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FIXED POINT THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper we obtain strong and weak convergence theorems for the Mann type doubly sequence iteration process with errors using asymptotically nonexpansive mappings in uniformly convex Banach spaces. Our new results improve, generalize and extend some recent results (see e.g. [6], [7] and [19]).

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1. INTRODUCTION

Let X be a Banach space with dual X^*, and let C be a nonempty subset of X. Also, we let $J : X \to 2^{X^*}$ denote the normalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x\ ,\ f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X. \tag{1}$$

A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\| \qquad \forall\, x, y \in C \text{ and each } n \ge 0. \tag{2}$$

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \qquad \forall\, x, y \in C.$$

A self mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty), k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n\|x - y\| \qquad \forall\, x, y \in C \text{ and each } n \ge 1. \tag{3}$$

Goebel and Kirk [11] proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point. One of the most important directions in the study of fixed points is the iteration technique. Iterative techniques for approximating fixed point of nonexpansive self-mappings have been studied by various authors (see e.g. [3], [4], [13], [14], [16], [17] and others) using the Mann iteration process or the Ishikawa iteration process. For nonexpansive mappings, some authors (see e.g.[12] and [18]) have studied the strong and weak convergence theorems in Hilbert spaces or uniformly convex Banach spaces. In 1991, Schu [17] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces. On the other hand, there are some attempts in the sense of doubly sequence settings (see e.g. [1] and [15]). The concept of non-self asymptotically nonexpansive mappings was introduced by Chidume [7] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The non-self asymptotically nonexpansive mapping is defined as follows:

Definition 1. [7] Let C be a nonempty subset of a real Banach space X and let $P : X \to C$ be the nonexpansive retraction of X onto C. A non-self mapping $T : C \to X$ is called asymptotically nonexpansive if there exist sequences $\{k_n\} \subset [1, \infty), k_n \to 1$ as $n \to \infty$ such that

$$(4)\quad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \qquad \forall\, x, y \in C \text{ and each } n \geq 1.$$

T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$(5)\quad \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \qquad \forall\, x, y \in C \text{ and each } n \geq 1.$$

By the following iteration process:

$$(6)\qquad x_1 \in C, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n),$$

Chidume [7] proved some strong and weak convergence theorems for non-self asymptotically nonexpansive mappings. For more information about fixed points by asymptotically nonexpansive mappings we refer to [5, 8, 10] and others. Wang [19], generalized the iteration scheme (6) by giving the following scheme:

$$(7)\qquad \begin{cases} x_1 \in C \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n) \\ y_n = P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), n \geq 1 \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1) with

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \quad 0 \leq \alpha_n,\ \beta_n \leq 1.$$

Recently, Moore [15] generalized the Mann type iteration in the doubly sequence setting. Very recently, we studied the main results of Moore [15] using the Mann type doubly sequence iterates with errors (see [1]). In the present paper we will extend the results of Chidume [7] and Wang [19] in the doubly sequence setting by adding the errors for their iteration schemes. Now suppose that X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X, which is also a nonexpansive retract of X with retraction P. Let $T_1, T_2 : C \to X$ are two non-self asymptotically nonexpansive mappings. For approximating the common fixed points of two non-self asymptotically nonexpansive mappings, we further generalize the iteration scheme

(7) as follows:

$$(8)\begin{cases} x_{1,1} \in C \\ x_{k,n+1} = P((1-\alpha_n)x_{k,n} + \alpha_n T_1(PT_1)^{n-1}y_{k,n} + \alpha_n v_{k,n}) \\ y_{k,n} = P((1-\beta_n)x_{k,n} + \beta_n T_2(PT_2)^{n-1}x_{k,n} + \beta_n u_{k,n}),\ k,\ n \geq 1 \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequence in [0,1). If $T_1 = T_2$, $\beta_n = 0$ and $v_{k,n} = u_{k,n} = 0$ for all $k, n \geq 1$, the iteration scheme (8) will be the generalization of the scheme (6) in double sequence setting.

2. PRELIMINARIES

For clearness , we start by the following concepts and results:

Definition 2. (see e.g [15]) Let $\mathbb{N}$ denote the set of all natural numbers and let X be a real Banach space. A double sequence in X is meant a function $f : \mathbb{N} \times \mathbb{N} \to X$ defined by $f(n,m) = x_{n,m} \in X$.

The double sequence $\{x_{n,m}\}$ is said to converge strongly to x^* if for a given $\epsilon > 0$ there exist integers $N, M > 0$ such that $\forall\, n \geq N,\ m \geq M,$ we have that

$$\|x_{n,m} - x^*\| < \epsilon.$$

If $\forall\, n, r \geq N,\ m, t \geq M,$ we have that

$$\|x_{n,r} - x_{m,t}\| < \epsilon,$$

then the double sequence is said to be Cauchy. Furthermore, if for each fixed n, we have that $x_{n,m} \to x_n^*$ as $m \to \infty$, then $x_n^* \to x^*$ as $n \to \infty$, so $x_{n,m} \to x^*$ as $n, m \to \infty$.
Let X be a Banach space with dimension $X \geq 2$. The modulus of convexity of X is a function δ from (0,2] into (0,1] defined by

$$\delta(\epsilon) = \inf\{1 - \frac{1}{2}\|x-y\|, \quad x, y \in X, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\},$$

the Banach space X is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon > 0$ and $\epsilon \in (0,2]$.
A subset C of X is said to be retract if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a Banach space is retract.
A mapping $P : X \to X$ is said to be retraction if $P^2 = P$.
Now we define the Opial's condition using doubly sequence sense.

Definition 3. A Banach space X is said to satisfy Opial's condition if for any sequence $\{x_{k,n}\}$ in X, $x_{k,n} \rightharpoonup x$ implies that

$$\lim_{k,n\to\infty} \sup \|x_{k,n} - x\| < \lim_{k,n\to\infty} \sup \|x_{k,n} - y\| \ \ \forall\, y \in X \ \text{ with } \ y \neq x,$$

where $x_{k,n} \rightharpoonup x$ denotes that $\{x_{k,n}\}$ converges weakly to x.

Definition 4. A mapping $T : C \to X$ is said to be semi-compact if, for any sequence $\{x_{k,n}\}$ in C such that $\|x_{k,n} - Tx_{k,n}\| \to 0$ $(n \to \infty)$, there exists subsequence $\{x_{k,n_j}\}$ of $\{x_{k,n}\}$ such that $\{x_{k,n_j}\}$ converges strongly to $x^* \in C$.

Definition 5. A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be demi-closed at p if whenever $\{x_{k,n}\}$ is a sequence in $D(T)$ such that $\{x_{k,n}\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_{k,n}\}$ converges strongly to p, then $Tx^* = p$.

Lemma 1. [18] *Let α_n and t_n be two nonnegative sequences satisfying*

$$\alpha_{n+1} \leq \alpha_n + t_n \quad \forall \ n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2. *Let X be a real uniformly convex Banach space and let $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\lim_{k,n\to\infty} \sup \|x_{k,n}\| \leq r$, $\lim_{k,n\to\infty} \sup \|y_{k,n}\| \leq r$ and $\lim_{k,n\to\infty} \|t_n x_{k,n} + (1-t_n)y_{k,n}\| = r$, hold for some $r \geq 0$, then*

$$\lim_{k,n\to\infty} \|x_{k,n} - y_{k,n}\| = 0.$$

Proof. The proof of this lemma is very similar to the correspondence one in [17].

Lemma 3. [7] *Let X be a real uniformly convex Banach space and let C be a non-empty closed subset of X. Suppose that $T : C \to X$ be a non-self asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demi-closed at zero.*

Let $S := \{x \in X : \|x\| = 1\}$ denote the unite sphere of the Banach space X. Then, X is said to have a Gâteaux differentiable norm if the limit

$$\lim_{n\to\infty} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$, and we call X smooth. Also, X is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$ the limit is attained uniformly for $(x, y) \in S \times S$. It is known that if X is smooth, then any duality mapping on X is single-valued, and if X has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-*weak** continuous (see [9]).
Let C be a nonempty closed convex and bounded subset of the Banach space X, and let the diameter of C be defined by

$$d(C) := \sup\{\|x - y\| : x, y \in C\}.$$

For each $x \in C$, let

$$r(x, C) := \sup\{\|x - y\| : x, y \in C\} \text{ and let } r(C) := \inf\{r(x, C) : x \in C\}$$

denote the Chebyshev radius of C relative to itself. The normal structure coefficient $N(X)$ of X is defined by

$N(X) := \inf\left\{\frac{d(C)}{r(C)} : d(C) > 0\right\}$, where C is a closed convex and bounded subset of X.
A space X such that $N(X) > 1$ is said to have a uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see [2]).

LEMMA 4. [6] *In a Banach space X, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y\ ,\ j(x+y)\rangle, \quad x, y \in X, \tag{9}$$

where $j(x+y) \in J(x+y)$.

LEMMA 5. [20] *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n \sum_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\left\{\sum_n\right\}_{n=0}^{\infty}$ are such that
(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) either $\limsup_{n\to\infty} \sum_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \sum_n| < \infty$.
Then $\{a_n\}_{n=0}^{\infty}$ converges to zero.

In 2004, Chidume et al [7] proved the following theorems:

THEOREM 1. *Let X be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, let C be a non-empty closed convex and bounded subset of X, $T : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1,\infty)$. Let $u \in C$ be fixed, $\{t_n\}_n \subset (0,1)$ be such that $\lim_{n\to\infty} t_n = 1$, and $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. Then,*
(i) for each integer $n \geq 0$, there is a unique $x_n \in C$ such that

$$x_n = (1-\frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n \ ;$$

and if, in addition, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then,
(ii) the sequence $\{x_n\}_n$ converges strongly to a fixed point of T.

THEOREM 2. *Let X be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, C a non-empty closed convex and bounded subset of $X, T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$. Let $u \in C$ be fixed, $\{t_n\}_n \subset (0,1)$ be such that $\lim_{n\to\infty} t_n = 1, t_n k_n < 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. Define the sequence $\{z_n\}_n$ iteratively by $z_0 \in C$,*

$$z_{n+1} = (1-\frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n z_n \quad n = 0,1,2,\ldots\ . \tag{10}$$

Then,
(i) for each integer $n \geq 0$, *there is a unique* $x_n \in C$ *such that*

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_n; \tag{11}$$

and if, in addition, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ *then,*
(ii) $\{z_n\}_n$ *converges strongly to a fixed point of* T.

3. NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPS

Suppose that C be a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space X. Let $T_1, T_2 : C \to X$, be two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = \lim_{n\to\infty} l_n = 1$ and

$$F(T_1) \cap F(T_2) = \{x \in C : T_1 x = T_2 x = x\} \neq \emptyset, \text{ respectively.}$$

Suppose $\{x_{k,n}\}$ is generated iterative by

$$\begin{cases} x_{1,1} \in C \\ x_{k,n+1} = P((1-\alpha_n)x_{k,n} + \alpha_n T_1 (PT_1)^{n-1} y_{k,n} + \alpha_n v_{k,n}) \\ y_{k,n} = P((1-\beta_n)x_{k,n} + \beta_n T_2 (PT_2)^{n-1} x_{k,n} + \beta_n u_{k,n}), \quad k, n \geq 1 \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1)$.
In this section :
(1) Strong convergence theorems of $\{x_{k,n}\}$ to some $q \in F(T_1) \cap F(T_2)$ are obtained under conditions that one of T_1 and T_2 is completely continuous or demi-compact and

$$\sum_{n=1}^{\infty} (k_n - 1) < \infty, \quad , \sum_{n=1}^{\infty} (l_n - 1) < \infty.$$

(2) If X is real uniformly convex Banach space satisfying Opial's condition, then the weak convergence of $\{x_{k,n}\}$ to some $q \in F(T_1) \cap F(T_2)$ is obtained.
Now, we will prove the following lemmas.

LEMMA 6. *Let C be a non-empty closed convex subset of a normed space X, and let $T_1, T_2 : C \to X$ are two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that*

$$\sum_{n=1}^{\infty} (k_n - 1) < \infty, \quad \sum_{n=1}^{\infty} (l_n - 1) < \infty, \quad k_n \to 1, \quad l_n \to 1 \text{ as } n \to \infty, \text{respectively.}$$

Suppose that $\{x_{k,n}\}$ is defined by (8), where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1)$. If $F(T_1) \cap F(T_2) \neq \emptyset$, then, $\lim_{k,n\to\infty} \|x_{k,n} - q\|$ exists for each $q \in F(T_1) \cap F(T_2)$.

Proof. Setting $k_n = 1 + \lambda_n,\;\; l_n = 1 + \gamma_n$. Since

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty,\; \sum_{n=1}^{\infty}(l_n - 1) < \infty,$$

then

$$\sum_{n=1}^{\infty}\lambda_n < \infty \;\text{ and }\; \sum_{n=1}^{\infty}\gamma_n < \infty.$$

For any $q \in F(T_1) \cap F(T_2)$, by (8) we have

$$\begin{aligned}\|x_{k,n+1} - q\| &= \|(1-\alpha_n)(x_{k,n} - q) + \alpha_n(T_1(PT_1)^{n-1}y_{k,n} - q) + \alpha_n v_{k,n}\|\\ &\le (1-\alpha_n)\|x_{k,n} - q\| + \alpha_n(1+\lambda_n)\|y_{k,n} - q\| + \alpha_n\|v_{k,n}\|\\ &= (1-\alpha_n)\|x_{k,n} - q\| + \alpha_n[(1+\lambda_n)\|y_{k,n} - q\| + \|v_{k,n}\|],\end{aligned}$$

where

$$\begin{aligned}\|y_{k,n} - q\| &= \|(1-\beta_n)(x_{k,n} - q) + \beta_n(T_2(PT_2)^{n-1}x_{k,n} - q) + \beta_n u_{k,n}\|\\ &\le (1-\beta_n)\|x_{k,n} - q\| + \beta_n(1+\gamma_n)\|x_{k,n} - q\| + \beta_n\|u_{k,n}\|\\ &\le (1+\beta_n\gamma_n)\|x_{k,n} - q\| + \beta_n\|u_{k,n}\|.\end{aligned}$$

Thus

$$\begin{aligned}&\|x_{k,n+1} - q\| \le (1-\alpha_n)\|x_{k,n} - q\| + \alpha_n(1+\lambda_n)\|y_{k,n} - q\| + \alpha_n\|v_{k,n}\|\\ \le\;& (1-\alpha_n)\|x_{k,n} - q\| + \alpha_n(1+\lambda_n)(1+\beta_n\gamma_n)\|x_{k,n} - q\|\\ +\;& \alpha_n\beta_n(1+\lambda_n)\|u_{k,n}\| + \alpha_n\|v_{k,n}\|\\ \le\;& [1 + \alpha_n(\lambda_n + \beta_n\gamma_n + \lambda_n\beta_n\gamma_n)]\|x_{k,n} - q\|\\ +\;& \alpha_n\beta_n(1+\lambda_n)\|u_{k,n}\| + \alpha_n\|v_{k,n}\|\\ <\;& \exp^{\sum_{n=1}^{\infty}(\lambda_n+\gamma_n+\lambda_n\gamma_n)}\|x_{1,1} - q\| + (1+\lambda_n)\|u_{k,n}\| + \|v_{k,n}\|,\end{aligned}$$

where $1 + x < e^x\,\forall\, x > 0$ and $\sum_{n=1}^{\infty}(\lambda_n + \gamma_n + \lambda_n\gamma_n) < \infty$, then $\{x_{k,n}\}$ is bounded. It implies that there exists a constant $M > 0$ such that $\|x_{k,n} - q\| \le M$ for all $n \ge 1$ so,

$$\|x_{k,n+1} - q\| = \|x_{k,n} - q\| + (\lambda_n + \gamma_n + \lambda_n\gamma_n)M + M_1 \tag{12}$$

$$\text{where,}\;\; M_1 = (1+\lambda_n)\|u_{k,n}\| + \|v_{k,n}\|,\; \sum_{n=1}^{\infty}u_{k,n} < \sum_{n=1}^{\infty}v_{k,n} < \infty.$$

It follows from Lemma 1 that $\lim_{k,n\to\infty}\|x_{k,n} - q\|$ exists. The proof is therefore completed.

Lemma 7. *Let C be a non-empty closed convex subset of a uniformly convex Banach space X, and let $T_1, T_2 : C \to X$ are two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty,\; \sum_{n=1}^{\infty}(l_n - 1) < \infty,\; k_n \to 1,\;\; l_n \to 1 \;\textit{as}\; n \to \infty, \;\;\textit{respectively.}$$

Suppose that $\{x_{k,n}\}$ *is defined by (8), where* $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequences in* $[0,1)$, $\lim_{n\to\infty}\beta_n = 1$. *If* $F(T_1)\cap F(T_2)\neq\emptyset$, *then*

$$\lim_{k,n\to\infty}\|x_{k,n}-T_1x_{k,n}\| = \|x_{k,n}-T_2x_{k,n}\| = 0.$$

Proof. Setting $k_n = 1+\lambda_n,\ \ l_n = 1+\gamma_n$, $q\in F(T_1)\cap F(T_2)$, then by using Lemma 6, we see that $\lim_{k,n\to\infty}\|x_{k,n}-q\|$ exists. Assume $\lim_{k,n\to\infty}\|x_{k,n}-q\| = c$. From (8), we have

$$\|y_{k,n}-q\| \le (1+\beta_n\gamma_n)\|x_{k,n}-q\| + \beta_n\|u_{k,n}\|. \tag{13}$$

Taking lim sup on both sides in (13), we obtain

$$\lim_{k,n\to\infty}\sup\|y_{k,n}-q\| = \lim_{k,n\to\infty}\sup[(1+\gamma_n)\|x_{k,n}-q\| + \beta_n\|u_{k,n}\|] < c. \tag{14}$$

Since T_1 is asymptotically nonexpansive mapping, then

$$\|T_1(PT_1)^{n-1}y_{k,n}-q\| \le k_n\|y_{k,n}-q\|,$$

taking lim sup on both sides in this inequality, we have

$$\lim_{k,n\to\infty}\sup\|T_1(PT_1)^{n-1}y_{k,n}-q\| \le c. \tag{15}$$

Since $\lim_{k,n\to\infty}\sup\|x_{k,n+1}-q\| = c$, then

$$\lim_{k,n\to\infty}\|(1-\alpha_n)(x_{k,n}-q) + \alpha_n(T_1(PT_1)^{n-1}y_{k,n}-q) + \alpha_n v_{k,n}\| \le c.$$

By Lemma 2, we have

$$\lim_{k,n\to\infty}\|x_{k,n}-T_1(PT_1)^{n-1}y_{k,n}\| = 0. \tag{16}$$

In addition,

$$\begin{aligned}\|x_{k,n}-q\| &\le \|x_{k,n}-T_1(PT_1)^{n-1}y_{k,n}\| + \|T_1(PT_1)^{n-1}y_{k,n}-q\|\\ &\le \|x_{k,n}-T_1(PT_1)^{n-1}y_{k,n}\| + (1+\lambda_n)\|y_{k,n}-q\|.\end{aligned}$$

Taking lim inf on both sides in the above inequality and using (16), we obtain

$$\lim_{k,n\to\infty}\inf\|y_{k,n}-q\| \ge c. \tag{17}$$

Thus, it follows from (14) and (17) that $\lim_{k,n\to\infty}\inf\|y_{k,n}-q\| = c$, which implies that

$$\lim_{k,n\to\infty}\|(1-\beta_n)(x_{k,n}-q) + \beta_n(T_2(PT_2)^{n-1}x_{k,n}-q) + \beta_n u_{k,n}\| = c.$$

Then by Lemma 2, we obtain

$$\lim_{k,n\to\infty}\|x_{k,n}-T_2(PT_2)^{n-1}x_{k,n}\| = 0. \tag{18}$$

Further, by (8), we have

$$\lim_{k,n\to\infty}\|y_{k,n}-T_2(PT_2)^{n-1}x_{k,n}\| = 0. \tag{19}$$

We now prove that

$$\lim_{k,n\to\infty} \|x_{k,n} - T_2(PT_2)^{n-1}x_{k,n}\| = 0.$$

Since,

$$\begin{aligned} & \|x_{k,n} - T_1(PT_1)^{n-1}x_{k,n}\| \\ = & \|x_{k,n} - T_1(PT_1)^{n-1}y_{k,n} + T_1(PT_1)^{n-1}y_{k,n} - T_1(PT_1)^{n-1}x_{k,n}\| \\ \leq & \|x_{k,n} - T_1(PT_1)^{n-1}y_{k,n}\| + k_n\|x_{k,n} - y_{k,n}\|. \\ \leq & \|x_{k,n} - T_1(PT_1)^{n-1}y_{k,n}\| + k_n\|\beta_n(x_{k,n} - T_2(PT_2)^{n-1}x_{k,n} - u_{k,n})\|. \end{aligned}$$

Thus by (16) and (18), we have

$$\lim_{k,n\to\infty} \|x_{k,n} - T_1(PT_1)^{n-1}x_{k,n}\| = 0. \tag{20}$$

Further, it follows from (18) and (20) that

$$\lim_{k,n\to\infty} \|T_1(PT_1)^{n-1}x_{k,n} - T_2(PT_2)^{n-1}x_{k,n}\| = 0. \tag{21}$$

By (8) and (16), we still have

$$\lim_{k,n\to\infty} \|x_{k,n+1} - T_1(PT_1)^{n-1}y_{k,n}\| = 0. \tag{22}$$

Since T_1 is asymptotically nonexpansive mapping, then T_1 is uniformly L-Lipschitzian for some $L > 0$. Hence

$$\begin{aligned} & \|x_{k,n} - T_1x_{k,n}\| = \|x_{k,n} - T_1(PT_1)^{n-1}x_{k,n} + T_1(PT_1)^{n-1}x_{k,n} - T_1x_{k,n}\| \\ \leq & \|x_{k,n} - T_1(PT_1)^{n-1}x_{k,n}\| \\ + & \|T_1(PT_1)^{n-1}x_{k,n} - T_1(PT_1)^{n-1}y_{k,n-1}\| \\ + & \|T_1(PT_1)^{n-1}y_{k,n-1} - T_1x_{k,n}\| \\ \leq & \|x_{k,n} - T_1(PT_1)^{n-1}x_{k,n}\| + k_n\|x_{k,n} - y_{k,n-1}\| \\ + & L\|T_1(PT_1)^{n-2}y_{k,n} - x_{k,n}\|. \end{aligned}$$

It follows from (22) that

$$\lim_{k,n\to\infty} \|T_1(PT_1)^{n-2}y_{k,n-1} - x_{k,n}\| = 0. \tag{23}$$

In addition,

$$\begin{aligned} & \|x_{k,n+1} - y_{k,n}\| = \|x_{k,n+1} - T_1(PT_1)^{n-1}y_{k,n} + T_1(PT_1)^{n-1}y_{k,n} - y_{k,n}\| \qquad (24) \\ \leq & \|x_{k,n+1} - T_1(PT_1)^{n-1}y_{k,n}\| + \|y_{k,n} - x_{k,n}\| + \|x_{k,n} - T_1(PT_1)^{n-1}y_{k,n}\| \\ \leq & \|x_{k,n+1} - T_1(PT_1)^{n-1}y_{k,n}\| + \beta_n\|T_2(PT_2)^{n-1}x_{k,n} - x_{k,n}\| \\ + & \|x_{k,n} - T_1(PT_1)^{n-1}y_{k,n}\| + \beta_n\|u_{k,n}\|. \end{aligned}$$

Using (16), (18) and (22), we obtain

$$\lim_{k,n\to\infty} \|x_{k,n+1} - y_{k,n}\| = 0. \tag{25}$$

By (20), (21) and (24), it follows from (18) that

$$\lim_{k,n\to\infty} \|x_{k,n} - T_1 x_{k,n}\| = 0.$$

Similarly, we may show that

$$\lim_{k,n\to\infty} \|x_{k,n} - T_2 x_{k,n}\| = 0.$$

The proof is completed. □

THEOREM 3. *Let C be a non-empty closed convex subset of a uniformly convex Banach space X, and let $T_1, T_2 : C \to X$ are two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty,\ \sum_{n=1}^{\infty}(l_n - 1) < \infty,\ k_n \to 1,\ l_n \to 1 \text{ as } n \to \infty, \text{ respectively.}$$

Suppose that $\{x_{k,n}\}$ is defined by (8), where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1)$. If one of T_1 and T_2 is completely continuous, and $F(T_1) \cap F(T_2) \neq \emptyset$, then, $\{x_{k,n}\}$ converges strongly to a common fixed point of T_1 and T_2.

Proof. By Lemma 6, $\{x_{k,n}\}$ is bounded. In addition, by Lemma 7,

$$\lim_{k,n\to\infty} \|x_{k,n} - T_1 x_{k,n}\| = 0$$

also, $\lim_{k,n\to\infty} \|x_{k,n} - T_2 x_{k,n}\| = 0$, then $\{T_1 x_{x,n}\}$, and $\{T_2 x_{x,n}\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1 x_{x,n_j}\}$ of $\{T_1 x_{x,n}\}$ such that $T_1 x_{x,n_j} \to p$ as $j \to \infty$. It follows from Lemma 7 that

$$\lim_{j\to\infty} \|x_{k,n_j} - T_1 x_{k,n_j}\| = \lim_{j\to\infty} \|x_{k,n_j} - T_2 x_{k,n_j}\| = 0.$$

So be the continuity of T_1 and Lemma 6, we get that $\lim_{k,n\to\infty} \|x_{k,n} - p\|$ exists. Thus $\lim_{k,n\to\infty} \|x_{k,n} - p\| = 0$. The proof is completed. □

THEOREM 4. *Let C be a non-empty closed convex subset of a uniformly convex Banach space X, and let $T_1, T_2 : C \to X$ are two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty,\ \sum_{n=1}^{\infty}(l_n - 1) < \infty,\ k_n \to 1,\ l_n \to 1 \text{ as } n \to \infty, \text{ respectively.}$$

Suppose that $\{x_{k,n}\}$ is defined by (8), where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1)$. If one of T_1 and T_2 is demi-compact, and $F(T_1) \cap F(T_2) \neq \emptyset$, then, $\{x_{k,n}\}$ converges strongly to a common fixed point of T_1 and T_2.

Proof. Since one of T_1 and T_2 is demi-compact, $\{x_{k,n}\}$ is bounded and

$$\lim_{k,n\to\infty} \|x_{k,n} - T_1 x_{k,n}\| = \lim_{k,n\to\infty} \|x_{k,n} - T_2 x_{k,n}\| = 0,$$

then there exists subsequence $\{x_{k,n_j}\}$ converges strongly to q. It follows from Lemma 3 that $q \in F(T_1) \cap F(T_2)$. Thus, $\lim_{k,n\to\infty} \|x_{k,n} - q\|$ exists by Lemma

6. Since the subsequence $\{x_{k,n_j}\}$ of $\{x_{k,n}\}$ such $\{x_{k,n_j}\}$ converges strongly to q, then $\{x_{k,n}\}$ converges strongly to a common fixed point $q \in F(T_1) \cap F(T_2)$. The proof is therefore completed. □

THEOREM 5. *Let C be a non-empty closed convex subset of a uniformly convex Banach space X, satisfying Opial's condition. Suppose $T_1, T_2 : C \to X$ are two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty, \ \sum_{n=1}^{\infty}(l_n - 1) < \infty, k_n \to 1, l_n \to 1 \text{ as } n \to \infty, \text{ respectively.}$$

Let $\{x_{k,n}\}$ be defined by (8), where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1)$. If $F(T_1) \cap F(T_2) \neq \emptyset$, then, $\{x_{k,n}\}$ converges weakly to a common fixed point of T_1 and T_2.

Proof. For any $q \in F(T_1) \cap F(T_2)$, it follows from Lemma 6 that the limit $\lim_{k,n\to\infty} \|x_{k,n} - q\|$ exists. We now prove that $\{x_{k,n}\}$ has a unique weak subsequential limit in $F(T_1) \cap F(T_2)$. Firstly, let q_1 and q_2 are weak limits of subsequences $\{x_{k,n_i}\}$ and $\{x_{k,n_j}\}$ and of $\{x_{k,n}\}$, respectively. By Lemmas 3 and 7, we know that $q_1, q_2 \in F(T_1) \cap F(T_2)$. Since one of T_1 and T_2 is demi-compact, $\{x_{k,n}\}$ is bounded and

$$\lim_{k,n\to\infty} \|x_{k,n} - T_1 x_{k,n}\| = \lim_{k,n\to\infty} \|x_{k,n} - T_2 x_{k,n}\| = 0,$$

then there exists a subsequence $\{x_{k,n_j}\}$ converges strongly to q. It follows from Lemma 3 that $q \in F(T_1) \cap F(T_2)$. Thus, $\lim_{k,n\to\infty} \|x_{k,n} - q\|$ exists by Lemma 6. Since the subsequence $\{x_{k,n_j}\}$ of $\{x_{k,n}\}$ such $\{x_{k,n_j}\}$ converges strongly to q, then $\{x_{k,n}\}$ converges strongly to a common fixed point $q \in F(T_1) \cap F(T_2)$. The proof is completed. □

4. ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

In this section we will prove, under appropriate conditions on C (where C be a nonempty closed convex and bounded subset of a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure) that a sequence defined iteratively by: $z_{0,0} \in C$ and

$$z_{k,n+1} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n z_{k,n} + \frac{t_n}{k_n} v_{k,n} \quad n = 0, 1, 2, ... \tag{26}$$

converges strongly to a fixed point of the asymptotically nonexpansive map T.

THEOREM 6. *Let X be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, C a nonempty closed convex and bounded subset of X, $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. Let $u \in C$ be fixed, $\{t_n\}_n \subset (0, 1)$*

be such that $\lim_{n\to\infty} t_n = 1$, *and* $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. *Then,*
(i) for each integer $n \geq 0$, *there is a unique* $x_{k,n} \in C$ *such that*

$$x_{k,n} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_{k,n} + \frac{t_n}{k_n} v_{k,n};$$

and if, in addition, $\lim_{k,n\to\infty} \|x_{k,n} - Tx_{k,n}\| = 0$, *then*
(ii) the sequence $\{x_{k,n}\}_{k,n}$ *converges strongly to a fixed point of* T.

Proof. In view of Theorem 2 of [21] and Theorem 3.1 of [7], our theorem is easy to prove. □

COROLLARY 1. *Let* X *be a real reflexive Banach space with a uniformly Gâteaux differentiable norm,* C *a nonempty closed convex and bounded subset of* X, $T : C \to C$ *a completely continuous, asymptotically nonexpansive mapping with sequence* $\{k_n\} \subset [1, \infty)$. *Let* $u \in C$ *be fixed,* $\{t_n\}_n \subset (0, 1)$ *be such that* $\lim_{n\to\infty} t_n = 1$, *and* $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. *Then,*
(i) for each integer $n \geq 0$, *there is a unique* $x_{k,n} \in C$ *such that*

$$x_{k,n} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_{k,n} + \frac{t_n}{k_n} v_{k,n}$$

and if, in addition, $\lim_{k,n\to\infty} \|x_{k,n} - Tx_{k,n}\| = 0$, *then,*
(ii) the sequence $\{x_{k,n}\}_{k,n}$ *converges strongly to a fixed point of* T.

Proof. For each integer $n \geq 0$, the mapping $f_n : C \to C$ defined for each $x \in C$ by $f_n x := (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x + \frac{t_n}{k_n} v_{k,n}$ is a contraction. It follows that there exists a unique $x_{k,n} \in C$ such that $f_n x_{k,n} = x_{k,n}$. Since T is completely continuous there is a subsequence $\{Tx_{k_i,n_j}\}_{i,j}$ of $\{Tx_{k,n}\}_{k,n}$ that converges strongly to some $y^* \in C$, and since $\|x_{k_i,n_j} - Tx_{k_i,n_j}\| \to 0$ as $i, j \to \infty$, we have that $y^* = Ty^*$. The rest of the proof follows as in the proof of Theorem 6. □

THEOREM 7. *Let* X *be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure,* C *be a nonempty closed convex and bounded subset of* X, $T : C \to C$ *be an asymptotically nonexpansive mapping with sequence* $\{k_n\}_n \subset [1, \infty)$. *Let* $u \in C$ *be fixed,* $\{t_n\}_n \subset (0, 1)$ *be such that* $\lim_{n\to\infty} t_n = 1$, $t_n k_n < 1$ *and* $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$. *Define the sequence* $\{z_{k,n}\}_{k,n}$ *iteratively by* $z_{0,0} \in C$,

$$z_{k,n+1} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n z_{k,n} + \frac{t_n}{k_n} v_{k,n} \quad n = 0, 1, 2, ... \tag{27}$$

Then,
(i) for the integers $k, n \geq 0$, *there is a unique* $x_{k,n} \in C$ *such that*

$$x_{k,n} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_{k,n} + \frac{t_n}{k_n} v_{k,n}; \tag{28}$$

and if, in addition,

$$\lim_{k,n\to\infty} \|x_{k,n} - Tx_{k,n}\| = 0, \quad \lim_{k,n\to\infty} \|z_{k,n} - Tz_{k,n}\| = 0$$

then,
(ii) the sequence $\{z_{k,n}\}_{k,n}$ *converges strongly to a fixed point of* T.

Proof. From (27), we have

$$x_{k,m} - z_{k,n} = (1 - \frac{t_m}{k_m})(u - z_{k,n}) + \frac{t_m}{k_m}(T^m x_{k,m} - z_{k,n}) + \frac{t_m}{k_m} v_{k,m}.$$

Applying inequality (9), we estimate as follows:

$$\begin{aligned}
& \|x_{k,m} - z_{k,n}\|^2 \\
= \;& \|(1 - \frac{t_m}{k_m})(u - z_{k,n}) + \frac{t_m}{k_m}(T^m x_{k,m} - z_{k,n}) + \frac{t_m}{k_m} v_{k,m}\|^2 \\
\leq \;& \frac{t_m^2}{k_m^2}\|T^m x_{k,m} - z_{k,n} + v_{k,m}\|^2 + 2(1 - \frac{t_m}{k_m})\langle u - z_{k,n}\ ,\ j(x_{k,m} - z_{k,n})\rangle \\
\leq \;& \frac{t_m^2}{k_m^2}\|T^m x_{k,m} - z_{k,n}\|^2 + 2\|T^m x_{k,m} - z_{k,n}\|\|v_{k,m}\| \\
+ \;& \|v_m\|^2 + 2(1 - \frac{t_m}{k_m})[\langle u - x_{k,m}\ ,\ j(x_{k,m} - z_{k,n})\rangle + k_m^2\|x_{k,m} - z_{k,n}\|^2] \\
\leq \;& \frac{t_m^2}{k_m^2}[\|T^m x_{k,m} - T^m z_{k,n}\| + \|T^m z_{k,m} - z_{k,n}\|]^2 \\
+ \;& 2\|T^m x_{k,m} - z_{k,n}\|\ \|v_{k,m}\| \\
+ \;& \|v_{k,m}\|^2 + 2(1 - \frac{t_m}{k_m})[\langle u - x_{k,m}\ ,\ j(x_{k,m} - z_{k,n})\rangle + k_m^2\|x_{k,m} - z_{k,n}\|^2],
\end{aligned}$$

which implies that

$$\begin{aligned}
& \|x_{k,m} - z_{k,n}\|^2 \leq \frac{t_m^2}{k_m^2}[k_m^2\|x_{k,m} - z_{k,n}\| \\
+ \;& 2k_m\|x_{k,m} - z_{k,n}\|\ \|T^m z_{k,n} - z_{k,n}\| \\
+ \;& \|T^m z_{k,n} - z_{k,n}\|^2] + 2\|T^m x_{k,m} - z_{k,n}\|\|v_{k,m}\| + \|v_{k,m}\|^2 \\
+ \;& 2(1 - \frac{t_m}{k_m})[\langle u - x_{k,m}\ ,\ j(x_{k,m} - z_{k,n})\rangle + k_m^2\|x_{k,m} - z_{k,n}\|^2] \\
\leq \;& (1 - (1 - \frac{t_m}{k_m}))^2 k_m^2\|x_{k,m} - z_{k,n}\|^2 + \|T^m z_{k,n} - z_{k,n}\|[2k_m\|x_{k,m} - z_{k,n}\| \\
+ \;& \|T^m z_{k,n} - z_{k,n}\| + 2\|v_m\|] + 2k_m\|x_{k,m} - z_{k,n}\|\ \|v_{k,m}\| \\
+ \;& 2(1 - \frac{t_m}{k_m})[\langle u + v_{k,m} - x_{k,m}\ ,\ j(x_{k,m} - z_{k,n})\rangle + k_m^2\|x_{k,m} - z_{k,n}\|^2] \\
\leq \;& (1 + (1 - \frac{t_m}{k_m})^2)k_m^2\|x_{k,m} - z_{k,n}\|^2 + \|T^m z_{k,n} - z_{k,n}\|M \\
+ \;& 2k_m\|x_{k,m} - z_{k,n}\|\ \|v_{k,m}\| + 2(1 - \frac{t_m}{k_m})\langle u - x_{k,m}\ ,\ j(x_{k,m} - z_{k,n})\rangle,
\end{aligned}$$

for some constant $M > 0$. It follows that

$$\begin{aligned}
&\lim_{k,n\to\infty} \sup\langle u - x_{k,m}, j(z_{k,n} - x_{k,m})\rangle \\
\leq\; & \frac{[k_m^2 - 1 + k_m^2(1 - \frac{t_m}{k_m})^2]}{2(1 - \frac{t_m}{k_m})} \lim_{k,n\to\infty} \sup \|x_{k,m} - z_{k,n}\| \\
+\; & \lim_{k,n\to\infty} \sup \frac{M\|z_{k,n} - T^m z_{k,n}\|}{(1 - \frac{t_m}{k_m})}.
\end{aligned}$$

Observe that $\frac{[k_m^2-1+k_m^2(1-\frac{t_m}{k_m})^2]}{2(1-\frac{t_m}{k_m})} = \frac{k_m(k_m+1)}{2}[\frac{k_m-1}{k_m-t_m}] + \frac{k_m^2}{2}(1-\frac{t_m}{k_m}) \to 0$, $m \to \infty$. Since $\{z_{k,n}\}$ and $\{x_{k,m}\}$ are bounded, $\{T^m z_{k,n}\}$ is bounded and $\|z_{k,n} - Tz_{k,n}\| \to 0$ as $k, n \to \infty$, it follows from the last inequality that

$$\lim_{k,m\to\infty} \sup \lim_{k,n\to\infty} \sup\langle u - x_{k,m} \,,\, j(z_{k,n} - x_{k,m})\rangle \leq 0. \tag{29}$$

But by Theorem 6 we have that $x_{k,m} \to x^* \in F(T)$ as $k, m \to \infty$. Moreover, j is norm to $weak^*$ uniformly continuous on bounded sets. Therefore, there exists $N > 0$ such that

$$|\langle x^* - x_{k,m} \,,\, j(z_{k,n} - x_{k,m})\rangle| < \frac{\epsilon}{2} \text{ and}$$
$$|\langle u - x^* \,,\, j(z_{k,n} - x_{k,m}) - j(z_{k,n} - x^*)\rangle| < \frac{\epsilon}{2},$$

for all $n, m \geq N$. This implies that

$$\begin{aligned}
&|\langle u - x_{k,m} \,,\, j(z_{k,n} - x_{k,m})\rangle - \langle u - x^* \,,\, j(z_{k,n} - x^*)\rangle| \\
&\leq |\langle u - x_{k,m} \,,\, j(z_{k,n} - x_{k,m})\rangle - \langle u - x^* \,,\, j(z_{k,n} - x^*)\rangle| \\
&+|\langle u - x^* \,,\, j(z_{k,n} - x_{k,m})\rangle - \langle u - x^* \,,\, j(z_{k,n} - x^*)\rangle| \\
&= |\langle x^* - x_{k,m} \,,\, j(z_{k,n} - x_{k,m})\rangle \\
&+\langle u - x^* \,,\, j(z_{k,n} - x_{k,m}) - j(z_{k,n} - x^*)\rangle| < \epsilon
\end{aligned} \tag{30}$$

for all $n, m \geq N$. Thus, from (29) and (30),

$$\begin{aligned}
&\lim_{k,n\to\infty} \sup\langle u - x^* \,,\, j(z_{k,n} - x^*)\rangle \\
&\leq \lim_{k,m\to\infty} \sup \lim_{k,n\to\infty} \sup\langle u - x_{k,m} \,,\, j(z_{k,n} - x_{k,m})\rangle + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain that

$$\lim_{k,n\to\infty} \sup\langle u - x^* \,,\, j(z_{k,n} - x^*)\rangle \leq 0.$$

Now from the iteration procedure (27) and inequality (9) we have that

$$\begin{aligned}
\|z_{k,n+1} - x^*\|^2 &\leq \frac{t_n^2}{k_n^2}\|T^n z_{k,n} - x^*\|^2 + 2(1 - \frac{t_n}{k_n})\langle u - x^* \,,\, j(z_{k,n+1} - x^*)\rangle. \\
&\leq \frac{t_n}{k_n}\|z_{k,n} - x^*\|^2 + 2(1 - \frac{t_n}{k_n})\langle u - x^* \,,\, j(z_{k,n+1} - x^*)\rangle, \\
&\leq (1 - \alpha_n)\|z_{k,n} - x^*\|^2 + 2\alpha_n\beta_n
\end{aligned}$$

where $\alpha_n = (1 - \frac{t_n}{k_n})$. So $\beta_n := \langle u - x^* \ , \ j(z_{k,n+1} - x^*)\rangle$, and hence $\limsup \alpha_n \beta_n \leq 0$. Now it follows from Lemma 1 that $z_{k,n} \to x^*$ as $k, n \to \infty$, completing the proof. □

Corollary 2. *Let X be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, and let C be a nonempty closed convex and bounded subset of X, and let $T : C \to C$ be a completely continuous, asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$. Let $u \in C$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim_{n\to\infty} t_n = 1$, $t_n k_n < 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{z_{k,n}\}_{k,n}$ iteratively by $z_{0,0} \in C$,*

$$z_{k,n+1} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n z_{k,n} + \frac{t_n}{k_n} v_{k,n} \quad n = 0, 1, 2, ... \tag{31}$$

Then,
(i) for the integers $k, n \geq 0$, there is a unique $x_{k,n} \in C$ such that

$$x_{k,n} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_{k,n} + \frac{t_n}{k_n} v_{k,n}$$

and if, in addition,

$$\lim_{k,n\to\infty} \|x_{k,n} - Tx_{k,n}\| = 0, \ \textit{and} \ \lim_{k,n\to\infty} \|z_{k,n} - Tz_{k,n}\| = 0, , \ \textit{then}$$

(ii) the sequence $\{z_{k,n}\}_{k,n}$ converges strongly to a fixed point of T.

Proof. As in the proof of Corollary 1 there exists $y^* \in C$ such that $Ty^* = y^*$. The rest of the proof follows as in Theorem 7. □

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