MATHEMATICA, Tome 50 (73), N° 2, 2008, pp. 149–158

THE REDUCTION OF A k-SYMPLECTIC MANIFOLD

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Abstract. The aim of this paper is to construct an analogue of Marsden-Weinstein reduction for k-symplectic manifolds.

MSC 2000. 53C15.
Key words. Momentum map, symplectic manifold, Marsden-Weinstein reduction.

1. INTRODUCTION

One of the main results in the geometrical mechanics is, of course, the symplectic reduction, which was built on the work of Jacob, Liouville, Arnold and Smale. More exactly, we begin with a symplectic manifold (M, ω). Let G be a Lie group acting by symplectic maps on M and $J : M \to \mathcal{G}^*$ be an equivariant momentum map for this action (where $\mathcal{G}$ is the Lie algebra of G). Let $G_\mu = \{g \in G \mid g\cdot\mu = \mu\}$ be the isotropy subgroup of $\mu \in \mathcal{G}^*$ with respect to the coadjoint action. As a consequence of the equivariance, G_μ leaves $J^{-1}(\mu)$ invariant. Assume, for simplicity, that μ is a regular value of J, so $J^{-1}(\mu)$ is a smooth manifold and assume that G_μ acts freely and properly on $J^{-1}(\mu)$, so $J^{-1}(\mu)/G_\mu =: M_\mu$ is a smooth manifold. Denote by $i_\mu : J^{-1}(\mu) \to M$ the inclusion map and by $\pi_\mu : J^{-1}(\mu) \to M_\mu$ the projection. Then Meyer [1973] and Marsden and Weinstein [1974] proved that M_μ is also a symplectic manifold. Precisely, we have:

THEOREM 1. *(Mayer-Marsden-Weinstein) There exists a unique symplectic structure ω_μ on M_μ such that*

$$\pi_\mu^*\omega_\mu = i_\mu^*\omega.$$

Moreover, if X_H is a Hamiltonian vector field on M, where the Hamiltonian $H : M \to \mathbb{R}$ is a G-invariant smooth function, then the trajectories of X_H project to those of the reduced Hamiltonian vector field X_{H_μ} on M_μ, where the reduced Hamiltonian is defined by $H = H_\mu \circ \pi_\mu$.

Next we will recall the notion of *k-symplectic manifold.*

DEFINITION 1. [2] Let M be an $(n + nk)$-dimensional differential manifold. *A k-symplectic structure on M* is a family $(\omega_A, V)_{1\leq A\leq k}$, where:

(1) ω_A is a closed 2-form, for every $1 \leq A \leq k$;

(2) $\bigcap_{A=1}^{k} \ker \omega_A = \{0\}$;

(3) V is an nk-dimensional distribution, such that

$$\omega_{A|_{V\times V}} = 0,$$

for every $1 \leq A \leq k$.

We call $(M, \omega_A, V)_{1\leq A\leq k}$ a *k-symplectic manifold.*

The purpose of this paper is to obtain an analogue of Marsden-Weinstein reduction for the case of k-symplectic manifolds. In order to do this, we need to define an appropriate momentum map and the corresponding reduced spaces M_μ.

We will prove that under certain conditions on the values μ of the momentum map, the reduced spaces M_μ are also k-symplectic manifolds. The difficult part is to verify the condition 3 in the Definition (1). We can not control the dimension of the reduced distribution V_μ. It has to be an $n_\mu k$-dimensional distribution, where $(n_\mu + n_\mu k)$ is the dimension of M_μ.

To overcame this difficulty, we will introduce, in the next section, an equivalent definition for k-sympletic manifolds which is more appropriate for the performed reduction.

2. K-SYMPLECTIC MANIFOLDS

For the case of the k-symplectic manifolds, there exists an analogue of Darboux theorem that will be used in the followings.

THEOREM 2. *Around any point $x \in M$ there exists a local coordinates system $(q^i, p_i^A)_{1\leq A\leq k,\ 1\leq i\leq n}$, such that:*

(1) $\omega_A = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i^A$, $1 \leq A \leq k$;

(2) $V = < \dfrac{\partial}{\partial p_i^A} >_{1\leq A\leq k,\ 1\leq i\leq n}$.

Now, we will introduce an equivalent definition for k-symplectic manifolds, more suitable for doing reduction.

DEFINITION 2. Let M be an $(n + nk)$-dimensional differential manifold. *A k-symplectic structure on M* is a family $(\omega_A, V)_{1\leq A\leq k}$, where:

(1) ω_A is a closed 2-form, for every $1 \leq A \leq k$;

(2) $\bigcap_{A=1}^{k} \ker \omega_A = \{0\}$;

(3) V is an integrable distribution, maximal with the property that

$$\omega_{A|_{V\times V}} = 0,$$

for every $1 \leq A \leq k$.

Now, we will prove the key result for our paper.

Lemma 1. *The two definitions of k-symplectic manifolds are equivalent.*

Proof. First, we check that the Definition (1) implies the Definition (2). Indeed, using a Darboux chart, we have that $\omega_A = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i^A$ for every $1 \leq A \leq k$. As the distribution V has the property that $\omega_{A|_{V\times V}} = 0$, for every $1 \leq A \leq k$, we can choose either $V =< \frac{\partial}{\partial q^i} >_{1\leq i\leq n}$ or $V =< \frac{\partial}{\partial p_i^A} >_{1\leq A\leq k,\ 1\leq i\leq n}$.

Because $\dim V = nk$, we obtain that $V =< \frac{\partial}{\partial p_i^A} >_{1\leq A\leq k,\ 1\leq i\leq n}$ and consequently, V is the maximal distribution on M with the property that $\omega_{A|_{V\times V}} = 0$, for every $1 \leq A \leq k$.

Conversely, let (V, φ) be a chart around an arbitrary point $x \in M$ such that $\varphi = (q^i, p_i^A)_{1\leq A\leq k,\ 1\leq i\leq n}$. From the Definition (2) we have that ω_A are closed 2-forms, for every $1 \leq A \leq k$ and due to the Poincaré Lemma, they are locally exact. Locally, we have that

$$\omega_A = d\left(\sum_{1\leq B\leq k, 1\leq i\leq n} f_B^{Ai}\mathrm{d}p_i^B + \sum_{i=1}^{n} g_i^A \mathrm{d}q^i\right)$$

or

$$\begin{aligned}\omega_A &= \frac{1}{2}\sum_{1\leq B,C\leq k,1\leq i,j\leq n}\left(\frac{\partial f_B^{Ai}}{\partial p_j^C} - \frac{\partial f_C^{Aj}}{\partial p_i^B}\right)\mathrm{d}p_j^C\wedge \mathrm{d}p_i^B\\ &+ \sum_{1\leq C\leq k,1\leq i,j\leq n}\left(\frac{\partial g_i^A}{\partial p_j^C} - \frac{\partial f_C^{Aj}}{\partial q^i}\right)\mathrm{d}p_j^C\wedge \mathrm{d}q^i\\ &+ \frac{1}{2}\sum_{1\leq C\leq k,1\leq i,j\leq n}\left(\frac{\partial g_i^A}{\partial q^j} - \frac{\partial g_j^A}{\partial q^i}\right)\mathrm{d}q^j\wedge \mathrm{d}q^i.\end{aligned}$$

We have three possibilities for choosing the distribution V:

(i) $V =< \frac{\partial}{\partial q^i} >_{1\leq i\leq r}$ is an r-dimensional distribution, where $r \leq n$;

(ii) $V =< \frac{\partial}{\partial p_i^A} >_{1\leq A\leq r_1,\ 1\leq i\leq r_2}$ is an $(r_1 \cdot r_2)$-dimensional distribution, where $r_1 \leq k$ and $r_2 \leq n$;

(iii) $V =< \frac{\partial}{\partial q^j}, \frac{\partial}{\partial p_i^A} >_{1\leq A\leq r_1,\ 1\leq i\leq r_2,\ 1\leq j\leq r_3}$ is an $(r_3 + r_1 \cdot r_2)$-dimensional distribution where $r_1 \leq k$ and $r_2, r_3 \leq n$.

In the first case (i), the condition 3 of the Definition (2) implies that $\frac{\partial g_i^A}{\partial q^j} - \frac{\partial g_j^A}{\partial q^i} = 0$, for every $1 \leq i,\ j \leq r$. Suppose that $r < n$. Then, from the local expression of ω_A, we obtain that $\omega_A(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^{r+1}}) = 0$, for every

$1 \leq i \leq r$. We also have that $\omega_A(\frac{\partial}{\partial q^{r+1}}, \frac{\partial}{\partial q^{r+1}}) = 0$. Using the maximality of V in the Definition (2) and repeating this argument, we obtain that in the case (i), $V =< \frac{\partial}{\partial q^i} >_{1 \leq i \leq n}$ is an n-dimensional distribution.

In an analogue way we obtain for the case (ii) that V is an nk-dimensional distribution.

The same argument as before shows that in the case (iii), V is an $(n+nk)$-dimensional distribution, fact which implies that $\omega_A \equiv 0$, for every $1 \leq A \leq k$. We rule out this case as a trivial one.

Using again the maximality of V in the Definition (2), we obtain that V is an nk-dimensional integrable distribution. □

Let $V_A := \bigcap_{B=1, B\neq A}^{k} \ker \omega_B$. Then $V_A \cap V_B = \bigcap_{C=1}^{k} \ker \omega_C = \{0\}$, for every $1 \leq A,\ B \leq k$. It is easy to see that

$$V := V_1 \oplus ... \oplus V_k$$

is the distribution that satisfies the conditions of the Definition (2).

Next we will comment on the condition 2 of Definition (1). In the symplectic case, the nondegeneracy of the symplectic form is expressed by $\omega(X,Y) = 0$, for any $Y \in \chi(M)$ implies $X = 0$. In the k-symplectic case, a similar situation would be $\sum_{A=1}^{k} \omega_A(X, Y_A) = 0$, for any $Y_A \in \chi(M)$ implies $X = 0$. This is **not** true, in spite of the condition 2 of the definition of a k-symplectic manifold. To see this, let $M = \mathbb{R}^6$ with the following k-symplectic structure $\omega_1 = \sum_{i=1}^{2} \mathrm{d}q^i \wedge \mathrm{d}p_i^1$ and $\omega_2 = \sum_{i=1}^{2} \mathrm{d}q^i \wedge \mathrm{d}p_i^2$, where $(q^1, q^2, p_1^1, p_2^1, p_1^2, p_2^2)$ are coordinates on $\mathbb{R}^6$. It is immediate to see that $i_X\omega_1 + i_X\omega_2 = 0$, where $X = \frac{\partial}{\partial p_1^1} + \frac{\partial}{\partial p_2^1} - \frac{\partial}{\partial p_1^2} - \frac{\partial}{\partial p_2^2}$.

3. MARSDEN - WEINSTEIN REDUCTION FOR K-SYMPLECTIC MANIFOLDS

Let $T_k^1 M = TM \oplus \cdots \oplus TM$ be the Whitney sum of k copies of TM. Consider the bundle morphism:

$$\Omega^{\#} : T_k^1 M \to T^*M$$

$$\Omega^{\#}(X_1, \ldots, X_k) := \sum_{A=1}^{k} i_{X_A}\omega_A.$$

In order to introduce the notion of momentum map, we need to specify what we understand by a Hamiltonian system in the case of the k-symplectic manifolds.

Definition 3. [10] *A k-Hamiltonian system* is an ordered k-tuples of vector fields $(X)_A := (X_1, \ldots, X_k) \in T^1_k M$ such that there exists a smooth function $H : M \to \mathbb{R}$ called the Hamiltonian of $(X)_A$, with the property that

$$\Omega^{\#}(X_1, \ldots, X_k) = dH.$$

We will denote by $(X)^H_A = (X^H_1, \ldots, X^H_k)$ a k-Hamiltonian system with the Hamiltonian H.

Definition 4. *A k-symplectic action* of a Lie group G on the manifold M is an action $\Phi : G \times M \to M$ such that

$$(\Phi_g)^* \omega_A = \omega_A,$$

for any $g \in G$ and $1 \leq A \leq k$, where $\Phi_g : M \to M$ is defined by $\Phi_g(x) = \Phi(g, x)$.

Now we can introduce the notion of momentum map for the case of the k-symplectic manifolds.

Let $\mathcal{G}^k = \mathcal{G} \times \overset{k}{\ldots} \times \mathcal{G}$ and $\mathcal{G}^{*^k} = \mathcal{G}^* \times \overset{k}{\ldots} \times \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual Lie algebra of the Lie algebra $\mathcal{G}$ of G.

Definition 5. *A momentum map* for a k-symplectic action $\Phi : G \times M \to M$ is a map $J : M \to \mathcal{G}^{*^k}$ defined by

$$J(x)(\xi_1, \ldots, \xi_k) := J^{(\xi_1, \ldots, \xi_k)}(x),$$

where $J^{(\xi_1, \ldots, \xi_k)} : M \to \mathbb{R}$ is the Hamiltonian function for the k-tuple $((\xi_1)_M, \ldots, (\xi_k)_M)$ of infinitesimal vector fields generated by $(\xi_1, \ldots, \xi_k) \in \mathcal{G}^k$.

There exist different definitions for a Hamiltonian system in the case of the k-symplectic manifolds and consequently, different notions of momentum map.

For $g \in G$, define ${Ad_g}^k : \mathcal{G}^k \to \mathcal{G}^k, {Ad_g}^k(\xi_1, \ldots, \xi_k) = (Ad_g \xi_1, \ldots, Ad_g \xi_k)$, where $Ad : G \to Aut(G)$ is the adjoint representation and $Ad_g = Ad(g)$, and $Ad^{*^k}_g : \mathcal{G}^{*^k} \to \mathcal{G}^{*^k}, Ad^{*^k}_g(\mu) = \mu \circ {Ad_g}^k$.

Definition 6. A momentum map $J : M \to \mathcal{G}^{*^k}$ is called Ad^{*^k}*-equivariant* if

$$J(\Phi_g(x)) = Ad^{*^k}_{g^{-1}} J(x), \qquad (\forall)\ g \in G \quad \text{and} \quad (\forall)\ x \in M.$$

As in the symplectic case, if the 2-forms ω_A, $1 \leq A \leq k$ are all exact, i.e. $\omega_A = -d\theta_A$ with θ_A invariant 1-forms with respect to the action Φ, then there is a particular simple formula for the momentum map. Moreover, it will be Ad^{*^k}-equivariant.

Remark that the definition of the momentum map given above does not coincide with the one given by Puta [10].

LEMMA 2. *If M is a k-symplectic manifold with $\omega_A = -d\theta_A$, $1 \leq A \leq k$, then $J : M \to {\mathcal{G}^*}^k$, $J(x)(\xi_1, \ldots, \xi_k) = \sum_{A=1}^{k} \theta_A(x)((\xi_A)_M(x))$ is an ${Ad^*}^k$-equivariant momentum map for a G-action which leaves invariant the 1-forms θ_A, $1 \leq A \leq k$.*

Proof. The invariance of the 1-forms θ_A implies that $L_{(\xi_A)_M}\theta_A = 0$, for any $\xi_A \in \mathcal{G}$. On the other hand, we have

$$\begin{aligned} L_{(\xi_A)_M}\theta_A &= i_{(\xi_A)_M}(d\theta_A) + d(i_{(\xi_A)_M}\theta_A) \\ &= -i_{(\xi_A)_M}\omega_A + d(i_{(\xi_A)_M}\theta_A). \end{aligned}$$

As a consequence, we obtain that $d(i_{(\xi_A)_M}\theta_A) = i_{(\xi_A)_M}\omega_A$.

Next we will determine the Hamiltonian function for a k-Hamiltonian system $((\xi_1)_M, \ldots, (\xi_k)_M)$ of infinitesimal vector fields. Precisely,

$$\begin{aligned} J^{(\xi_1,\ldots,\xi_k)}(x) &= J(x)(\xi_1, \ldots, \xi_k) \\ &= \sum_{A=1}^{k} \theta_A(x)((((\xi_A)_M(x))) \\ &= \sum_{A=1}^{k} \theta_A((\xi_A)_M)(x), \end{aligned}$$

for any $x \in M$, which shows that $J^{(\xi_1,\ldots,\xi_k)} = \sum_{A=1}^{k} \theta_A((\xi_A)_M)$.

Remains to prove that $J^{(\xi_1,\ldots,\xi_k)}$ is the correct Hamiltonian function for the k-Hamiltonian system $((\xi_1)_M, \ldots, (\xi_k)_M)$. Indeed, we have

$$\begin{aligned} dJ^{(\xi_1,\ldots,\xi_k)} &= d(\sum_{A=1}^{k} \theta_A((\xi_A)_M)) \\ &= \sum_{A=1}^{k} d(\theta_A((\xi_A)_M)) \\ &= \sum_{A=1}^{k} d(i_{(\xi_A)_M}\theta_A) \\ &= \sum_{A=1}^{k} i_{(\xi_A)_M}\omega_A \\ &= \Omega^{\#}((\xi_1)_M, \ldots, (\xi_k)_M). \end{aligned}$$

The Ad^{*^k}-equivariance of J is a trivial extension of the Proposition 4.1.26 in [1]. □

Let $J : M \to \mathcal{G}^{*^k}$ be an Ad^{*^k}-equivariant momentum map and $\mu \in \mathcal{G}^{*^k}$ a regular value of J. Then $J^{-1}(\mu)$ is a differential manifold and we will denote by $i_\mu : J^{-1}(\mu) \to M$ the canonical inclusion. Denote by $G_\mu = \{g \in G : Ad^{*^k}_{g^{-1}}(\mu) = \mu\} \subset G$ the isotropy subgroup of μ with respect to the k-coadjoint action. It is easy to see that, like in the symplectic case, G_μ leaves invariant $J^{-1}(\mu)$. If G_μ acts freely and properly on $J^{-1}(\mu)$, then the quotient space $M_\mu := J^{-1}(\mu)/G_\mu$ is also a differential manifold, with the canonical projection $\pi_\mu : J^{-1}(\mu) \to M_\mu$ a surjective submersion.

In order to prove the main result of the paper, we need the further preparations. Recall the following result, which is the Proposition 4.1.28 in [1].

LEMMA 3. *Let $f : M \to N$ be equivariant with respect to the actions Φ and Ψ of G on M and N, respectively. Then for any $\xi \in \mathcal{G}$,*

$$Tf \circ \xi_M = \xi_N \circ f$$

where ξ_M and ξ_N denote the infinitesimal generators on M and N, respectively.

Now we can prove the following lemma.

LEMMA 4. *For every $x \in J^{-1}(\mu)$, we have*

(i) $T_x(G_\mu \cdot x) = T_x(G \cdot x) \cap T_x(J^{-1}(\mu))$;

(ii) $$T_x(J^{-1}(\mu)) = \{X_x \in T_xM \mid \sum_{A=1}^{k} \omega_A(x)((\xi_A)_M(x), X_x) = 0,\ \textit{for any } (\xi_A)_{1\le A\le k} \in \mathcal{G}\}.$$

Proof. Let $\xi \in \mathcal{G}$. Then $\xi_M(x) \in T_x(G \cdot x)$. The assertion (i) is that $\xi_M(x) \in T_x(J^{-1}(\mu))$ if and only if $\xi \in \mathcal{G}_\mu$, the Lie algebra of G_μ.

Since J is Ad^{*^k}-equivariant and using the Lemma (3), we obtain that $T_xJ(\xi_M(x)) = (\xi_{\mathcal{G}^*}(\mu_1), \dots, \xi_{\mathcal{G}^*}(\mu_k))$, where $\mu = (\mu_1, \dots, \mu_k) \in \mathcal{G}^{*^k}$. Then $\xi_M(x) = T_x(J^{-1}(\mu)) = \ker T_xJ \iff (\xi_{\mathcal{G}^*}(\mu_1), \dots, \xi_{\mathcal{G}^*}(\mu_k)) = (0, \overset{k}{\dots}, 0)$, which is equivalent to $\xi \in \mathcal{G}_\mu$.

For (ii), we differentiate the relation that defines the momentum map $J(x)(\xi_1, \dots, \xi_k) = J^{(\xi_1,\dots,\xi_k)}(x)$. We have

$$dJ(x)(\xi_1, \dots, \xi_k) = dJ^{(\xi_1,\dots,\xi_k)}(x) = \sum_{A=1}^{k} i_{(\xi_A)_M}\omega_A(x).$$

□

Now we can prove a reduction result for k-symplectic manifolds.

Theorem 3. *Let $(M, \omega_A, V)_{1\leq A\leq k}$ be a k-symplectic manifold on which we have an action of a Lie group G such that the 2-forms ω_A, $1 \leq A \leq k$ are invariant and there exists an Ad^{*^k}-equivariant momentum map $J : M \to \mathcal{G}^{*^k}$. Assume that $\mu \in \mathcal{G}^{*^k}$ is a regular value of J and that the isotropy group G_μ under the Ad^{*^k}-action on $\mathcal{G}^{*^k}$ acts freely and properly on $J^{-1}(\mu)$. Then $M_\mu = J^{-1}(\mu)/G_\mu$ has a unique k-symplectic structure $(M_\mu, (\omega_\mu)_A, V_\mu)_{1\leq A\leq k}$ with the property that*

$$\pi_\mu^*(\omega_\mu)_A = i_\mu^*\omega_A, \quad (\forall)\ 1 \leq A \leq k,$$

(where $\pi_\mu : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \to M$ is the inclusion), if $\pi_{\mu_*}X \in \ker(\omega_\mu)_A$ implies that $i_{\mu_*}X \in \ker\omega_A$, for every $X \in \chi(J^{-1}(\mu)), 1 \leq A \leq k$.

Proof. As in the symplectic case, define $(\omega_\mu)_A$ by

$$(\omega_\mu)_A[x]([v], [w]) := \omega_A(x)(v, w), \quad (\forall)\ v, w \in T_x(J^{-1}(\mu)),$$

where $[v] = \pi_{\mu_{*x}}(v)$, $[w] = \pi_{\mu_{*x}}(w)$ and $[x] = \pi_\mu(x)$ with $x \in J^{-1}(\mu)$. Because π_μ and π_{μ_*} are surjective, the 2-forms $(\omega_\mu)_A$, $1 \leq A \leq k$ are unique.

The next thing to be proved is that the 2-forms $(\omega_\mu)_A$, $1 \leq A \leq k$ are well defined. Let $\xi_A, \eta_A \in T_x(G_\mu \cdot x)$ and by Lemma (4) (i) we obtain that $\xi_A, \eta_A \in T_x(J^{-1}(\mu))$. Using again Lemma (4) (ii), we have that $\sum_{k=1}^{A} \omega_A(\xi_A, w) = 0$, for every $(\xi_A)_{1\leq A\leq k}$ and $\sum_{k=1}^{A} \omega_A(\eta_A, v) = 0$, for every $(\eta_A)_{1\leq A\leq k}$. In particular, if we take $(\xi_A)_{1\leq A\leq k} = (0, \ldots, \xi_A, \ldots, 0)$ and $(\eta_A)_{1\leq A\leq k} = (0, \ldots, \eta_A, \ldots, 0)$, we obtain that $\omega_A(\xi_A, w) = 0$ and $\omega_A(\eta_A, v) = 0$. The same argument also gives us that $\omega_A(\xi_A, \eta_A) = 0$. The equality

$$\begin{aligned} \omega_A(x)(v + \xi_A, w + \eta_A) &= \omega_A(x)(v, w) + \omega_A(x)(\xi_A, w) \\ &+ \omega_A(x)(v, \eta_A) + \omega_A(x)(\xi_A, \eta_A) \end{aligned}$$

shows us that the 2-forms $(\omega_\mu)_A$, $1 \leq A \leq k$ are well defined.

In order to prove that $(\omega_\mu)_A$ are closed, $1 \leq A \leq k$, remark that from the definition of $(\omega_\mu)_A$ we have $\pi_\mu^*(\omega_\mu)_A = i_\mu^*\omega_A$. Taking the differential, we have $d\pi_\mu^*(\omega_\mu)_A = di_\mu^*\omega_A$ or $\pi_\mu^*(d(\omega_\mu)_A) = i_\mu^* d\omega_A = 0$. Consequently, since π_μ is surjective, $d(\omega_\mu)_A = 0$, for every $1 \leq A \leq k$.

Next we will prove the condition 2 in the Definition (2). Let $\widetilde{X} \in \bigcap_{A=1}^{k} \ker(\omega_\mu)_A$ or equivalent $(\omega_\mu)_A(\widetilde{X}, \widetilde{Y}) = 0$, for every $1 \leq A \leq k$ and every $\widetilde{Y} \in \chi(M_\mu)$. Using the definition of $(\omega_\mu)_A$, we have that

$$0 = (\omega_\mu)_A(\widetilde{X}, \widetilde{Y}) = (\omega_\mu)_A(\pi_{\mu_*}X, \pi_{\mu_*}Y) = \omega_A(i_{\mu_*}X, i_{\mu_*}Y),$$

for every $1 \le A \le k$ and every $\widetilde{Y} \in \chi(M_\mu)$, where $\widetilde{Y} = \pi_{\mu_*} Y$ and $\widetilde{X} = \pi_{\mu_*} X$. This implies that $i_{\mu_*} X \in \bigcap_{A=1}^{k} \ker \omega_A = \{0\}$ and consequently, $X = 0$ and $\widetilde{X} = \pi_{\mu_*} X = 0$.

Using the forms $(\omega_\mu)_A$, $1 \le A \le k$ we will construct the distribution V_μ needed for the definition of a k-symplectic structure on M_μ.

Let $(V_\mu)_A := \bigcap_{B=1, B\neq A}^{k} \ker\, (\omega_\mu)_B$. It is immediate to see that $(V_\mu)_A \cap (V_\mu)_B = \bigcap_{C=1}^{k} \ker\, (\omega_\mu)_C = \{0\}$, for every $1 \le A,\ B \le k$. Define

$$V_\mu := (V_\mu)_1 \oplus ... \oplus (V_\mu)_k.$$

We will prove that V_μ is the distribution that verifies the conditions of the Definition (2). Take $\widetilde{Z} = (\widetilde{Z}_1, \ldots, \widetilde{Z}_k) \in V_\mu$ with $\widetilde{Z} \in (V_\mu)_A$. Then $\widetilde{Z}_A \in \bigcap_{B=1, B\neq A}^{k} \ker\, (\omega_\mu)_B$, or equivalent $\pi_{\mu_*} Z_A \in \bigcap_{B=1, B\neq A}^{k} \ker\, (\omega_\mu)_B$, where $\widetilde{Z}_A = \pi_{\mu_*} Z_A$, implies that $i_{\mu_*} Z_A \in \bigcap_{B=1, B\neq A}^{k} \ker\, \omega_B$.

This shows that for any $\widetilde{Z} \in V_\mu$, there exists $Z \in \chi(J^{-1}(\mu))$ with $i_{\mu_*} Z \in V$.

It is easy to see that V_μ has the same properties in the Definition (2) as V. □

THEOREM 4. *Assume that the hypothesis in Theorem 3 hold. Let* $(X)_A^H = (X_1^H, \ldots, X_k^H)$ *be a k-Hamiltonian system with the Hamiltonian function* $H : M \to \mathbb{R}$. *If* X_A^H, $1 \le A \le k$ *are G-invariant vector fields, H is a G-invariant function and J is an integral for all* X_A^H, $1 \le A \le k$, *then on* M_μ *we obtain a reduced k-Hamiltonian system* $(\widetilde{X})_A^{H_\mu} = (\widetilde{X}_1^{H_\mu}, \ldots, \widetilde{X}_k^{H_\mu})$, *where* $H \circ i_\mu = H_\mu \circ \pi_\mu$ *and* $\widetilde{X}_A^{H_\mu} \circ \pi_\mu = \pi_{\mu_*} X_A^H$.

Proof. The condition that J is an integral for all X_A^H, $1 \le A \le k$ implies that the flow F_t^A of every vector field X_A^H leaves the manifold $J^{-1}(\mu)$ invariant. These flows F_t^A, $1 \le A \le k$ are also G invariant and consequently, they induce the reduced flows H_t^A on M_μ. If we denote by $\widetilde{X}_A^{H_\mu}$ the vector field generated by H_t^A, where we define $H_\mu : M_\mu \to \mathbb{R}$ by $H \circ i_\mu = H_\mu \circ \pi_\mu$, since H is a G-invariant function, we have the obvious relation $\widetilde{X}_A^{H_\mu} \circ \pi_\mu = \pi_{\mu_*} X_A^H$. Since we also have $i_\mu^* \omega_A = \pi_\mu^* (\omega_\mu)_A$, it is immediate to see that $(\widetilde{X})_A^{H_\mu} = (\widetilde{X}_1^{H_\mu}, \ldots, \widetilde{X}_k^{H_\mu})$ is a k-Hamiltonian system on M_μ with the Hamiltonian $H_\mu : M_\mu \to \mathbb{R}$. □

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Received October 11, 2007

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