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# VARIATIONAL INCLUSIONS FOR A NONCONVEX SECOND-ORDER DIFFERENTIAL INCLUSION

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**Abstract.** We establish several variational inclusions for mild solutions of a nonconvex second-order differential inclusion on a separable Banach space.
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## 1. INTRODUCTION

In Control Theory, mainly, if we want to obtain necessary optimality conditions, it is essential to have several "differentiability" properties of solutions with respect to initial conditions. One of the most powerful result in the theory of differential equations, the classical Bendixson-Picard-Lindelőf theorem states that the maximal flow of a differential equation is differentiable with respect to initial conditions and its derivatives verify the variational equation. This result has been generalized in various ways to differential inclusions by considering several variational inclusions and proving corresponding theorems that extend Bendixson-Picard-Lindelőf theorem.

The present paper is concerned with second-order differential inclusions of the form

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set valued map and $A$ is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\ t \in \mathbb{R}\}$ on a Banach space $X$. The aim of this paper is to extend the results concerning the differentiability of solutions of differential inclusions with respect to initial conditions to the mild solutions of problem (1.1). The results we extend known as the contingent, the intermediate (quasitangent) and the circatangent variational inclusion are obtained in the "classical case" of first-order differential inclusions. For this results and for a complete discussion on this topic we refer to [1].

The proofs of our results follows by similar approach to the classical case of differential inclusions [1,6] and use a recent result [2] concerning the existence of mild solutions of problem (1.1).

The paper is organized as follows: in Section 2 we present preliminary results to be used in the next section and in Section 3 we prove our main results.

## 2. PRELIMINARIES

In this short section we recall some basic notations and concepts concerning differential inclusions.

Let $Y$ be a normed space, $X \subset Y$ and $x \in \overline{X}$ (the closure of X).

From the multitude of the tangent cones in the literature (e.g. [1]) we recall only the contingent, the quasitangent and Clarke's tangent cones, defined, respectively by:

$$K_xX = \{v \in Y; \quad \exists s_m \to 0+, \ \exists v_m \to v: \ x + s_m v_m \in X\},$$

$$Q_xX = \{v \in Y; \quad \forall s_m \to 0+, \ \exists v_m \to v: \ x + s_m v_m \in X\},$$

$$C_xX = \{v \in Y; \forall (x_m, s_m) \to (x, 0+), \ x_m \in X, \ \exists y_m \in X: \ \frac{y_m - x_m}{s_m} \to v\}.$$

This cones are related as follows: $C_xX \subset Q_xX \subset K_xX$.

Corresponding to each type of tangent cone, say $\tau_xX$, one may introduce a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset Y \to \mathcal{P}(Y)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows

$$\tau_y G(x; v) = \{w \in Y \mid (v, w) \in \tau_{(x,y)} Graph(G)\}, \quad v \in \tau_x X.$$

Let denote by I the interval $[0, T]$, $T > 0$ and let $X$ be a real Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot, \cdot)$. Denote by $B$ the closed unit ball in $X$ and by $B(X)$ the Banach space of bounded linear operators from $X$ into $X$.

We recall that a family $\{C(t); \ t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following conditions are satisfied

(i) $C(0) = I$, where $I$ is the identity operator in $X$,

(ii) $C(t + s) + C(t - s) = 2C(t)C(s) \ \forall t, s \in R$,

(iii) the map $t \to C(t)y$ is strongly continuous $\forall y \in X$.

The strongly continuous sine family $\{S(t) \mid t \in \mathbb{R}\}$ associated to a strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$S(t)y := \int_0^t C(s)y \mathrm{d}s, \quad y \in X, \ t \in \mathbb{R}.$$

Let $M \geq 0$ be such that $|C(t)| \leq M \ \forall t \in I$ (e.g., [7]).

Note that $|S(t)| \leq Mt \ \forall t \in I$.

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t); \ t \in \mathbb{R}\}$ is defined by

$$Ay = (\frac{\mathrm{d}^2}{\mathrm{d}t^2})C(t)y|_{t=0}.$$

Fore more details on strongly continuous cosine and sine family of operators we refer to [4,7].

In what follows $A$ is infinitesimal generator of a cosine family $\{C(t); \ t \in \mathbb{R}\}$ on a real separable Banach space $X$ and $F(\cdot, \cdot) : I \times X \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values, which define the following Cauchy problem associated to a second-order differential inclusion

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = x_1. \tag{2.1}$$

A continuous mapping $x(\cdot) \in C(I,X)$ is called a *mild solution* of problem (2.1) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I,X)$ such that:

$$f(t) \in F(t, x(t)) \quad a.e.\,(I) \tag{2.2}$$

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-u)f(u)\mathrm{d}u \quad \forall t \in I, \tag{2.3}$$

i.e., $f(\cdot)$ is a (Bochner) integrable selection of the set-valued map $F(.,x(\cdot))$ and $x(\cdot)$ is the mild solution of the Cauchy problem

$$x'' = Ax + f(t), \quad x(0) = x_0, \quad x'(0) = x_1. \tag{2.4}$$

We shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (2.1) if $f(\cdot)$ verifies (2.2) and $x(\cdot)$ is a mild solution of (2.4).

We shall use the following notations for the solution sets of (2.1).

(2.5)

$$\mathcal{S}(x_0, x_1) = \{(x(\cdot), f(\cdot)); \quad (x(\cdot), f(\cdot)) \text{ is a trajectory-selection pair of } (2.1)\}.$$

In what follows $y_0, y_1 \in X$, $g(\cdot) \in L^1(I,X)$ and $y(\cdot) \in C(I,X)$ is a mild solution of the Cauchy problem

$$y'' = Ay + g(t) \quad y(0) = y_0, \quad y'(0) = y_1. \tag{2.6}$$

HYPOTHESIS 2.1. i) *$A$ is infinitesimal generator of a given strongly continuous bounded cosine family $\{C(t) \mid t \in \mathbb{R}\}$ on the separable Banach space $X$.*

ii) *$F(\cdot,\cdot) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and for every $x \in X$, $F(\cdot, x)$ is measurable.*

iii) *There exist $\beta > 0$ and $L(\cdot) \in L^1(I,(0,\infty))$ such that for almost all $t \in I, F(t,\cdot)$ is $L(t)$-Lipschitz on $y(t) + \beta B$ in the sense that*

$$d_H(F(t,x_1), F(t,x_2)) \le L(t)|x_1 - x_2| \quad \forall \ x_1, x_2 \in y(t) + \beta B,$$

*where $d_H(A,C)$ is the Pompeiu-Hausdorff distance between $A, C \subset X$*

$$d_H(A,C) = \max\{d^*(A,C), d^*(C,A)\}, \quad d^*(A,C) = \sup\{d(a,C); a \in A\}.$$

iv) *The function $t \to \gamma(t) := d(g(t), F(t,y(t)))$ is integrable on $I$.*

Set $m(t) = e^{MT\int_0^t L(u)\mathrm{d}u}$, $t \in I$.

On $C(I,X) \times L^1(I,X)$ we consider the following norm

$$|(x,f)|_{C\times L} = |x|_C + |f|_1 \quad \forall\ (x,f) \in C(I,X) \times L^1(I,X),$$

where, as usual, $|x|_C = \sup_{t\in I} |x(t)|$, $x \in C(I,X)$ and $|f|_1 = \int_0^T |f(t)|\mathrm{d}t$, $f \in L^1(I,X)$.

The next result [2] is an extension of Filippov's theorem concerning the existence of solutions to a Lipschitzian differential inclusion [5] to second-order differential inclusions of the form (2.1).

THEOREM 2.2. [2] *Consider $\delta \geq 0$, assume that Hypothesis* 2.1 *is satisfied and set,* $\eta(t) = m(t)(\delta + MT\int_0^t \gamma(s)\mathrm{d}s)$.

*If $\eta(T) \leq \beta$, then for any $x_0, x_1 \in X$ with $M(|x_0 - y_0| + T|x_1 - y_1|) \leq \delta$ and any $\varepsilon > 0$ there exists $(x(\cdot), f(\cdot)) \in \mathcal{S}(x_0, x_1)$ such that*

$$|x(t) - y(t)| \leq \eta(t) + \varepsilon MTtm(t) \quad \forall t \in I,$$

$$|f(t) - g(t)| \leq L(t)(\eta(t) + \varepsilon MTtm(t)) + \gamma(t) + \varepsilon \quad a.e.\ (I).$$

## 3. THE MAIN RESULTS

Let $(y(\cdot), g(\cdot))$ be a trajectory-selection pair of problem (2.1). We wish to "linearize" (2.1) along $(y(\cdot), g(\cdot))$ by replacing it by several second-order variational inclusions.

Consider, first, the quasitangent variational inclusion

$$\begin{cases} w''(t) \in Aw(t) + Q_{g(t)}(F(t,\cdot))(y(t); w(t)) & a.e.\ (I) \\ w(0) = u, \quad w'(0) = v, \end{cases} \tag{3.1}$$

where $u, v \in X$.

THEOREM 3.1. *Consider the solution map $\mathcal{S}(\cdot,\cdot)$ as a set valued map from $X \times X$ into $C(I, X) \times L^1(I, X)$ and assume that Hypothesis* 2.1 *is satisfied.*

*Then, for any $u, v \in X$ and any trajectory-selection pair $(w, \pi)$ of the linearized inclusion* (3.1) *one has*

$$(w, \pi) \in Q_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v)).$$

*Proof.* Let $u, v \in X$ and let $(w, \pi) \in C(I, X) \times L^1(I, X)$ be a trajectory-selection pair of (3.1). By the definition of the quasitangent derivative and from the Lipschitzianity of $F(t, \cdot)$, for almost all $t \in I$, we have

$$\lim_{h \to 0+} d(\pi(t), \frac{F(t, y(t) + hw(t)) - g(t)}{h}) = 0. \tag{3.2}$$

Moreover, since $g(t) \in F(t, y(t))$ a.e. $(I)$, from Hypothesis 2.1, for all enough small $h > 0$ and for almost all $t \in I$, one has

$$d(g(t) + h\pi(t), F(t, y(t) + hw(t))) \leq h(|\pi(t)| + L(t)|w(t)|)$$

By standard arguments (e.g., Lemmas 1.4 and 1.5 in [6]) the function $t \mapsto d(g(t)+h\pi(t), F(t, y(t)+hw(t)))$ is measurable. Therefore, using the Lebesgue dominated convergence theorem we infer

$$\int_0^T d(g(t) + h\pi(t), F(t, y(t) + hw(t))) = o(h), \tag{3.3}$$

where $\lim_{h \to 0+} \frac{o(h)}{h} = 0$.

We apply Theorem 2.2 with $\varepsilon = h^2$ and by (3.3) we deduce the existence of $M \geq 0$ and of trajectory-selections pairs $(y_h(\cdot), g_h(\cdot))$ of the second-order differential inclusion in (2.1) satisfying

$$|y_h - y - hw|_C + |g_h - g - h\pi|_1 \leq M(o(h) + h^2),$$
$$y_h(0) = y(0) + hu, \quad y_h'(0) = y'(0) + hv,$$

which implies

$$\lim_{h\to 0+} \frac{y_h - y}{h} = w \quad \text{in} \quad C(I,X),$$
$$\lim_{h\to 0+} \frac{g_h - g}{h_n} = \pi \quad \text{in} \quad L^1(I,X).$$

Therefore

$$\lim_{h\to 0+} d_{C\times L}\left((w,\pi), \frac{\mathcal{S}((y(0)+hu, y'(0)+hv)) - (y,g)}{h}\right) = 0$$

and the proof is complete. □

We consider next the variational inclusion defined by the Clarke directional derivative of the set-valued map $F(t,\cdot)$, i.e., the so called circatangent variational inclusion

$$\text{(3.4)} \qquad \begin{cases} w''(t) \in Aw(t) + C_{g(t)}(F(t,\cdot))(y(t); w(t)) & a.e.\ (I) \\ w(0) = u, \quad w'(0) = v, \end{cases}$$

Theorem 3.2. *Consider the solution map $\mathcal{S}(\cdot,\cdot)$ as a set valued map from $X \times X$ into $C(I,X) \times L^1(I,X)$ and assume that Hypothesis 2.1 is satisfied.*

*Then, for any $u, v \in X$ and any trajectory-selection pair $(w,\pi)$ of the linearized inclusion (3.4) one has*

$$(w,\pi) \in C_{(y,g)}\mathcal{S}((y(0), y'(0); (u,v)).$$

*Proof.* Let $u, v \in X$, let $(w,\pi) \in C(I,X)\times L^1(I,X)$ be a trajectory-selection pair of (3.4), let $(y_n, g_n)$ be a sequence of trajectory-selection pairs of (2.1) that converges to $(y,g) \in C(I,X) \times L^1(I,X)$ and let $h_n \to 0+$. Then there exists a subsequence $g_j(\cdot) := g_{n_j}(\cdot)$ such that

$$\text{(3.5)} \qquad \lim_{j\to\infty} g_j(t) = g(t) \quad a.e.\ (I).$$

Denote $\lambda_j := h_{n_j}$. From (3.4) and from the definition of the Clarke directional derivative, for almost all $t \in I$ we have

$$\text{(3.6)} \qquad \lim_{j\to\infty} d\left(\pi(t), \frac{F(t, y_j(t) + \lambda_j w(t)) - g_j(t)}{\lambda_j}\right) = 0.$$

Since $g_j(t) \in F(t, y_j(t))$ a.e. $(I)$, for almost all $t \in I$, we get

$$d(g_j(t) + \lambda_j\pi(t), F(t, y_j(t) + \lambda_j w(t))) \leq \lambda_j(|\pi(t)| + L(t)|w(t)|).$$

The last inequality together with Lebesgue's dominated convergence theorem implies

$$\int_0^T d(g_j(t) + \lambda_j \pi(t), F(t, y_j(t) + \lambda_j w(t))) = o(\lambda_j), \tag{3.7}$$

where $\lim_{j\to\infty} \frac{o(\lambda_j)}{\lambda_j} = 0$.

We apply Theorem 2.2 with $\varepsilon = \lambda_j^2$ and by (3.7) we deduce the existence of $M \geq 0$ and of trajectory-selections pairs $(\overline{y}_j(\cdot), \overline{g}_j(\cdot))$ of the second-order differential inclusion in (2.1) satisfying

$$|\overline{y}_j - y_j - \lambda_j w|_C + |\overline{g}_j - g_j - \lambda_j \pi|_1 \leq M(o(\lambda_j) + \lambda_j^2),$$
$$\overline{y}_j(0) = y(0) + \lambda_j u, \quad \overline{y}_j'(0) = y'(0) + \lambda_j v.$$

It follows that

$$\lim_{j\to\infty} \frac{\overline{y}_j - y}{\lambda_j} = w \quad \text{in} \quad C(I, X),$$
$$\lim_{j\to\infty} \frac{\overline{g}_j - g}{\lambda_j} = \pi \quad \text{in} \quad L^1(I, X),$$

which completes the proof. □

Finally, we consider the contingent variational inclusion

$$\begin{cases} w''(t) \in Aw(t) + \overline{co} K_{g(t)}(F(t, \cdot))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v, \end{cases} \tag{3.8}$$

THEOREM 3.3. *Consider the solution map $\mathcal{S}(\cdot, \cdot)$ as a set valued map from $X \times X$ into $C(I, X) \times L^\infty(I, X)$, with $L^\infty(I, X)$ supplied with the weak-* topology and assume that Hypothesis* 2.1 *is satisfied.*

*Then for any $u, v \in X$ one has*

$$K_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v)) \subset$$
$$\{(w, \pi); (w, \pi)$$

*is a trajectory-selection pair of* (3.8).

*Proof.* Let $u, v \in X$ and let $(w, \pi) \in K_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v))$. According to the definition of the contingent derivative there exist $h_n \to 0+, u_n \to u, v_n \to v$, $w_n(\cdot) \to w(\cdot)$ in $C(I, X)$, $\pi_n(\cdot) \to \pi(\cdot)$ in weak-* topology of $L^\infty(I, X)$ and $c > 0$ such that

$$\begin{aligned} &|\pi_n(t)| \leq c \quad a.e.\ (I), \\ &g(t) + h_n \pi_n(t) \in F(t, y(t) + h_n w_n(t)) \quad a.e.\ (I), \\ &w_n(0) = u_n, w_n'(0) = v_n. \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} &w_n(\cdot) \quad \text{converges pointwise to} \quad w(\cdot) \\ &\pi_n(\cdot) \quad \text{converges weak in } L^1(I, X) \text{ to} \quad \pi(\cdot) \end{aligned} \tag{3.10}$$

We apply Mazur's theorem (e.g. [3]) and we find that there exists

$$v_m(t) = \sum_{p=m}^{\infty} a_m^p \pi_p(t)$$

$v_m(\cdot) \to \pi(\cdot)$ (strong) in $L^1(I, X)$, where $a_m^p \geq 0$, $\sum_{p=m}^{\infty} a_m^p = 1$ and for any $m$, $a_m^p \neq 0$ for a finite number of $p$.

Therefore, a subsequence (again denoted) $v_m(\cdot)$ converges la $\pi(\cdot)$ a.e.. From (3.9) for any $p$ and for almost all $t \in I$

$$w_p'(t) \in \frac{1}{h_p}(F(t, y(t) + h_p w_p(t)) - g(t)) \cap cB$$

Let $t \in I$ be such that $v_m(t) \to \pi(t)$ and $g(t) \in F(t, y(t))$. Fix $n \geq 1$ and $\epsilon > 0$. From (3.9) there exists $m$ such that $h_p \leq 1/n$ and $|w_p(t) - w(t)| \leq 1/n$ for any $p \geq m$.

If we denote

$$\phi(z, h) := \frac{1}{h}(F(t, y(t) + hz) - g(t)) \cap cB$$

then

$$v_m(t) \in co(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h))$$

and if $m \to \infty$, we get

$$\pi(t) \in \overline{co}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h)).$$

Since, $\phi(z, h) \subset cB$, we infer that

$$\pi(t) \in \overline{co} \cap_{\epsilon > 0, n \geq 1} (\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B).$$

On the other hand,

$$\cap_{\epsilon > 0, n \geq 1}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B) \subset K_{g(t)} F(t, \cdot)(y(t); w(t))$$

and the proof is complete. □

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