z_2$. In view of

$$p_3 = (kx_2 + (1-k)a_2, ky_2 + (1-k)b_2)$$

we have that

$$\begin{aligned}C_1 &= \|p_3 - z_2\| = \|k(p_2 - z_2)\| = \max\{\|k(x_2 - a_2)\|, \|k(y_2 - b_2)\|\} \\ &= |k| \max\{\|x_2 - a_2\|, \|y_2 - b_2\|\} = |k|\,\|y_2 - b_2\| = \|ky_2 - kb_2\| \\ &= \|y_3 - b_2\|.\end{aligned}$$

Set $p_4 = (x_1, y_3)$ and let $w = (w_1, w_2) \in M$. Since

$$\begin{aligned}\|x_1 - a_1\| &= \|p_1 - q_M(p_1)\| = \sup\{\|p_1 - t\| : \ t = (w_1, w_2) \in M\} \\ &= \sup\big\{\max\{\|x_1 - w_1\|, \|y_1 - w_2\|\} : \ t = (w_1, w_2) \in M\big\}\end{aligned}$$

and since $\|y_3 - w_2\| \le \|p_3 - q_{_M}(p_3)\| = \|p_3 - z_2\| = \|y_3 - b_2\|$, we have

$$\begin{aligned}\|p_4 - w\| &= \max\{\|x_1 - w_1\|, \|y_3 - w_2\|\} \\ &\le \max\{\|x_1 - a_1\|, \|y_3 - b_2\|\} = \max\{C_1, C_1\} = C_1.\end{aligned}$$

But

$$\|p_4 - z_1\| = \max\{\|x_1 - a_1\|, \|y_3 - b_1\|\} \ge \|x_1 - a_1\| = C_1$$

and

$$\|p_4 - z_2\| = \max\{\|x_1 - a_2\|, \|y_3 - b_2\|\} \ge \|y_3 - b_2\| = C_1,$$

therefore $q_{_M}(P_4) = z_1 = z_2$, since M is uniquely remotal. □

LEMMA 2.8. *Let M be a subset of X and let*

$$Q_M = \{t \in M : \ \exists x \in X \text{ with } q_{_M}(x) = t\}.$$

Then M is singleton if and only if Q_M is singleton.

Proof. If $Q_M = \{t_0\}$ is singleton then $q_{_M}(t_0) = t_0$. Then

$$\sup\{\|t_0 - t\| : t \in M\} = \|t_0 - q_{_M}(t_0)\| = 0,$$

whence we conclude that M is singleton. Conversely suppose that $M = \{t_0\}$, then obviously $Q_M = \{t_0\}$. □

THEOREM 2.9. *Let $(X, \|.\|), (Y, \|.\|)$ be arbitrary non-Archimedean normed spaces over a nontrivial non-Archimedean field K. Then all uniquely remotal sets in $Z = X \times Y$ are singleton.*

Proof. Suppose that M is a uniquely remotal set in $(Z, \|.\|)$ which is not singleton. Then there exist $p_1 = (x_1, y_1),\ p_2 = (x_2, y_2) \in Z$ and $z_1, z_2 \in M$ such that $q_{_M}(p_1) = z_1 = (a_1, b_1) \ne (a_2, b_2) = z_2 = q_{_M}(p_2)$. Using Lemma 2.7, we can assume that

$$\|p_i - z_i\| = \|x_i - a_i\| \quad (i = 1, 2).$$

Let $p = (x, y)$ be an arbitrary element of Z with $q_{_M} = (a, b)$. If $\|p - q_{_M}(p)\| = \|y - b\|$, then $z_1 = q_{_M}(p) = z_2$, a contradiction. Hence $\|p - q_{_M}(p)\| = \|x - a\|$. Fix $0 \ne y_0 \in Y$ and let $\sup_{z \in M} \|z\| = \lambda$. Since K is nontrivial, there exists an element $n_0 \in K$ such that $|n_0| < 1$. Hence $\frac{\|y_0\|}{|n_0|^m} > \lambda$ for some m. Taking $p = (0, \frac{y_0}{n_0^m})$ and $q_{_M}(p) = (a_0, b_0)$, we have

$$\|p - q_{_M}(p)\| = \|0 - a_0\| \le \|(a_0, b_0)\| \le \sup_{z \in M} \|z\|,$$

whence

$$\begin{aligned}\sup_{z \in K} \|z\| &< \|\frac{y_0}{n_0^m}\| = \max\{0, \|\frac{y_0}{n_0^m}\|\} = \|p\| \\ &\le \max\{\|p - q_{_M}(p)\|, \|q_{_M}(p)\|\} \le \sup_{z \in K} \|z\|,\end{aligned}$$

a contradiction. Hence M is singleton. □

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Ferdowsi University of Mashhad
Department of Mathematics
Centre of Excellence in Analysis
on Algebraic Structures (CEAAS)
P. O. Box 1159
Mashhad 91775, Iran
E-mail: moslehian@ferdowsi.um.ac.ir
E-mail: niknam@math.um.ac.ir,
dassamankin@yahoo.co.uk
E-mail: shadkam.s@wali.um.ac.ir