MATHEMATICA, Tome 50 (73), N° 2, 2008, pp. 119–138

GENERALIZATION OF CERTAIN SUBCLASS OF CONVEX FUNCTIONS AND A CORRESPONDING SUBCLASS OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Making use of Salagean operators, D^n and D^{n+m} ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m \in \mathbb{N} = \{1, 2, ...\}$), we define the class $T_j(n, m, \alpha, \beta)$ ($n \in \mathbb{N}_0$, $j, m \in \mathbb{N}, -1 \le \alpha < 1, \beta \ge 0$). In this paper, we obtain coefficient estimates, distortion theorem , closure theorems and radii of close - to - convexity , starlikeness and convexity for functions belonging to the class $T_j(n, m, \alpha, \beta)$. We consider integral operators associated with functions belonging to the class $T_j(n, m, \alpha, \beta)$. We also obtain several results for the modified Hadamard products of functions belonging to the class $T_j(n, m, \alpha, \beta)$. Finally, distortion theorems for the fractional calculus of functions in the class $T_j(n, m, \alpha, \beta)$ are obtained.

MSC 2000. 30C45
Key words. Analytic, Salagean operator, uniformly convex, closure theorems, modified Hadamard products.

1. INTRODUCTION

Let A_j denote the class of functions of the form:

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} = \{1.2, ...\}) \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in A_j we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [16]. It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in \mathbb{N}_0 = N \cup \{0\}. \tag{1.5}$$

With the help of the differential operator D^n, for $-1 \le \alpha < 1$, $\beta \ge 0$, $j, m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we let $S_j(n, m, \alpha, \beta)$ denote the subclass of A_j consisting of

The work of this author was supported by the Grant 2-CEx 06-11-10 / 25.07.2006

functions f(z) of the form (1.1) and satisfying the analytic condition :

$$\text{(1.6)} \qquad \operatorname{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)} - \alpha\right\} > \beta\left|\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right| \ , z \in U \ .$$

The operator D^{n+m} was studied by Sekine [18], Aouf et al. ([3] and [4]) and Hossen et all. [7] and Aouf [2].

We note that :

(i) $S_j(1,1,\alpha,\beta) = \beta - UCV(\alpha, j)$, is the class of $\beta-$ uniformly convex functions of order α, $-1 \leq \alpha < 1$, $\beta \geq 0$, $j \in \mathbb{N}$, that is the class

$$\text{(1.7)} \qquad \left\{ f(z) \in A_j : \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in U \right\}.$$

The class $\beta-UCV(0,1) = \beta-UCV$ was introduced by Kanas and Wisniowska [10]. The class $1 - UCV(0,1) = UCV$ was introduced by Ma and Minda [11] and Ronning [13].

(ii) $S_j(0,1,\alpha,\beta) = \beta - S_p(\alpha, j)$, is the class of $\beta-$ starlike functions of order α, $-1 \leq \alpha < 1$, $\beta \geq 0$, $j \in \mathbb{N}$, that is the class

$$\text{(1.8)} \qquad \left\{ f(z) \in A_j : \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \ \ z \in U \right\}.$$

The class $\beta - S_p(0,1) = \beta - S_p$ was introduced by Kanas and Wisniowska [9]. The class $1 - S_p(0,1) = \beta - S_p$ was introduced by Ronning [14].

We denote by T_j the subclass of A_j consisting of functions of the form :

$$\text{(1.9)} \qquad f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k (a_k \geq 0 ; \ k \geq j+1; \ j \in \mathbb{N}).$$

Further, we define the class $T_j(n, m, \alpha, \beta)$ by

$$\text{(1.10)} \qquad T_j(n, m, \alpha, \beta) = S_j(n, m, \alpha, \beta) \cap T_j.$$

Also we not that :

(i) $T_1(n,1,\alpha,\beta) = TS(n,\alpha,\beta)$ $(-1 \leq \alpha < 1, \ \beta \geq 0 \ , n \in \mathbb{N}_0)$ (Rosy and Murugusndaramoorth [15]);

(ii) $T_1(0,1,\alpha,0) = T^*(\alpha)$ and $T_1(1,1,\alpha,0) = C(\alpha)$ $(0 \leq \alpha < 1)$ (Silverman [19]);

(iii) $T_j(0,1,\alpha,0) = T_\alpha(j)$ and $T_j(1,1,\alpha,0) = C_\alpha(j)$ $(0 \leq \alpha < 1, j \in \mathbb{N})$ (Chatterjea [5] and Srivastava et al . [20]);

(iv) $T_j(n,m,\alpha,0) = T_j(n,m,\alpha)$ $(0 \leq \alpha < 1; \ j \in \mathbb{N})$ (Sekine [18] and Hossen et al. [7]);

(v) $T_1(n,1,\alpha,0) = T^*(n,\alpha)$ $(0 \leq \alpha < 1 \ , \ n \in \mathbb{N}_0)$ (Hur and Oh [8]);

(vi) $T_j(n,1,0,\beta) = T_j(n,m,\alpha)$ $(0 \leq \alpha < 1; \ j \in \mathbb{N}, \ n \in \mathbb{N}_0, \ \beta \geq 0)$ (Dixit and Pathak [6]).

2. COEFFICIENT ESTIMATES

LEMMA 1. *Let the function $f(z)$ be defined by* (1.1) *with* $j = 1$. *Then* $f(z) \in S(n, m, \alpha, \beta)$ *if*

$$\sum_{k=2}^{\infty} k^n[k^m(1+\beta) - (\alpha+\beta)]\,|a_k| \leq 1 - \alpha\,. \tag{2.1}$$

Proof. It suffices to show that

$$\beta\left|\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right| - \mathrm{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right\} \leq 1 - \alpha.$$

We have

$$\beta\left|\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right| - \mathrm{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right\} \leq (1+\beta)\left|\frac{D^{n+m}f(z)}{D^n f(z)} - 1\right|$$

$$\leq \frac{(1+\beta)\sum\limits_{k=2}^{\infty} k^m(k^m - 1)\,|a_k|}{1 - \sum\limits_{k=2}^{\infty} k^n\,|a_k|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty} k^n[k^m(1+\beta) - (\alpha+\beta)]\,|a_k| \leq 1 - \alpha,$$

and hence the proof is complete.

THEOREM 1. *A necessary and sufficient condition for $f(z)$ of the form* (1.9) *(with $j = 1$) to be in the class $T_1(n, m, \alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty} k^n[k^m(1+\beta) - (\alpha+\beta)]a_k \leq 1 - \alpha. \tag{2.2}$$

Proof. In view of Lemma 1, we need only to prove the necessity.
If $f(z) \in T_1(n, m, \alpha, \beta)$ and z is real, then (1.6) yields

$$\frac{1 - \sum\limits_{k=2}^{\infty} k^{n+m}a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} k^n a_k z^{k-1}} - \alpha \geq \beta\left|\frac{\sum\limits_{k=2}^{\infty} k^n(k^m - 1)a_k z^{k-1}}{1 - \sum\limits_{k=2}^{\infty} k^n a_k z^{k-1}}\right|.$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty} k^n[k^m(1+\beta) - (\alpha+\beta)]a_k \leq 1 - \alpha. \tag{2.3}$$

Theorem 2. *Let the function $f(z)$ be defined by* (1.9). *Then $f(z) \in T_j(n,\alpha,m,\beta)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n[k^m(1+\beta)-(\alpha+\beta)]a_k \le 1-\alpha.$$

Proof. Putting $a_k = 0$ $(k = 2,3,...,j)$ in Theorem 1, we can prove the assertion of Theorem 2 .

Corollary 1. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n,m,\alpha,\beta)$. Then*

$$a_k \le \frac{1-\alpha}{k^n[k^m(1+\beta)-(\alpha+\beta)]} \quad (k \ge j+1). \tag{2.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)}{k^n[k^m(1+\beta)-(\alpha+\beta)]}\, z^k \quad (k \ge j+1). \tag{2.5}$$

3. GROWTH AND DISTORTION THEOREM

Theorem 3. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n,m,\alpha,\beta)$. Then*

$$\left|D^i f(z)\right| \ge |z| - \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]}\,|z|^{j+1} \tag{3.1}$$

and

$$\left|D^i f(z)\right| \le |z| + \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]}\,|z|^{j+1} \tag{3.2}$$

for $z \in U$, where $0 \le i \le n$. The equalities in (3.1) *and* (3.2) *are attained for the function $f(z)$ given by*

$$f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]} z^{j+1}. \tag{3.3}$$

Proof. Note that $f(z) \in T_j(n,m,\alpha,\beta)$ if and only if $D^i f(z) \in T_j(n-i,m,\alpha,\beta)$ and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{3.4}$$

Using Theorem 2, we know that

$$(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)] \sum_{k=j+1}^{\infty} k^i a_k \tag{3.5}$$
$$\le \sum_{k=j+1}^{\infty} k^n[k^m(1+\beta)-(\alpha+\beta)]a_k \le 1-\alpha$$

that is, that

$$\sum_{k=2}^{\infty} k^i a_k \leq \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]}. \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$\begin{aligned} \left|D^i f(z)\right| &\geq |z| - |z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\geq |z| - \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]} |z|^{j+1} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \left|D^i f(z)\right| &\leq |z| + |z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\leq |z| + \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]} |z|^{j+1}. \end{aligned} \tag{3.8}$$

Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]} z^{j+1} (z \in U). \tag{3.9}$$

This completes the proof of Theorem 3.

Corollary 2. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n, m, \alpha, \beta)$. Then*

$$|f(z)| \geq |z| - \frac{1-\alpha}{(j+1)^{n}[(j+1)^m(1+\beta)-(\alpha+\beta)]} |z|^{j+1}. \tag{3.10}$$

and

$$|f(z)| \leq |z| + \frac{1-\alpha}{(j+1)^{n}[(j+1)^m(1+\beta)-(\alpha+\beta)]} |z|^{j+1}. \tag{3.11}$$

for $z \in U$. The equalities in (3.10) *and* (3.11) *are attained for the function $f(z)$ given by* (3.3).

Proof. Taking $i = 0$ in Theorem 3, we can easily show (3.10) and (3.11).

Corollary 3. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n, m, \alpha, \beta)$. Then*

$$\left|f'(z)\right| \geq 1 - \frac{1-\alpha}{(j+1)^{n-1}[(j+1)^m(1+\beta)-(\alpha+\beta)]} |z|^{j} \tag{3.12}$$

and

$$\left|f'(z)\right| \leq 1 + \frac{1-\alpha}{(j+1)^{n-1}[(j+1)^m(1+\beta)-(\alpha+\beta)]} |z|^{j} \tag{3.13}$$

for $z \in U$. *The equalities in* (3.12) *and* (3.13) *are attained for the function* $f(z)$ *given by* (3.3).

Proof. Note that $D^1 f(z) = zf'(z)$. Hence, taking $i = 1$ in Theorem 3, we have Corollary 3.

4. CONVEX LINEAR COMBINATIONS

In this section, we shall prove that the class $T_j(n, m, \alpha, \beta)$ is closed under convex linear combinations.

THEOREM 4. *The class* $T_j(n, m, \alpha, \beta)$ *is a convex set.*

Proof. Let the functions

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{k,\nu}\, z^k \quad (a_{k,\nu} \geq 0;\ \nu = 1, 2) \tag{4.1}$$

be in the class $T_j(n, m, \alpha, \beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1 - \lambda) f_2(z) \quad (0 \leq \lambda \leq 1) \tag{4.2}$$

is also in the class $T_j(n, m, \alpha, \beta)$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} \{\lambda a_{k,1} + (1 - \lambda) a_{k,2}\}\, z^k, \tag{4.3}$$

with the aid of Theorem 2, we have

$$\sum_{k=2}^{\infty} k^n [k^m(1+\beta) - (\alpha+\beta)]\, \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} \leq 1 - \alpha \tag{4.4}$$

which implies that $h(z) \in T_j(n, m, \alpha, \beta)$. Hence $T_j(n, m, \alpha, \beta)$ is a convex set.

THEOREM 5. *Let* $f_j(z) = z$ *and*

$$f_k(z) = z - \frac{1-\alpha}{k^n[k^m(1+\beta) - (\alpha+\beta)]} z^k \quad (k \geq j+1,\ n \in \mathbb{N}_0,\ m \in \mathbb{N}), \tag{4.5}$$

for $-1 \leq \alpha < 1$ *and* $\beta \geq 0$. *Then* $f(z)$ *is in the class* $T_j(n, m, \alpha, \beta)$ *if and only if it can be expressed in the form:*

$$f(z) = \sum_{k=j}^{\infty} \lambda_k f_k(z), \tag{4.6}$$

where $\lambda_k \geq 0$ $(k \geq j)$ *and* $\sum_{k=j}^{\infty} \lambda_k = 1$.

Proof. Assume that

$$f(z)=\sum_{k=j}^{\infty}\lambda_k f_k(z)=z-\sum_{k=j+1}^{\infty}\frac{1-\alpha}{k^n[k^m(1+\beta)-(\alpha+\beta)]}\lambda_k z^k.$$

Then it follows that

$$\sum_{k=j+1}^{\infty}\frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha}\frac{1-\alpha}{k^n[k^m(1+\beta)-(\alpha+\beta)]}\lambda_k \tag{4.7}$$
$$=\sum_{k=j+1}^{\infty}\lambda_k=1-\lambda_j\le 1.$$

So by Theorem 2, $f(z)\in T_j(n,m,\alpha,\beta)$. Conversely, assume that the function $f(z)$ defined by (1.9) belongs to the class $T_j(n,m,\alpha,\beta)$.Then

$$a_k\le\frac{1-\alpha}{k^n[k^m(1+\beta)-(\alpha+\beta)]}\quad(k\ge j+1;\ \ n\in\mathbb{N}_0;\ \ m\in\mathbb{N}). \tag{4.8}$$

Setting

$$\lambda_k=\frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha}a_k\quad(k\ge j+1;\ \ n\in\mathbb{N}_0;\ \ m\in\mathbb{N}) \tag{4.9}$$

and

$$\lambda_j=1-\sum_{k=j+1}^{\infty}\lambda_k, \tag{4.10}$$

we can see that $f(z)$ can be expressed in the form (4.6). This completes the proof of Theorem 5.

Corollary 4. *The extreme points of the class $T_j(n,m,\alpha,\beta)$ are the functions $f_k(z)$ $(k\ge j)$ given by Theorem 5.*

5. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem 6. *Suppose that the function $f(z)$ defined by (1.9) is in the class $T_j(n,m,\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0\le\rho<1)$ in $|z|<r_1$, where*

$$r_1=r_1(n,m,\alpha,\beta,\rho)=\inf_k\left\{\frac{(1-\rho)k^{n-1}[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha}\right\}^{\frac{1}{k-1}}, \tag{5.1}$$
$$(k\ge j+1).$$

The result is sharp, the extremal function $f(z)$ being given by (2.5).

Proof. We must show that

$$\left|f'(z)-1\right|\le 1-\rho\ \text{ for }\ \ |z|<r_1(n,m,\alpha,\beta,\rho),$$

where $r_1(n,m,\alpha,\beta,\rho)$ is given by (5.1). Indeed we find from the definition (1.9) that

$$\left|f'(z)-1\right| \le \sum_{k=j+1}^{\infty} ka_k \left|z\right|^{k-1}.$$

Thus

$$\left|f'(z)-1\right| \le 1-\rho$$

if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\rho}\right)a_k \left|z\right|^{k-1} \le 1. \tag{5.2}$$

But, by Theorem 2, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right)\left|z\right|^{k-1} \le \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha},$$

that is, if

$$|z| \le \left\{\frac{(1-\rho)k^{n-1}[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha}\right\}^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{5.3}$$

Theorem 6 follows easily from (5.3).

THEOREM 7. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n,m,\alpha,\beta)$. Then the function $f(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_2$, where*
(5.4)

$$r_2 = r_2(n,m,\alpha,\beta,\rho) = \inf_k \left\{\frac{(1-\rho)k^n[k^m(1+\beta)-(\alpha+\beta)]}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}} \quad (k \ge j+1).$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho \ \text{ for } \ |z| < r_2(n,m,\alpha,\beta,\rho),$$

where $r_2(n,m,\alpha,\beta,\rho)$ is given by (5.4). Indeed we find, again from the definition (1.9) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum\limits_{k=j+1}^{\infty}(k-1)a_k\left|z\right|^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}a_k\left|z\right|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho$$

if

$$\sum_{k=j+1}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k\,|z|^{k-1}\leq 1. \tag{5.5}$$

But, by Theorem 2, (5.5) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1}\leq \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha},$$

that is, if

$$|z|\leq\left\{\frac{(1-\rho)k^n[k^m(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\right\}^{\frac{1}{k-1}}\quad(k\geq j+1). \tag{5.6}$$

Theorem 7 follows easily from (5.6).

Corollary 5. *Suppose that the function* $f(z)$ *defined by* (1.9) *is in the class* $T_j(n,m,\alpha,\beta)$. *Then* $f(z)$ *is convex of order* ρ $(0\leq\rho<1)$ *in* $|z|<r_3$, *where*

$$r_3=r_3(n,m,\alpha,\beta,\rho)=\inf_k\left\{\frac{(1-\rho)k^{n-1}[k^m(1+\beta)-(\alpha+\beta)]}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}} \tag{5.7}$$

$$(k\geq j+1).$$

The result is sharp, with the extremal function $f(z)$ *given by* (2.5).

6. A FAMILY OF INTEGRAL OPERATORS

Theorem 8. *Suppose that the function* $f(z)$ *defined by* (1.9) *is in the class* $T_j(n,m,\alpha,\beta)$ *and let* c *be a real number such that* $c>-1$. *Then the function* $F(z)$ *defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\mathrm{d}t\quad(c>-1) \tag{6.1}$$

also belongs to the class $T_j(n,m,\alpha,\beta)$.

Proof. From the representation (6.1) of $F(z)$, it follows that

$$F(z)=z-\sum_{k=j+1}^{\infty}b_kz^k,$$

where

$$b_k=\left(\frac{c+1}{c+k}\right)a_k.$$

Therefore, we have

$$\sum_{k=j+1}^{\infty}k^n[k^m(1+\beta)-(\alpha+\beta)]b_k=\sum_{k=j+1}^{\infty}k^n[k^m(1+\beta)-(\alpha+\beta)]\left(\frac{c+1}{c+k}\right)a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n[k^m(1+\beta)-(\alpha+\beta)]a_k \leq 1-\alpha,$$

since $f(z) \in T_j(n,m,\alpha,\beta)$. Hence, by Theorem 2, $F(z) \in T_j(n,m,\alpha,\beta)$.

THEOREM 9. *Let the function* $F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0)$ *be in the class* $T_j(n,m,\alpha,\beta)$, *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *given by* (6.1) *is univalent in* $|z| < R^*$, *where*

$$R^* = \inf_k \left\{ \frac{k^{n-1}[k^m(1+\beta)-(\alpha+\beta)](c+1)}{(1-\alpha)(c+k)} \right\}^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{6.2}$$

The result is sharp.

Proof. Form (6.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k.$$

In order to obtain the required result, it suffices to show that

$$\left|f'(z)-1\right| < 1 \text{ wherever } \quad |z| < R^*,$$

where R^* is given by (6.2). Now

$$\left|f'(z)-1\right| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{(c+1)} a_k \, |z|^{k-1}.$$

Thus $|f'(z)-1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{(c+1)} a_k \, |z|^{k-1} < 1. \tag{6.3}$$

But Theorem 2 confirms that

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} a_k \leq 1. \tag{6.4}$$

Hence (6.3) will be satisfied if

$$\frac{k(c+k)}{(c+1)} |z|^{k-1} < \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha},$$

that is , if

$$|z| < \left\{ \frac{k^{n-1}[k^m(1+\beta)-(\alpha+\beta)](c+1)}{(1-\alpha)(c+k)} \right\}^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{6.5}$$

Therefore, the function $f(z)$ given by (6.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n[k^m(1+\beta)-(\alpha+\beta)](c+1)} z^k \quad (k \geq j+1). \tag{6.6}$$

7. MODIFIED HADAMARD PRODUCTS

Let the functions $f_\nu(z)$ $(\nu = 1,2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 \circledast f_2)(z) = z - \sum_{k=j+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{7.1}$$

THEOREM 10. *Let each of the functions* $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (4.1) *be in the class* $T_j(n,m,\alpha,\beta)$. *Then* $(f_1 \circledast f_2)(z) \in T_j(n,m,\delta(j,n,m,\alpha,\beta),\beta)$, *where*

$$\delta(j,n,m,\alpha,\beta) = 1 - \frac{[(j+1)^m - 1](1+\beta)(1-\alpha)^2}{(j+1)^n[(j+1)^m(1+\beta)-(\alpha+\beta)]^2 - (1-\alpha)^2}. \tag{7.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [17], we need to find the largest $\delta = \delta(j,n,m,\alpha,\beta)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\delta+\beta)]}{1-\delta} a_{k.1} a_{k,2} \leq 1. \tag{7.3}$$

Since

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} a_{k.1} \leq 1, \tag{7.4}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} a_{k.2} \leq 1, \tag{7.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} \sqrt{a_{k.1} a_{k.2}} \leq 1. \tag{7.6}$$

Thus it is sufficient to show that

$$\frac{k^n[k^m(1+\beta)-(\delta+\beta)]}{1-\delta} a_{k.1} a_{k.2} \leq \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \quad (k \geq j+1), \tag{7.7}$$

that is, that

$$\sqrt{a_{k.1} a_{k.2}} \leq \frac{[k^m(1+\beta)-(\alpha+\beta)](1-\delta)}{[k^m(1+\beta)-(\delta+\beta)](1-\alpha)} \quad (k \geq j+1). \tag{7.8}$$

Note that

$$\sqrt{a_{k.1}a_{k.2}} \leq \frac{(1-\alpha)}{k^n[k^m(1+\beta)-(\alpha+\beta)](1-\alpha)} \quad (k \geq j+1). \tag{7.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n[k^m(1+\beta)-(\alpha+\beta)]} \leq \frac{[k^m(1+\beta)-(\alpha+\beta)](1-\delta)}{[k^m(1+\beta)-(\delta+\beta)](1-\alpha)} \quad (k \geq j+1), \tag{7.10}$$

or, equivalently, that

$$\delta \leq 1 - \frac{(k^m-1)(1+\beta)(1-\alpha)^2}{k^n[k^m(1+\beta)-(\alpha+\beta)]^2-(1-\alpha)^2} \quad (k \geq j+1). \tag{7.11}$$

Since

$$\Phi(k) = 1 - \frac{(k^m-1)(1+\beta)(1-\alpha)^2}{k^n[k^m(1+\beta)-(\alpha+\beta)]^2-(1-\alpha)^2} \tag{7.12}$$

is an increasing function of $k (k \geq j+1)$, letting $k = j+1$ in (7.12), we obtain

$$\delta \leq \Phi(j+1) = 1 - \frac{[(j+1)^m-1](1+\beta)(1-\alpha)^2}{(j+1)^n[(j+1)^m(1+\beta)-(\alpha+\beta)]^2-(1-\alpha)^2}, \tag{7.13}$$

which proves the main assertion of Theorem 10.

Finally, by taking the functions $f_\nu(z)$ $(\nu = 1, 2)$ given by

$$f_\nu(z) = z - \frac{1-\alpha}{(j+1)^n[(j+1)^m(1+\beta)-(\alpha+\beta)]} z^{j+1} \quad (\nu = 1, 2), \tag{7.14}$$

we can see that the result is sharp.

THEOREM 11. *Suppose that the function $f_1(z)$ defined by* (4.1) *is in the class $T_j(n, m, \alpha, \beta)$, and the function $f_2(z)$ defined by* (4.1) *is in the class $T_j(n, m, \gamma, \beta)$. Then $(f_1 \circledast f_2)(z) \in T_j(n, m, \xi(j, n, m, \alpha, \gamma, \beta), \beta)$, where*

$$\begin{aligned} \xi(j,n,m,\alpha,\gamma,\beta) &= 1 \\ &-[(j+1)^m-1](1+\beta)(1-\alpha)(1-\gamma) \\ &\times\{(j+1)^n[(j+1)^m(1+\beta) \\ &-(\alpha+\beta)][(j+1)^m(1+\beta)-(\gamma+\beta)]-(1-\alpha)(1-\gamma)\}^{-1}. \end{aligned} \tag{7.15}$$

The result is the best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n[(j+1)^m(1+\beta)-(\alpha+\beta)]} z^{j+1} \tag{7.16}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n[(j+1)^m(1+\beta)-(\gamma+\beta)]} z^{j+1}. \tag{7.17}$$

Proof. Proceeding as in the proof of Theorem 10, we get

$$\xi \leq 1 - \frac{(k^m - 1)(1+\beta)(1-\alpha)(1-\gamma)}{k^n[k^m(1+\beta) - (\alpha+\beta)][k^m(1+\beta) - (\gamma+\beta)] - (1-\alpha)(1-\gamma)}$$

$$(k \geq j+1). \tag{7.18}$$

Since the right-hand side of (7.18) is an increasing function of k , setting $k = j+1$ in (7.18), we obtain (7.15). This completes the proof of Theorem 11.

Corollary 6. *Let the functions $f_\nu(z)$ $(\nu = 1,2,3)$ defined by (4.1) be in the class $T_j(n,m,\alpha,\beta)$. Then $(f_1 \circledast f_2 \circledast f_3)(z)$ belongs to the class $T_j(n,m,\zeta(j,n,m,\alpha,\beta),\beta)$, where*

$$\zeta(j,n,m,\alpha,\beta) = 1 - \frac{[(j+1)^m - 1](1+\beta)(1-\alpha)^3}{(j+1)^{2n}[(j+1)^m(1+\beta) - (\alpha+\beta)]^3 - (1-\alpha)^3}. \tag{7.19}$$

The result is the best possible for the functions $f_\nu(z)$ $(\nu = 1,2,3)$ given by

$$f_\nu(z) = z - \frac{1-\alpha}{(j+1)^n[(j+1)^m(1+\beta) - (\alpha+\beta)]} z^{j+1} \quad (\nu = 1,2,3). \tag{7.20}$$

Proof. From Theorem 10, we have $(f_1 \circledast f_2)(z) \in T_j(n,m,\delta(j,n,m,\alpha,\beta),\beta)$, where δ is given by (7.2). Now, using Theorem 11, we get $(f_1 \circledast f_2 \circledast f_3)(z) \in T_j(n,m,\zeta(j,n,m,\alpha,\beta),\beta)$, where

$$\zeta(j,n,m,\alpha,\beta) = 1 - \frac{[(j+1)^m - 1](1+\beta)(1-\alpha)^3}{(j+1)^{2n}[(j+1)^m(1+\beta) - (\alpha+\beta)]^3 - (1-\alpha)^3}.$$

This completes the proof of Corollary 6.

Theorem 12. *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (4.1) be in the class $T_j(n,m,\alpha,\beta)$. Then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{7.21}$$

belongs to the class $T_j(n,m,\tau(j,n,m,\alpha,\beta),\beta)$, where

$$\tau(j,n,m,\alpha,\beta) = 1 - \frac{2[(j+1)^m - 1](1+\beta)(1-\alpha)^2}{(j+1)^n[(j+1)^m(1+\beta)(\alpha+\beta)]^2 - 2(1-\alpha)^2}. \tag{7.22}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (7.14).

Proof. By virtue of Theorem 2, we obtain

$$\sum_{k=j+1}^{\infty} \left\{ \frac{k^n[k^m(1+\beta) - (\alpha+\beta)]}{1-\alpha} \right\}^2 a_{k,1}^2$$

$$\leq \left\{ \sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} a_{k,1} \right\}^2 \leq 1 \tag{7.23}$$

and

$$\sum_{k=j+1}^{\infty} \left\{ \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} \right\}^2 a_{k,2}^2$$

$$\leq \left\{ \sum_{k=j+1}^{\infty} \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} a_{k,2} \right\}^2 \leq 1. \tag{7.24}$$

It follows from (7.23) and (7.24) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2} \left\{ \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} \right\}^2 \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1. \tag{7.25}$$

Therefore, we need to find the largest $\tau = \tau(j, n, m, \alpha, \beta)$ such that

$$\frac{k^n[k^m(1+\beta)-(\tau+\beta)]}{1-\tau} \leq \frac{1}{2} \left\{ \frac{k^n[k^m(1+\beta)-(\alpha+\beta)]}{1-\alpha} \right\}^2 \quad (k \geq j+1), \tag{7.26}$$

that is,

$$\tau \leq 1 - \frac{2(k^m-1)(1+\beta)(1-\alpha)^2}{k^n[k^m(1+\beta)-(\alpha+\beta)]^2 - 2(1-\alpha)^2} (k \geq j+1). \tag{7.27}$$

Since

$$D(k) = 1 - \frac{2(k^m-1)(1+\beta)(1-\alpha)^2}{k^n[k^m(1+\beta)-(\alpha+\beta)]^2 - 2(1-\alpha)^2} \tag{7.28}$$

is an increasing function of $k(k \geq j+1)$,we readily have

$$\tau \leq D(j+1) = 1 - \frac{2[(j+1)^m-1](1+\beta)(1-\alpha)^2}{(j+1)^n[(j+1)^m(1+\beta)-(\alpha+\beta)]^2 - 2(1-\alpha)^2}, \tag{7.29}$$

and Theorem 12 follows at once.

8. APPLICATIONS OF FRACTIONAL CALCULUS

We begin with the statements of the following definitions of fractional calculus (that is , fractional derivative and fractional integral) which were defined by Owa [12].

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta (\mu > 0), \tag{8.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2. The fractional derivative of order μ is defined , for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \le \mu < 1), \tag{8.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\mu$ is defined by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \quad (0 \le \mu < 1\,;\ n \in \mathbb{N}_0). \tag{8.3}$$

Theorem 13. *Suppose that the function* $f(z)$ *defined by* (1.9) *is in the class* $T_j(n,m,\alpha,\beta)$. *Then*

$$\left|D_z^{-\mu}(D^i f(z))\right| \ge \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)} |z|^j\right\} \tag{8.4}$$

and

$$\left|D_z^{-\mu}(D^i f(z))\right| \le \frac{|z|^{1+\mu}}{\Gamma(2+2\mu)} \left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)} |z|^j\right\} \tag{8.5}$$

$(\mu > 0\ ;\ 0 \le i \le n\ ;\ z \in U)$.

The result is sharp

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\mu) z^{-\mu} D_z^{-\mu}(D^i f(z)) \\ &= z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} k^i a_k z^k = z - \sum_{k=j+1}^{\infty} \Psi(k) k^i a_k z^k, \end{aligned} \tag{8.6}$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} \quad (k \ge j+1). \tag{8.7}$$

Since

$$0 < \Psi(k) \le \Psi(j+1) = \frac{\Gamma(j+2)\Gamma(2+\mu)}{\Gamma(j+2+\mu)}, \tag{8.8}$$

Therefore, by using (3.6) and (8.8) , we see that

$$\begin{aligned} |F(z)| &\ge |z| - \Psi(j+1)\,|z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\ge |z| - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^{j+1} \end{aligned} \tag{8.9}$$

and

$$\begin{aligned} |F(z)| &\le |z| + \Psi(j+1)\,|z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\ge |z| + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^{j+1}, \end{aligned} \tag{8.10}$$

which proves the inequalities (8.4) and (8.5) of Theorem 13. The equalities in (8.4) and (8.5) are attained for the function $f(z)$ given by

$$\begin{aligned} D_z^{-\mu}(D^i f(z)) &= \frac{z^{1+\mu}}{\Gamma(2+\mu)}\{1- \\ &\frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)} z^j \Big] \end{aligned} \tag{8.11}$$

or , equivalently , by

$$D^i f(z) = z - \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]} z^{j+1}. \tag{8.12}$$

Thus we complete the proof of Theorem 13 .

Taking $i = 0$ in Theorem 13, we have

Corollary 7. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n, m, \alpha, \beta)$. Then*

$$\begin{aligned} \left|D_z^{-\mu} f(z)\right| &\ge \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \\ &\times \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^j\right\} \end{aligned} \tag{8.13}$$

and

$$(8.14)\qquad \left|D_z^{-\mu} f(z)\right| \le \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \times \left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^j\right\}$$

$$(\mu > 0\ ;\quad z \in U).$$

The equalities in (8.13) *and* (8.14) *are attained for the function* $f(z)$ *given by* (3.3).

Remark 1. We note that the results obtained by Rosy and Murugusundarmoothy [14, Theorem 4 and Corollary 2] are not correct. The correct results are given by Theorem 13 and Corollary 7 after putting $j = m = 1$.

Theorem 14. *Suppose that the function* $f(z)$ *defined by* (1.9) *is in the class* $T_j(n,m,\alpha,\beta)$. *Then*

$$(8.15)\qquad \left|D_z^{-\mu}(D^i f(z))\right| \ge \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^j\right\}$$

and

$$(8.16)\qquad \left|D_z^{-\mu}(D^i f(z))\right| \le \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^j\right\}$$

$$(0 \le \mu < 1;\ 0 \le i \le n-1\,;\ z \in U).$$

The result is sharp .

Proof. Let

$$(8.17)\quad G(z) = \Gamma(2-\mu) z^\mu D_z^\mu (D^i f(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} k^i a_k z^k$$

$$= z - \sum_{k=j+1}^{\infty} \theta(k) k^{i+1} a_k z^k,$$

where

$$(8.18)\qquad \theta(k) = \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} (k \ge j+1).$$

It is easily seen from (8.18) that

$$(8.19)\qquad 0 < \theta(k) \le \theta(j+1) = \frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+2-\mu)}.$$

Consequently , with the aid of (3.6)and (8.19) , we have

$$(8.20) \qquad |G(z)| \geq |z| - \theta(j+1)\,|z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1} a_k$$
$$\geq |z| - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^{j+1}$$

and

$$(8.21) \qquad |G(z)| \leq |z| + \theta(j+1)\,|z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1} a_k$$
$$\leq |z| + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^{j+1}.$$

Now (8.15) and (8.16) follows from (8.20) and (8.21), respectively .

Since the equalities in (8.15) and (8.16) are attained for the function $f(z)$ given by

$$(8.22) \qquad \left|D_z^{\mu}(D^i f(z))\right| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}$$
$$\times \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^{j}\right\}$$

or for the function $\mathrm{D}^i f(z)$ given by (8.12), the proof of Theorem 14 is thus completed

Taking $i = 0$ in Theorem 14, we have

Corollary 8. *Suppose that the function $f(z)$ defined by* (1.9) *is in the class $T_j(n, m, \alpha, \beta)$. Then*

$$(8.23) \qquad |D_z^{\mu} f(z)| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}$$
$$\times \left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2-\mu)}\,|z|^{j}\right\}$$

and

$$(8.24) \qquad |D_z^{\mu} f(z)| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}$$
$$\left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}[(j+1)^m(1+\beta)-(\alpha+\beta)]\Gamma(j+2+\mu)}\,|z|^{j}\right\}$$
$$(0 \leq \mu < 1 \ ; \quad z \in U).$$

The equalities in (8.23) *and* (8.24) *are attained for the function $f(z)$ given by* (3.3).

Remark 2. We note that the results obtained by Rosy and Murugusundarmoothy [14, Theorem 5 and Corollary 3] are not correct. The correct results are given by Theorem 14 and Corollary 8, respectively, after putting $j = m = 1$.

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Received December 12, 2007

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