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COEFFICIENT ESTIMATES AND THE CONVEX HULL PROBLEM FOR MEROMORPHIC FUNCTIONS

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Abstract. We consider the class $S(p)$ of meromorphic univalent functions in the unit disk $\mathbb{D}$ having a simple pole at $p \in (0,1)$. Let $\Sigma^s(p, w_0)$ consist of functions $f \in S(p)$ for which $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a starlike set with respect to a point $w_0 \neq 0, \infty$. In this paper, we find a sharp estimate for the real part of the constant coefficient in the Laurent expansion of functions in $S(p)$. Also we prove a result on the closed convex hull of $\Sigma^s(p, w_0)$. Lastly, we obtain certain coefficient estimates in the Laurent expansion for functions in $\Sigma^s(p, w_0)$.

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1. INTRODUCTION

Let $\mathbb{D} := \{z : |z| < 1\}$ be the open unit disk. Let $\mathcal{S}$ denote the class of analytic univalent functions f in $\mathbb{D}$ with standard normalization $f(0) = f'(0) - 1 = 0$. The class $S(p)$ of meromorphic and univalent functions in $\mathbb{D}$, having a simple pole at $z = p \in (0,1)$ with the standard normalization at the origin and its subclasses have renewed their interest in function theory. We refer to [1, 2, 3, 4, 10] for the latest development. Another related class of interest lies in $\Sigma^s(p, w_0)$, the class of meromorphically starlike functions f satisfying

(i) $f \in S(p)$,
(ii) $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a starlike set with respect to a point $w_0 \neq 0, \infty$.

Characterization and results about $\Sigma^s(p, w_0)$ can be obtained from [3, 4, 5, 7, 8, 9]. Clearly, each $f \in S(p)$ has the Laurent expansion

$$f(z) = \frac{a_{-1}}{z-p} + \sum_{n=0}^{\infty} a_n(f)(z-p)^n, \quad |z-p| < 1-p. \tag{1}$$

We now recall a familiar result of Zemyan [11] on the set of variability of the residue a_{-1} for functions in $S(p)$.

Theorem A. *Let* $\Omega_p = \{a_{-1} : a_{-1} = \mathrm{Res}_{z=p} f(z),\ f \in S(p)\}$. *Then*

$$\Omega_p = \{-p^2(1-p^2)^{\epsilon} :\ |\epsilon| \leq 1\}. \tag{2}$$

A function f belongs to the class $Co(p)$, called the class of *concave functions*, if and only if

(i) $f \in S(p)$,
(ii) $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a convex set.

Results about Taylor and Laurent coefficients, and the closed convex hull of the family of concave functions can be obtained from [1, 2, 3, 4, 7, 9, 10].

Theorem B. [7, Theorem 4] *If f is a member of $Co(p)$ with expansion* (1) *then*

$$\left| p + \frac{a_0(f)(1-p^2)}{a_{-1}(f)} \right| \leq \frac{1+p^2}{p} \tag{3}$$

and the inequality is sharp.

We will indicate in the proof of Theorem 1 that the estimate in Theorem B holds for $f \in S(p)$ as well.

In [3, Theorem 3.1], the following representation formula for functions in the class $\Sigma^s(p, w_0)$ has been obtained.

Theorem C. *For $0 < p < 1$, let $f \in \Sigma^s(p, w_0)$. Then there exists a function ω holomorphic in $\mathbb{D}$ such that $\omega(\mathbb{D}) \subset \overline{\mathbb{D}}$, $\omega(0) = -\frac{1}{2}\left(\frac{1}{w_0} + p + \frac{1}{p}\right)$ and*

$$f(z) = w_0 + \frac{pw_0(1 + z\omega(z))^2}{(z-p)(1-zp)}, \quad z \in \mathbb{D}. \tag{4}$$

Now we recall the lower bound for the modulus of the residue for functions in $\Sigma^s(p, w_0)$.

Theorem D. [3, Theorem 3.3] *If $f \in \Sigma^s(p, w_0)$ and has the Laurent expansion* (1)*, then we have*

$$|a_{-1}| \geq \frac{p(1-p)}{1+p}|w_0|. \tag{5}$$

The inequality is sharp for the function

$$g(z) = \frac{-zp}{(z-p)(1-pz)} = w_0 + \frac{pw_0}{(z-p)(1-pz)}(1-z)^2 \in \Sigma^s(p, w_0)$$

where $w_0 = \frac{-p}{(1-p)^2}$.

The present article is organized as follows: In Section 2 we use Theorems A and B to obtain a sharp estimate for the real part of $a_0(f)$ for functions in $S(p)$ for certain values of p in $(0, 1)$. In Section 3 we prove that, for all $p \in (0, 1)$ and for certain values of w_0, the closed convex hull of $\Sigma^s(p, w_0)$ is a proper subset of the closed convex hull of the family of functions defined by the representation formula (4) in the topology of uniform convergence on compact subsets of $\mathbb{D} \setminus \{p\}$ (see [12]).

2. AN ESTIMATE FOR THE REAL PART OF $a_0(f)$, $f \in s(p)$

Theorem 1. *Let $f \in S(p)$ have the expansion* (1)*. Then*

$$\mathrm{Re}\,(a_0(f)) \geq \frac{-p}{(1-p^2)^2}, \quad p \in (0, \sqrt{1-\mathrm{e}^{-\pi}}), \quad \sqrt{1-\mathrm{e}^{-\pi}} \approx 0.97.$$

Furthermore, the above inequality is sharp.

Proof. For $f \in S(p)$ let

$$h(z) = \frac{-a_{-1}}{(1-p^2)f\left(\frac{p-z}{1-pz}\right)}.$$

Then h can easily seen to be a member of $S(p)$. Keeping in account the fact that h is analytic in $\mathbb{D} \setminus \{p\}$ with simple pole at $z = p$, it is a simple exercise to see that

$$h(z) = z + \left(p + \frac{(1-p^2)a_0}{a_{-1}}\right) z^2 + \cdots, \quad |z| < p.$$

Now by Jenkin's inequality (see [6]), we have

$$|h''(0)| \leq \frac{2(1+p^2)}{p}.$$

This shows that the estimate (3) of Theorem B continues to hold for functions in $S(p)$. Consequently, for any $f \in S(p)$ there exists a number $\tau \in \overline{\mathbb{D}}$ such that

$$a_0(f) = \frac{a_{-1}(f)}{1-p^2}\Big(-p + \tau \frac{1+p^2}{p}\Big). \tag{6}$$

It suffices to consider the points τ on the boundary of unit disk. Set $\tau = \mathrm{e}^{\mathrm{i}\phi}$ and $\epsilon = r\mathrm{e}^{\mathrm{i}\theta}$, $r \in (0,1]$, in Theorem A. Then, by (2), (6) can be rewritten as

$$a_0(f) = YJ, \tag{7}$$

where

$$Y = \frac{-p^2(1-p^2)^{r\cos\theta}}{(1-p^2)}, \; J = (1-p^2)^{\mathrm{i}r\sin\theta}\Big(-p + \mathrm{e}^{\mathrm{i}\phi}\,\frac{1+p^2}{p}\Big).$$

It follows easily that

$$\frac{-p^2}{(1-p^2)^2} \leq Y \leq -p^2.$$

Now, we need to compute extremum of the real part of J. To this end we have

$$\begin{aligned} \operatorname{Re} J \;&=\; \Big(-p + \frac{1+p^2}{p}\cos\phi\Big)\cos\big(\big(\log(1-p^2)\big)\, r\sin\theta\big) \\ &-\; \frac{1+p^2}{p}\sin\phi\sin\big(\big(\log(1-p^2)\big)\, r\sin\theta\big). \end{aligned} \tag{8}$$

Now, let $x = \Big(\log(1-p^2)\Big) r\sin\theta$, $\theta \in [0, 2\pi]$. Then $x \in [-\alpha, \alpha]$, where

$$\alpha = \log\left(\frac{1}{1-p^2}\right) > 0.$$

From (8) we obtain that

$$\operatorname{Re} J = Q(x,\phi) = \frac{1+p^2}{p}\cos(x+\phi) - p\cos x, \quad \phi \in [0, 2\pi].$$

In view of this simple form, we need to find the extremum for the function $Q(x, \phi)$. To do this, consider the expression

$$R(a, b) = \frac{1+p^2}{p}a - pb,$$

where $a = \cos(x + \phi)$ and $b = \cos x$. As $\phi \in [0, 2\pi]$ and cosine is a periodic function of period 2π, we see that the variables a and b are independent. Clearly, $-1 \leq b \leq 1$. Now, let for a fixed $\alpha > 0$, the minimum value of b be "t". Hence, the corners of the rectangle where (a, b) varies are $A(1, 1), B(1, t), C(-1, t), D(-1, 1)$ (see Fig. 1).

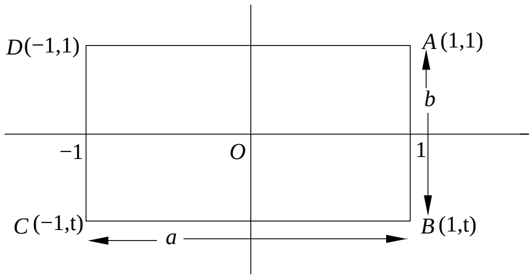

Fig. 1

Here we note that the maximum is attained at the corner B for certain values of p in $(0, 1)$. A little calculation shows that $t = -1$ is possible only for the interval $p \in [\sqrt{1 - e^{-\pi}}, 1]$. For the maximum of $R(a, b)$ in the remaining interval, we have

$$\max R(a, b) = R(1, 1) = \frac{1}{p} \quad \text{for } p \in (0, \sqrt{1 - e^{-\pi}}).$$

Using this, we get from (7)

$$\mathrm{Re}\,(a_0(f)) \geq \frac{-p}{(1-p^2)^2} \quad \text{for } p \in (0, \sqrt{1 - e^{-\pi}}).$$

The above estimate is sharp for the function

$$f(z) = \frac{-zp}{(z-p)(1-pz)}.$$

□

Remark 1. Since the equality $t = -1$ can hold only for $p \in [\sqrt{1 - e^{-\pi}}, 1)$, we have

$$\max R(a, b) = R(1, -1) = \frac{1 + 2p^2}{p}, \quad p \in [\sqrt{1 - e^{-\pi}}, 1).$$

We also see that the minimum is attained at the corner D. Hence,

$$\min R(a, b) = R(-1, 1) = -\frac{1 + 2p^2}{p}, \quad p \in (0, 1).$$

Hence we get

$$\mathrm{Re}\,(a_0(f)) \geq \frac{-p(1+2p^2)}{(1-p^2)^2}, \quad p \in [\sqrt{1-\mathrm{e}^{-\pi}}, 1).$$

Now using the estimate for minimum of $R(a,b)$ and (7) we get

$$\mathrm{Re}\,(a_0(f)) \leq p(1+2p^2), \quad p \in (0,1).$$

Remark 2. We note that the bounds obtained above for the real part of $a_0(f)$ are the same as the bounds for the real part of $a_0(f)$ in any direction, i.e., the bounds which are true for $\mathrm{Re}\,(a_0(f))$ are also valid for $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\beta}a_0(f))$ for some fixed parameter $\beta \in [0, 2\pi)$.

Corollary 1. *Let $f \in S(p)$ have the expansion* (1). *Then*

$$\frac{-p(1+2p^2)}{(1-p^2)^2} \leq \mathrm{Im}\,(a_0(f)) \leq p(1+2p^2), \quad p \in [\sqrt{1-\mathrm{e}^{-\pi/2}}, 1).$$

The strict inequality holds in the above estimate for $p \in (0, \sqrt{1-\mathrm{e}^{-\pi/2}})$.

Proof. We have from (7) that $a_0(f) = YJ$. Now a little computation of the imaginary part of J reveals that

$$\mathrm{Im}\,J = -p\sin x + \frac{1+p^2}{p}\sin(x+\phi),$$

where x and ϕ are as in Theorem 1. We observe that $-1 \leq \sin x \leq 1$ whenever $p \in [\sqrt{1-\mathrm{e}^{-\pi/2}}, 1)$, and that on the complement part $p \in (0, \sqrt{1-\mathrm{e}^{-\pi/2}})$, we have $-1 < \sin x < 1$. Now the proof follows easily. □

Corollary 2. *If $g \in \mathcal{S}$ has the expansion*

$$g(z) = \sum_{n=0}^{\infty} b_n (z-p)^n, \quad |z-p| < 1-p,$$

then

$$\mathrm{Re}\left(b_2\left(\frac{b_0}{b_1}\right)^2 - b_0\right) \geq \frac{-p}{(1-p^2)^2}, \quad p \in (0, \sqrt{1-\mathrm{e}^{-\pi}}).$$

The above estimate is sharp for the Koebe function $\frac{z}{(1-z)^2}$.

Proof. If $g \in \mathcal{S}$, then

$$f(z) = \frac{g(p)g(z)}{g(p)-g(z)}$$

is in $S(p)$. Now using the Laurent expansion (1) of f at p, we easily get that

$$a_0(f) = \frac{g(p)^2 g''(p)}{2g'(p)^2} - g(p).$$

Noting that $g(p) = b_0, g'(p) = b_1, g''(p) = 2b_2$ and using the estimate for the real part of $a_0(f)$ from Theorem 1, we get the desired estimate for functions in $\mathcal{S}$. It is not difficult to see that the estimate is sharp for the Koebe function. □

3. CLOSED CONVEX HULL AND THE LAURENT COEFFICIENTS OF $\Sigma^s(p, w_0)$

The closed convex hull of the family of functions defined by the representation formula (4) in Theorem C consists of all functions each of which are limits (in the topology of uniform convergence) of functions of the form

$$\left\{ w_0 + \frac{pw_0}{(z-p)(1-zp)} \sum_{i=1}^{n} t_i (1 + z\omega_i(z))^2 : \omega_i : \mathbb{D} \to \overline{\mathbb{D}},\ \omega_i \text{ is holomorphic} \right.$$
$$\left. \text{and } \omega_i(0) = -\frac{1}{2}\left(p + \frac{1}{p} + \frac{1}{w_0}\right),\ n = 1, 2, \cdots \right\}.$$

In the next theorem we prove a containment relation between the closed convex hull of $\Sigma^s(p, w_0)$ and the closed convex hull of family of functions defined by (4). We will use Theorem D to get this result .

THEOREM 2. *Let* $p \in (0,1)$ *and* $w_0 \in \left[\frac{-p}{(1-p)^2}, \frac{-p}{(1+p)^2}\right]$. *Then the closed convex hull of* $\Sigma^s(p, w_0)$ *is a proper subset of the closed convex hull of the family of functions defined by* (4).

Proof. First we observe that the coefficients $a_{-1}(f)$ of the functions f in the closed convex hull of $\Sigma^s(p, w_0)$ satisfy the inequality (5) of Theorem D. Next let us consider the following Taylor expansion for ω at $z = p$

$$\omega(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \quad |z-p| < 1-p. \tag{9}$$

Now a computation of $a_{-1}(f)$, using the representation formula (4) and the expansions (1) and (9), yields

$$a_{-1}(f) = \frac{pw_0}{1-p^2}[1 + p^2 c_0^2 + 2pc_0]. \tag{10}$$

We insert into (4) the functions

$$\omega_x(z) = \frac{-\left(\frac{z-p}{1-pz}\right) - x}{1 + x\left(\frac{z-p}{1-pz}\right)}, \quad z \in \mathbb{D}, \tag{11}$$

$x \in (0,1)$ fixed. The Taylor expansion of ω_x, at the point p, gives $c_0 = -x$. Using this value of c_0, we get from (10)

$$a_{-1}(f) = \frac{p(1-p)}{(1+p)}\left(\frac{1-px}{1-p}\right)^2 w_0.$$

It is easy to see that

$$\left(\frac{1-px}{1-p}\right)^2 > 1$$

for all $p \in (0,1)$ and $x \in (0,1)$. Hence for $w_0 \in \left[\frac{-p}{(1-p)^2}, \frac{-p}{(1+p)^2}\right]$, we have

$$a_{-1}(f) < -\frac{p(1-p)}{(1+p)}|w_0|.$$

So the functions f in (4) got by inserting $\omega_x(z)$ do not belong to the closed convex hull of $\Sigma^s(p, w_0)$. This finishes the proof. □

In view of the refined estimate [3, (3.6)], we can formulate the corrected version of the corollary after [7, Theorem 9]:

Corollary 3. *Let* $p \in (0,1)$ *and* $f \in \Sigma^s(p, w_0)$ *have the expansion* (1). *Then*

(i) $|a_0 - w_0| \leq \frac{p(2+p)}{(1-p)^2}|w_0|$,

(ii) $|a_1| \leq \frac{p|w_0|}{(1-p)^3(1+p)}$.

The estimate (ii) *is sharp for*

$$f(z) = \frac{-zp}{(z-p)(1-pz)}$$

with $w_0 = \frac{-p}{(1+p)^2}$.

Remark 3. We observe that the extremal function f of [7, Theorem 8] belongs to $\Sigma^s(p, w_0)$ if and only if $w_0 = \frac{-p}{(1-p)^2}$. This is a direct consequence of the fact that

$$f'(z) = -(1-p)^2 \left(\frac{1+z}{1-z}\right) \frac{f(z) - w_0}{(z-p)(1-zp)},$$

and $f'(0) = 1$. Hence, the inequality (i) for the above corollary cannot be sharp since the inequality [3, (3.6)] is sharp for $(-zp)/\left((1-zp)(z-p)\right)$ with $w_0 = \frac{-p}{(1+p)^2}$. But this is not the case with estimate (ii) of the above corollary, as the estimate [7, (5.5)] is not involved with the point w_0.

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