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INTEGRAL INCLUSIONS THROUGH METRICAL FIXED POINT THEOREMS FOR φ-CONTRACTIONS

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Abstract. The aim of this paper is to present existence results for integral inclusions of Fredholm and Volterra type by using some fixed point theorems for multivalued operator in complete metric space. Our results extend and complement some theorems given by Petruşel in [5], [6] and Biles, Robinson and Spraker in [3].

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1. INTRODUCTION

In this paper we will present some existence results for integral inclusions. We consider the following integral inclusions

$$x(t) \in \int_a^t K(t,s,x(s)\mathrm{d}s + g(t),\ t \in [a,b], \tag{1}$$

and

$$x(t) \in \int_a^b K(t,s,x(s)\mathrm{d}s + g(t),\ t \in [a,b]. \tag{2}$$

Petruşel proved in [5] an existence result for the second integral inclusion using Wegrzyk's fixed point theorem. In [6] he also proved some fixed point theorems for the sum of two multivalued operators and then he gave several applications to integral inclusions.

In [3] Biles et al. enounced an existence result for the integral inclusions (1) and (2) by using metric and topological fixed point results.

The purpose of this paper is to prove some existence results for (1) and (2) by applying some fixed point theorems for multivalued φ-contractions. Our results extend and complement some theorems given by Petruşel [5], [6] and Biles, Robinson and Spraker [3].

Finally, I would like to mention that the paper is self-contained.

2. NOTATIONS AND AUXILIARY RESULTS

In this section we present some notions and symbols used in the sequel paper (see [5], [6]).

We consider the following families of subsets of a metric space (X,d):

$$\begin{aligned} P(X) &:= \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}, \\ P_b(X) &:= \{Y \in P(X) \mid Y \text{ is bounded }\}, \\ P_{cp}(X) &:= \{Y \in P(X) \mid Y \text{ is compact }\}, \\ P_{cl}(X) &:= \{Y \in P(X) \mid Y \text{ is closed }\}, \\ P_{b,cl}(X) &:= P_b(X) \cap P_{cl}(X). \end{aligned}$$

Define the following generalized functionals:

(1) $D\colon P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}$; D is called the gap functional between A and B. In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$.

(2) $\delta\colon P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\}$.

(3) $\rho\colon P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\rho(A,B) = \sup\{D(a,B) \mid a \in A\}$; ρ is called the (generalized) excess functional.

(4) $H\colon P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$; H is the (generalized) Pompeiu-Hausdorff functional.

(5) $\delta\colon P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\delta A := \sup\{d(a,b) \mid a, b \in A\}$.

Definition 1. Let (X,d) be a metric space. If $F\colon X \to P(X)$ is a multivalued operator, then $x \in X$ is called a fixed point for F if and only if $x \in F(x)$. The set $FixF := \{x \in X \mid x \in F(x)\}$ is called the fixed point set of F. Also, a sequence of successive approximations of F starting from $x \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X with $x_0 = x$, $x_{n+1} \in F(x_n)$, for $n \in \mathbb{N}$.

Definition 2. Let X, Y be Hausdorff topological spaces and $F\colon X \to P(Y)$ a multivalued operator. Then F is said to be continuous in $x_0 \in X$ if and only if it is lower semicontinuous and upper semicontinuous in $x_0 \in X$.

3. MAIN RESULTS

We begin this section by presenting some auxiliary results. We need first some definitions.

Definition 3. (Rus [7]) Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping. Then:

(1) φ is called a comparison function if φ is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$, for all $t > 0$.

(2) φ is called a strict comparison function if φ is monotone increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for all $t > 0$.

Here are some examples for strict comparison functions:

- $\varphi(t) = a \cdot t$, $t \in \mathbb{R}_+$ (where $a \in [0,1[$), is a strict comparison function.
- $\varphi(t) = \frac{t}{1+t}$, $t \in \mathbb{R}_+$, is a strict comparison function.
- $\varphi(t) = \ln(1+t)$, $t \in \mathbb{R}_+$, is a comparison function, but not a strict comparison function.

Definition 4. Let (X,d) be a metric space and $F:X\to P(X)$ a multi-valued operator. Then F is a $\varphi-$*contraction* if $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a comparison function, and

$$H(F(x_1),F(x_2))\le\varphi(d(x_1,x_2)),\ \text{for all } x_1,x_2\in X.$$

Definition 5. Let (Ω,Σ), (Φ,Γ) be two measurable spaces and X be a topological space. Then a mapping $F:\Omega\times\Phi\to P(X)$ is said to be jointly measurable if for every closed subset B of X, $F^{-1}(B)\in\Sigma\bigotimes\Gamma$, where $\Sigma\bigotimes\Gamma$ denotes the smallest σ-algebra on $\Omega\times\Phi$, which contains all the sets $A\times B$ with $A\in\Sigma$ and $B\in\Gamma$.

Recall now the following fixed point result.

Theorem 1. (Wegrzyk [9]) *Let (X,d) be a complete metric space and $T:X\to P_{cl}(X)$ be such that $H(T(x),T(y))\le\varphi(d(x,y))$, for each $x,y\in X$. Assume that $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ is a strict comparison function. Then $FixT$ is nonempty and for any $x_0\in X$ there exists a sequence of successive approximations of T starting from x_0 which converges to a fixed point of T.*

We also need:

Lemma 1. (Rybinsky [8]) *Let S be a complete measurable space, X be a Polish space (a complete, separable metric space) and Y be a separable Banach space. Suppose that $F(t,x)\colon S\times X\to P_{cl,cv}(Y)\setminus\emptyset$ is jointly measurable and lower semi-continuous in x for each fixed $t\in S$. Then, there exists $f\colon S\times X\to Y$ such that $f(t,x)\in F(t,x)$ for every $(t,x)\in S\times X$, f is jointly measurable and f is continuous in x for each fixed $t\in S$.*

Lemma 2. (Gorniewicz [4]) *Let X and Y be metric spaces. Let $\phi:X\to P(Y)$ be lower semi-continuous and let $f\colon X\to Y$, $\lambda\colon X\to(0,\infty)$ be continuous. Define $\psi\colon X\to P(Y)$ by $\psi(x)=\overline{B_{\lambda(x)}(f(x))}$. Assume also for all $x\in X$ that $\phi(x)\bigcap B_{\lambda(x)}(f(x))\ne\emptyset$. Then $\phi\bigcap\psi$ is lower semi-continuous.*

Theorem 2. (Aubin, Frankowska [1]) *Let X be a separable Banach space and $F:\Omega\rightsquigarrow X$ be a measurable set-valued map with nonempty closed images. If F is integrably bounded, then $\int\limits_\Omega\overline{\mathrm{co}}F\mathrm{d}\mu=\overline{\int\limits_\Omega F\mathrm{d}\mu}$. Furthermore, when X is reflexive and F has convex images and is integrably bounded, then the integral $\int\limits_\Omega F\mathrm{d}\mu$ is closed.*

Lemma 3. (Deimling [2]) *Let $[a,b]\in\mathbb{R}$, X be a separable Banach space and let $F,G\colon[a,b]\to P_{cl}(X)$ be measurable with closed values such that at least one of them has compact values and $F(t)\bigcap G(t)\ne\emptyset$ on $[a,b]$. Then $F\bigcap G$ is strongly measurable.*

The first main result of the paper is:

Theorem 3. *Let $K\colon[a,b]\times[a,b]\times\mathbb{R}^{\mathrm{n}}\to P_{cl,cv}(\mathbb{R}^{\mathrm{n}})$ and $g\colon[a,b]\to\mathbb{R}^{\mathrm{n}}$ satisfy the following conditions:*

(1) *For all* $t \in [a,b]$ *and* $x \in C[a,b]$ *there exists an integrable function* $M\colon [a,b] \to \mathbb{R}_+$ *such that*

$$K(t,\cdot,x(\cdot)) \subseteq M(\cdot)B_1(0) \text{ a.e. on } [a,b].$$

(2) *For all* $x \in C[a,b]$*, the operator*

$$K(t,s,x(s))\colon [a,b]\times[a,b] \to P_{cl,cv}(\mathbb{R}^{\mathrm{n}})$$

is jointly measurable.

(3) *For all* $(s,u) \in [a,b]\times\mathbb{R}^{\mathrm{n}}$ $K(\cdot,s,u)\colon [a,b] \to P_{cl,cv}(\mathbb{R}^{\mathrm{n}})$ *is lower semicontinuous.*

(4) *For all* $(t,s) \in [a,b]\times[a,b]$ *and for all* $u,v \in \mathbb{R}^{\mathrm{n}}$ *we have*

$$H(K(t,s,u),K(t,s,v)) \le \varphi(t,s,\|u-v\|_{\mathbb{R}^{\mathrm{n}}}),$$

where $\varphi\colon [a,b]\times[a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ *satisfies:*

(a) φ *is monotone increasing;*
(b) φ *is continuous;*
(c) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = \varphi(b,b,t)$ *is a strict comparison function;*
(d) $\varphi(t,s,p\cdot u) \le p\cdot\varphi(t,s,u)$, *for all* $p \ge 1$ *and for all* $(t,s,u) \in [a,b]\times[a,b]\times\mathbb{R}_+$.

(5) g *is continuous.*

Then there exists a continuous solution for the integral inclusion

$$x(t) \in \int_a^t K(t,s,x(s))\mathrm{d}s + g(t), \text{ for all } t \in [a,b].$$

Proof. Define the multivalued operator $T\colon C[a,b] \to P(C[a,b])$ by

$$T(x) = \{v \in C[a,b] :\ v(t) \in \int_a^t K(t,s,x(s))\mathrm{d}s + g(t),\ t \in [a,b]\}.$$

We consider the following steps:

1) For all $x \in C[a,b]$ there exists $k\colon [a,b]\times[a,b] \to \mathbb{R}^{\mathrm{n}}$ such that $k(t,s)$ is an integrable selection of $K(t,s,x(s))$, for all t, and $k(t,s)$ is continuous in t.

By assumptions (1)-(3) we can apply Lemma 1 to the operator

$$K(t,s,x(s)) : [a,b]\times[a,b] \to P_{cl,cv}(\mathbb{R}^{\mathrm{n}})$$

to obtain the desired k.

2) $T(x) \ne \emptyset$, for all $x \in C[a,b]$.

This follows from statement 1 and from (5).

3) $T(x)$ is closed for all $x \in C[a,b]$.

This follows from (1) and (2) via Theorem 2.

4) For all $x_1, x_2 \in C[a,b]$, for all $\gamma > 0$ and for all integrable selection $k_1(t,s)$ for $K(t,s,x_1(s))$ with $k_1(t,s)$ continuous in t, for all $s \in [a,b]$, there exists an integrable selection $k_2(t,s)$ for $K(t,s,x_2(s))$ with $k_2(t,s)$ continuous in t, for all $s \in [a,b]$, such that for all $t \in [a,b]$ and for all $s \in [a,b]$ we have:

$$\|k_1(t,s) - k_2(t,s)\|_{\mathbb{R}^{\mathrm{n}}} \le \varphi(t,s,\|x_1(s) - x_2(s)\|_{\mathbb{R}^{\mathrm{n}}}) + \gamma.$$

To prove statement 4, we define

$$F\colon [a,b]\times[a,b]\to P(\mathbb{R}^{\mathrm{n}}),\quad F(t,s):=\overline{B_C(k_1(t,s))}\bigcap K(t,s,x_2(s)),$$

where

$$C:=\varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^{\mathrm{n}}})+\gamma).$$

Next we will show that $F(t,s)$ is nonempty, closed and convex valued. Fix (t,s). We have that $k_1(t,s)\in K(t,s,x_1(s))$. We also know that

$$H(K(t,s,x_1(s)),K(t,s,x_2(s)))\le\varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^{\mathrm{n}}}).$$

So there exists $k_2^*\in K(t,s,x_2(s))$ such that

$$\|k_1(t,s)-k_2^*\|_{\mathbb{R}^{\mathrm{n}}}\le\varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^{\mathrm{n}}})+\gamma\le\varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^{\mathrm{n}}}).$$

Thus $k_2^*\in\overline{B_C(k_1(t,s))}$. But $k_2^*\in K(t,s,x_2(s))$. So $F(t,s)$ is nonempty. Since $\overline{B_C(k_1(t,s))}$ and $K(t,s,x_2(s))$ are closed and convex valued the intersection is also closed and convex valued.

Next we prove that $F(t,s)$ is jointly measurable in (t,s). From Lema 3 we know that $k_1(t,s)$ is jointly measurable. So it follows that $\overline{B_C(k_1(t,s))}$ is jointly measurable. Using (2), we can apply Lemma 3 to obtain the conclusion.

From Lemma 2 follows that $F(t,s)$ is lower semi-continuous in t for fixed $s\in[a,b]$. Applying now Lemma 1 to $F(t,s)$, statement 4 is proved.

5) We show that T is ψ-contraction, that is,

$$H(T(x_1),T(x_2))\le\psi(\|x_1-x_2\|_B),$$

for all $x_1,x_2\in C[a,b]$, where ψ a strict comparison function.

Let $x_1,x_2\in C[a,b]$ and $v_1\in T(x_1)$. It follows that $v_1\in C[a,b]$ and $v_1(t)\in\int_a^t K(t,s,x_1(s))\mathrm{d}s+g(t)$, $t\in[a,b]$. By statement 1 we have that there exists $k_1(t,s)\in K(t,s,x_1(s))$, for all $(t,s)\in[a,b]\times[a,b]$, such that $v_1(t)=\int_a^t k_1(t,s)\mathrm{d}s+g(t)$. Statement 4 implies that there exists $k_2(t,s)$ such that $k_2(\cdot,s)$ is continuous and $k_2(t,s)$ is a selection for $K(t,s,x_2(s))$.

Define $v_2(t):=\int_a^t k_2(t,s)\mathrm{d}s+g(t)\in T(x_2)$. Next we will estimate $\|v_1(t)-v_2(t)\|_{\mathbb{R}^{\mathrm{n}}}$. We have

$$\begin{aligned}\|v_1(t)-v_2(t)\|_{\mathbb{R}^{\mathrm{n}}}&\le\int_a^t\|k_1(t,s)-k_2(t,s)\|_{\mathbb{R}^{\mathrm{n}}}\mathrm{d}s\\&\le\int_a^t(\varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^{\mathrm{n}}})+\gamma)\mathrm{d}s.\end{aligned}$$

Taking $\gamma = \frac{\delta}{b-a}$ with arbitrary $\delta > 0$, we have

$$\begin{aligned}
&\left(\int_a^t \varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^n}) + \gamma\right)\mathrm{d}s \\
&\quad \le \int_a^t \varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^n}\mathrm{e}^{-\tau(s-a)}\mathrm{e}^{\tau(s-a)})\mathrm{d}s + \delta \\
&\quad \le \int_a^t \varphi(t,s,\|x_1-x_2\|_B\mathrm{e}^{\tau(s-a)})\mathrm{d}s + \delta \\
&\quad = \varphi(t,b,\|x_1-x_2\|_B)\int_a^t \mathrm{e}^{\tau(s-a)}\mathrm{d}s + \delta \le \varphi(t,b,\|x_1-x_2\|_B)\frac{\mathrm{e}^{\tau(t-a)}}{\tau} + \delta \\
&\quad \le [\varphi(t,b,\|x_1(s)-x_2(s)\|_B)\frac{1}{\tau} + \delta]\cdot \mathrm{e}^{\tau(t-a)}.
\end{aligned}$$

Hence we deduce

$$\|v_1(t)-v_2(t)\|_{\mathbb{R}^n}\mathrm{e}^{-\tau(t-a)} \le \varphi(t,b,\|x_1-x_2\|_B)\frac{1}{\tau} + \delta, \text{ for all } t \in [a,b].$$

Knowing that φ is monotone increasing and taking the supremum of $t \in [a,b]$ and $\tau > 1$ we have

$$\|v_1 - v_2\|_B \le \varphi(b,b,\|x_1-x_2\|_B) + \delta, \text{ for all } \delta > 0.$$

For $\delta \searrow 0$ we obtain

$$\|v_1 - v_2\|_B \le \varphi(b,b,\|x_1-x_2\|_B).$$

With $\varphi(b,b,\|x_1-x_2\|_B) := \psi(\|x_1-x_2\|_B)$ we have

$$H(T(x_1),T(x_2)) \le \psi(\|x_1-x_2\|_B).$$

Using statements 2, 3 and 5, and applying Wegrzyk's theorem, the conclusion follows. □

The second main result of the paper is:

THEOREM 4. *Let* $K\colon [a,b]\times[a,b]\times\mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *and* $g\colon [a,b] \to \mathbb{R}^n$ *satisfy the following conditions:*

(1) *For all* $t \in [a,b]$ *and* $x \in C[a,b]$ *there exists an integrable function* $M\colon [a,b] \to \mathbb{R}_+$ *such that*

$$K(t,\cdot,x(\cdot)) \subseteq M(\cdot)B_1(0) \text{ a.e. on } [a,b].$$

(2) *For all* $x \in C[a,b]$ *the operator*

$$K(t,s,x(s))\colon [a,b]\times[a,b] \to P_{cl,cv}(\mathbb{R}^n)$$

is jointly measurable.

(3) *For all* $(s,u) \in [a,b]\times\mathbb{R}^n$ *the operator*

$$K(\cdot,s,u)\colon [a,b] \to P_{cl,cv}(\mathbb{R}^n)$$

is lower semi-continuous.

(4) *For all* $(t,s) \in [a,b] \times [a,b]$ *and for all* $u, v \in \mathbb{R}^{\mathrm{n}}$ *we have*

$$H(K(t,s,u), K(t,s,v)) \leq \varphi(t,s,\|u-v\|_{\mathbb{R}^{\mathrm{n}}}),$$

where $\varphi : [a,b] \times [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies:*
(a) φ *is monotone increasing;*
(b) φ *is continuous;*
(c) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = (b-a)\varphi(b,b,t)$ *is a strict comparison function;*
(d) $\varphi(t,b,p \cdot u) \leq p \cdot \varphi(t,s,u)$, *for all* $p \geq 1$ *and for all* $(t,s,u) \in [a,b] \times [a,b] \times \mathbb{R}_+$.

(5) g *is continuous.*

Then there exists a continuous solution for the integral inclusion

$$x(t) \in \int_a^b K(t,s,x(s))\mathrm{d}s + g(t), \text{ for all } t \in [a,b].$$

Proof. Define the multivalued operator $T\colon C[a,b] \to P(C[a,b])$ by

$$T(x) = \{v \in C[a,b] : \ v(t) \in \int_a^b K(t,s,x(s))\mathrm{d}s + g(t), \ t \in [a,b]\}.$$

The proof follows the same steps as in the theorem above. The steps 1-4 are the same. The difference appears in the fifth step, which we will present next.

We have to show that T is ψ-contraction, that is

$$H(T(x_1), T(x_2)) \leq \psi(\|x_1 - x_2\|_C),$$

for all $x_1, x_2 \in C[a,b]$, where ψ a strict comparison function.

Let $x_1, x_2 \in C[a,b]$ and $v_1 \in T(x_1)$. It follows that $v_1 \in C[a,b]$ and we have that $v_1(t) \in \int_a^b K(t,s,x_1(s))\mathrm{d}s + g(t)$, $t \in [a,b]$. From statement 1 we have that there exists $k_1(t,s) \in K(t,s,x_1(s))$, for all $(t,s) \in [a,b] \times [a,b]$, such that $v_1(t) = \int_a^b k_1(t,s)\mathrm{d}s + g(t)$. Statement 4 implies that there exists $k_2(t,s)$ such that $k_2(\cdot,s)$ is continuous and $k_2(t,s)$ is selection for $K(t,s,x_2(s))$.

Define $v_2(t) := \int_a^b k_2(t,s)\mathrm{d}s + g(t) \in T(x_2)$. Next we will estimate $\|v_1(t) - v_2(t)\|_{\mathbb{R}^{\mathrm{n}}}$. We have

$$\begin{aligned}\|v_1(t) - v_2(t)\|_{\mathbb{R}^{\mathrm{n}}} &\leq \int_a^b \|k_1(t,s) - k_2(t,s)\|_{\mathbb{R}^{\mathrm{n}}}\mathrm{d}s \\ &\leq \int_a^b \varphi(t,s,\|x_1(s) - x_2(s)\|_{\mathbb{R}^{\mathrm{n}}}) + \gamma\mathrm{d}s.\end{aligned}$$

Taking $\gamma = \frac{\delta}{b-a}$ with arbitrary $\delta > 0$ we have

$$\begin{aligned}\int_a^b (\varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^n})+\gamma)\mathrm{d}s &\le \int_a^b \varphi(t,s,\|x_1(s)-x_2(s)\|_{\mathbb{R}^n})\mathrm{d}s+\delta \\ &\le \int_a^b \varphi(t,s,\|x_1-x_2\|_C)\mathrm{d}s+\delta \\ &= \varphi(t,b,\|x_1-x_2\|_C)\int_a^b \mathrm{d}s+\delta \\ &\le \varphi(t,b,\|x_1-x_2\|_C)(b-a)+\delta.\end{aligned}$$

Hence we deduce

$$\|v_1(t)-v_2(t)\|_C \le \varphi(t,b,\|x_1-x_2\|_C)(b-a)+\delta, \text{ for all } t\in[a,b].$$

For $\delta \searrow 0$ we obtain

$$\|v_1-v_2\|_C \le (b-a)\varphi(b,b,\|x_1-x_2\|_C).$$

Since $\psi(\|x_1-x_2\|_C) := (b-a)\varphi(b,b,\|x_1-x_2\|_C)$, we have

$$H(T(x_1),T(x_2)) \le \psi(\|x_1-x_2\|_C).$$

Using statements 2, 3 and 5 and applying Wegrzyk's theorem, the conclusion follows. □

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