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ON QUASI-HADAMARD PRODUCTS OF SOME FAMILIES OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

M.K. AOUF

Abstract. The objective of the present paper is to show quasi-Hadamard products of some families of starlike functions with negative coefficients in the open unit disc. Our results generalize corresponding results of Aouf and Srivastava.

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Key words. Analytic, quasi-Hadamard product, negative coefficients.

1. INTRODUCTION

Let $A(j)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \ (j \in N = \{1, 2, ..\}), \tag{1}$$

which are analytic in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$. For a function $f(z)$ in $A(j)$, we define

$$D^0 f(z) = f(z) \ , \tag{2}$$

$$D^1 f(z) = Df(z) = zf^{'}(z) \ , \tag{3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \ (n \in N) \ . \tag{4}$$

The differential operator D^n was introduced by Salagean [6]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to $A(j)$ is in the class $C(j, \lambda, \alpha, n)$ if and only if

$$\text{Re}\left\{ \frac{(1-\lambda)z(D^n f(z))^{'} + \lambda z(D^{n+1} f(z))^{'}}{(1-\lambda)D^n f(z) + \lambda D^{n+1} f(z)} \right\} > \alpha \ (n \in N_o = N \cup \{0\}) \tag{5}$$

for some $\alpha(0 \le \alpha < 1)$ and $\lambda(0 \le \lambda \le 1)$, and for all $z \in U$.

Let $T(j)$ denote the subclass of $A(j)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \ (a_k \ge 0; j \in N) \ . \tag{6}$$

Further, we define the class $P(j, \lambda, \alpha, n)$ by

$$P(j, \lambda, \alpha, n) = C(j, \lambda, \alpha, n) \cap T(j) \ . \tag{7}$$

The class $P(j, \lambda, \alpha, n)$ was studied by Aouf and Srivastava [2].

We note that, by specializing the parameters j, λ, α and n, we obtain the following subclasses studied by various authors:

(i) $P(j, \lambda, \alpha, 0) = P(j, \lambda, \alpha)$ (Altintas [1]);

(ii) $P(1, 0, \alpha, 0) = T^*(\alpha)$ and $P(1, 1, \alpha, 0) = P(1, 0, \alpha, 1) = C(\alpha)$ (Silverman [7]);

(iii) $P(j, 0, \alpha, 0) = T_\alpha(j)$ and $P(j, 1, \alpha, 0) = P(j, 0, \alpha, 1) = C_\alpha(j)$ (Chatterjea [3] and Srivastava et al. [8]);

(iv) $P(j, 0, \alpha, n) = P(j, \alpha, n)$ and $P(j, 1, \alpha, n) = P(j, \alpha, n + 1)$ (Aouf and Srivastava [2]).

Let $f_\ell(z)(\ell = 1, 2)$ be defined by

$$f_\ell(z) = z - \sum_{k=j+1}^{\infty} a_{k,\ell} z^k \ (a_{k,\ell} \geq 0). \tag{8}$$

Then the quasi-Hadamard product $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is given by

$$(f_1 * f_2)(z) = z - \sum_{k=j+1}^{\infty} a_{k,1} a_{k,2} z^k \ . \tag{9}$$

For the quasi-Hadamard product, Aouf and Srivastava [2] have shown that:

Theorem A. *If $f_\ell(z) \in P(j, \lambda, \alpha, n)(\ell = 1, 2)$, then $(f_1 * f_2)(z) \in P(j, \lambda, \beta, n)$, where*

$$\beta = 1 - \frac{j(1-\alpha)^2}{(j+1)^n(\lambda j + 1)(j + 1 - \alpha)^2 - (1-\alpha)^2} \ . \tag{10}$$

The result is sharp.

Corollary A. *If $f_\ell(z) \in P(j, \lambda, \alpha, n)(\ell = 1, 2, 3)$, then $(f_1 * f_2 * f_3)(z) \in P(j, \lambda, \beta, n)$, where*

$$\beta = 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(\lambda j + 1)^2(j + 1 - \alpha)^3 - (1-\alpha)^3} \ . \tag{11}$$

The result is sharp.

In the present paper, we generalize Theorem A and Corollary A using the technique of Owa [5].

2. QUASI-HADAMARD PRODUCTS

To prove our main result of quasi-Hadamard products, we need the following lemma given by Aouf and Srivastava [2].

Lemma. *A function $f(z)$ given by (6) is in the class $P(j, \lambda, \alpha, n)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n (k - \alpha)(\lambda k - \lambda + 1) a_k \leq 1 - \alpha \ . \tag{12}$$

Applying the above lemma, we derive:

THEOREM 1. *If $f_\ell(z)(\ell = 1,2,...,m)$ belong to the class $P(j,\lambda,\alpha_\ell,n)$ for each $\ell = 1,2,...,m$, then $(f_1 * f_2 * ... * f_m)(z) \in P(j,\lambda,\beta,n)$, where*

$$\beta = 1 - \frac{j\prod_{\ell=1}^{m}(1-\alpha_\ell)}{(j+1)^{n(m-1)}(\lambda j+1)^{m-1}\prod_{\ell=1}^{m}(j+1-\alpha_\ell) - \prod_{\ell=1}^{m}(1-\alpha_\ell)} \ . \tag{13}$$

The result is sharp for the functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{(j+1)^n(\lambda j+1)(j+1-\alpha_\ell)}\, z^{j+1}(\ell = 1,2,...,m). \tag{14}$$

Proof. For $m = 1$, we have that $\beta = \alpha_1$. For $m = 2$, Lemma gives that

$$\sum_{k=j+1}^{\infty}\frac{k^n(k-\alpha_\ell)(\lambda k-\lambda+1)}{1-\alpha_\ell}\, a_{k,\ell} \le 1(\ell = 1,2). \tag{15}$$

Note that, from (15),

$$\sum_{k=j+1}^{\infty}\left\{k^n(\lambda k-\lambda+1)\sqrt{\prod_{\ell=1}^{2}\left(\frac{k-\alpha_\ell}{1-\alpha_\ell}\right)a_{k,\ell}}\right\} \le 1 \ (\ell = 1,2) \ . \tag{16}$$

To prove the case when $m = 2$, we have to find the largest β such that

$$\sum_{k=j+1}^{\infty}\frac{k^n(k-\beta)(\lambda k-\lambda+1)}{1-\beta}a_{k,1}a_{k,2} \le 1 \tag{17}$$

or, such that

$$\left(\frac{k-\beta}{1-\beta}\right)\sqrt{a_{k,1},a_{k,2}} \ \le \sqrt{\prod_{\ell=1}^{2}\left(\frac{k-\alpha_\ell}{1-\alpha_\ell}\right)} \ \ (k \ge j+1) \ . \tag{18}$$

Further, by using (16), we need to find the largest β such that

$$\frac{k-\beta}{1-\beta} \le k^n(\lambda k-\lambda+1)\prod_{\ell=1}^{2}\left(\frac{k-\alpha_\ell}{1-\alpha_\ell}\right) \ (k \ge j+1) \ , \tag{19}$$

which is equivalent to

$$\begin{aligned}\beta &\leq \frac{k^n(\lambda k-\lambda+1)\prod_{\ell=1}^{2}(k-\alpha_\ell)-k\prod_{\ell=1}^{2}(1-\alpha_\ell)}{k^n(\lambda k-\lambda+1)\prod_{\ell=1}^{2}(k-\alpha_\ell)-\prod_{\ell=1}^{2}(1-\alpha_\ell)}\\ &= 1-\frac{(k-1)\prod_{\ell=1}^{2}(1-\alpha_\ell)}{k^n(\lambda k-\lambda+1)\prod_{\ell=1}^{2}(k-\alpha_\ell)-\prod_{\ell=1}^{2}(1-\alpha_\ell)}\ (k\geq j+1)\,.\end{aligned}$$

Defining the function $\varphi(k)$ by

$$\varphi(k)=1-\frac{(k-1)\prod_{\ell=1}^{2}(1-\alpha_\ell)}{k^n(\lambda k-\lambda+1)\prod_{\ell=1}^{2}(k-\alpha_\ell)-\prod_{\ell=1}^{2}(1-\alpha_\ell)}\ (k\geq j+1)\ , \tag{20}$$

we see that $\varphi^{'}(k)\geq 0$ for $k\geq j+1$. This implies that

$$\beta\leq\varphi(j+1)=1-\frac{j\prod_{\ell=1}^{2}(1-\alpha_\ell)}{(j+1)^n(\lambda j+1)\prod_{\ell=1}^{2}(j+1-\alpha_\ell)-\prod_{\ell=1}^{2}(1-\alpha_\ell)}\ . \tag{21}$$

Therefore, the result is true for $m=2$. Next, suppose that the result is true for any positive integer m. Then we have

$$(f_1*f_2*...*f_m*f_{m+1})(z)\in P(j,\lambda,\gamma,n)\ ,$$

where

$$\gamma=1-\frac{j(1-\beta)(1-\alpha_{m+1})}{(j+1)^n(\lambda j+1)(j+1-\beta)(j+1-\alpha_{m+1})-(1-\beta)(1-\alpha_{m+1})}\ , \tag{22}$$

where β is given by (13). It follows from (22) that

$$\gamma=1-\frac{j\prod_{\ell=1}^{m+1}(1-\alpha_\ell)}{(j+1)^{nm}(\lambda j+1)^m\prod_{\ell=1}^{m+1}(j+1-\alpha_\ell)-\prod_{\ell=1}^{m+1}(1-\alpha_\ell)}\ . \tag{23}$$

Thus, the result is true for $m+1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer m.

Finally, taking the functions $f_\ell(z)$ given by (14), we see that

$$\begin{aligned}(f_1 * f_2 * ... * f_m)(z) &= z - \left\{\prod_{\ell=1}^{m}\left(\frac{1-\alpha_\ell}{(j+1)^n(\lambda j+1)(j+1-\alpha_j)}\right)\right\} z^{j+1} \\ &= z - A_{j+1}\, z^{j+1}\,,\end{aligned}$$

where

$$A_{j+1} = \prod_{\ell=1}^{m}\left(\frac{1-\alpha_\ell}{(j+1)^n(\lambda j+1)(j+1-\alpha_j)}\right) .$$

Thus, we know that

$$\begin{aligned}&\sum_{k=j+1}^{\infty}\frac{k^n(k-\beta)(\lambda k-\lambda+1)}{1-\beta}\, A_k \\ &= \frac{(j+1)^n(\lambda j+1)(j+1-\beta)}{1-\beta}\, \left\{\prod_{\ell=1}^{m}(\frac{1-\alpha_\ell}{(j+1)^n(\lambda j+1)(j+1-\alpha_j)})\right\} = 1\,.\end{aligned}$$

Consequently, the result is sharp for functions $f_\ell(z)$ given by (14). □

Letting $\alpha_\ell = \alpha$ $(\ell = 1, 2, ..., m)$, we have:

Corollary 1. *If $f_\ell(z) \in P(j, \lambda, \alpha, n)$ $(\ell = 1, 2, ..., m)$, then $(f_1 * f_2 * ... * f_m)(z) \in P(j, \lambda, \beta, n)$, where*

$$\beta = 1 - \frac{j(1-\alpha)^m}{(j+1)^{n(m-1)}(\lambda j+1)^{m-1}(j+1-\alpha)^m - (1-\alpha)^m} . \tag{24}$$

The result is sharp for functions

$$f_\ell(z) = z - \frac{1-\alpha}{(j+1)^n(\lambda j+1)(j+1-\alpha)}\, z^{j+1}\ (\ell = 1, 2, ..., m) . \tag{25}$$

Putting $j = 1$, we have:

Corollary 2. *If $f_\ell(z) \in P(1, \lambda, \alpha_\ell, n)$ $(\ell = 1, 2, ..., n)$, then $(f_1 * f_2 * ... * f_m)(z) \in P(1, \lambda, \beta, n)$, where*

$$\beta = 1 - \frac{\prod_{\ell=1}^{m}(1-\alpha_\ell)}{2^{n(m-1)}(\lambda+1)^{m-1}\prod_{\ell=1}^{m}(2-\alpha_\ell) - \prod_{\ell=1}^{m}(1-\alpha_\ell)} . \tag{26}$$

The result is sharp for functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{2^n(\lambda+1)(2-\alpha_\ell)}\, z^2\ (\ell = 1, 2, ..., m) . \tag{27}$$

Putting $\lambda = \frac{1}{j}$, we have:

Corollary 3. *If* $f_\ell(z) \in P(j, \frac{1}{j}, \alpha_\ell, n)$ *for all* $\ell = 1, 2, ..., m$, *then* $(f_1 * f_2 * ... * f_m)(z) \in P(j, \frac{1}{j}, \beta, n)$, *where*

$$\beta = 1 - \frac{j \prod_{\ell=1}^{m} (1-\alpha_\ell)}{2^{m-1}(j+1)^{n(m-1)} \prod_{\ell=1}^{m} (j+1-\alpha_\ell) - \prod_{\ell=1}^{m} (1-\alpha_\ell)} . \tag{28}$$

The result is sharp for functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{2(j+1)^n (j+1-\alpha_\ell)} \, z^{j+1} \; (\ell = 1, 2, ..., m) . \tag{29}$$

Putting $\lambda = 0$, we have:

Corollary 4. *If* $f_\ell(z) \in P(j, 0, \alpha_\ell, n) = P(j, \alpha_\ell, n)(\ell = 1, 2, ..., m)$, *then* $(f_1 * f_2 * ... * f_m)(z) \in P(j, \beta, n)$, *where*

$$\beta = 1 - \frac{j \prod_{\ell=1}^{m} (1-\alpha_\ell)}{(j+1)^{n(m-1)} \prod_{\ell=1}^{m} (j+1-\alpha_\ell) - \prod_{\ell=1}^{m} (1-\alpha_\ell)} . \tag{30}$$

The result is sharp for functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{(j+1)^n (j+1-\alpha_\ell)} \, z^{j+1} \; (\ell = 1, 2, ..., m) . \tag{31}$$

Putting $\lambda = 1$, we have:

Corollary 5. *If* $f_\ell(z) \in P(j, 1, \alpha_\ell, n) = P(j, \alpha_\ell, n+1)(\ell = 1, 2, ..., m)$, *then* $(f_1 * f_2 * ... * f_m)(z) \in P(j, \beta, n+1)$, *where*
(2.24)

$$\beta = 1 - \frac{j \prod_{\ell=1}^{m} (1-\alpha_\ell)}{(j+1)^{(m-1)(n+1)} \prod_{\ell=1}^{m} (j+1-\alpha_\ell) - \prod_{\ell=1}^{m} (1-\alpha_\ell)} \; (\ell = 1, 2, ..., m).$$

The result is sharp for functions

$$f_\ell(z) = z - \frac{1-\alpha_\ell}{(j+1)^{n+1} (j+1-\alpha_\ell)} \, z^{j+1} \; (\ell = 1, 2, ..., m) . \tag{32}$$

Remark 1. (i) Putting $n = 0$ in Theorem 1, Corollary 1, Corollary 2, and Corollary 3, we obtain the results obtained by Kim et al. [4];

(ii) Putting $m = 2$ in Corollary 1, then we have Theorem A obtained by Aouf and Srivastava [2].

(iii) Putting $m = 3$ in Corollary 1, then we have Corollary A obtained by Aouf and Srivastava [2].

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Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail: mkaouf127@yahoo.com